CONIC STATIONARY SOLUTIONS OF ONE RESTRICTED THREE-BODY PROBLEM

Antonio Elipe, Manuel Palacios and Halina Prętka-Ziomek

Abstract. The equations of motion of one three-body problem composed of a dumb-bell (two masses at fixed distance) moving around a central mass under gravitational effects have been established. Conic stationary solutions of these equations have been studied and sufficient conditions for stability have been found in term of Lyapunov’s stability functions.

Keywords: Three-body problem, dumb-bell problem, stationary solutions.
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§1. Introduction

The interest of the study of the motion of a system composed of three material points $M_1, M_2$ and $M_3$ interacting by Newtonian law, in the assumption that the distance between $M_2$ and $M_3$ is constant, i.e., the points $M_2$ and $M_3$ form a dumb-bell, derives from the fact that it is the simplest problem about translational-rotary motion of a satellite in a gravitational field and gives the generic connections between the solution of this restricted three body problem and the classical one [2]. Particular cases of this problem can be equivalent to the classical restricted three bodies problem or to the generalized two fixed centres [1]. Not far from this is the problem of the motion of a point in the gravitational field created by a massic segment as an approximation to an elongate body [7, 8], as it is the case in some asteroids. The purpose of this paper is the study of the so call conic stationary solutions of the problem for arbitrary masses of the bodies and arbitrary size of the dumbbell. Other particular cases as the linear and isosecules cases have already been studied by the authors [4]. The conic motions is a solutions in which the dumb-bell axis describes a conic circular surface with axis orthogonally disposed with respect to the parallel planes in which the points move around. The constant semiangle $\theta$ of the conic surface is the same as the angle between axis of the dumb-bell and the $Gz$ axis, $G$ being the center of masses; the distance $z$ between the planes in which the motion of $M_1$ and the center of masses of the dumb-bell is performed remains constant. The values of $z$ and $\theta$ are not independent and result from the roots of an algebraic equation. These solutions describe the effect of the displacement of the center of masses with respect to the angle $\theta$ and other parameters of the problem. We also give sufficient conditions for stability [6].

§2. Formulation of the problem

The system of study is composed of three material points $M_1, M_2$ and $M_3$, of masses $m_1, m_2$ and $m_3$, mutually attracted by the Newtonian gravitational forces. It will be assumed that
the points $M_2$ and $M_3$ are rigidly connected by a segment of constant length $l$ and negligible mass, i.e., they form a dumb-bell.

Let $C$ be the center of masses of the dumb-bell and $l_2$, $l_3$ the distances from $M_2$ and $M_3$ to $C$.

The simplest way to study the problem of motion of that system is to consider it referred to an inertial heliocentric frame $S(M_1, s_1, s_2, s_3)$ and to use Hamiltonian formulation [4]. The center of masses of the dumb-bell is defined by the cylindrical coordinates $(r, z, \lambda)$ and the attitude of the dumb-bell in $S$ is given by two angles, namely nutation $\theta$ and precession $\phi$ (see the figure 1).

We can define an orthonormal rotating frame $(C; b_1, b_2, b_3)$ (see figure 1) made of the principal axes of inertia, where $\cos \theta = s_3 \cdot b_3$ and $b_1 = \frac{s_3 \times b_3}{|s_3 \times b_3|}$.

In these heliocentric coordinates, the Hamiltonian may be expressed as (see [9, 4])

$$H = \frac{1}{2m} \left( P_r^2 + \frac{(P_\omega - P_\psi)^2}{r^2} + P_z^2 \right) + \frac{1}{2A} \left( \frac{P_\psi^2}{\sin^2 \theta} + P_\theta^2 \right) + U(r, z, \psi, \theta),$$

where the potential function is

$$U = -G m_1 \left( \frac{m_2}{r_{12}} + \frac{m_3}{r_{13}} \right),$$

the mutual distances $r_{1j}$, for $j = 2, 3$, are

$$r_{1j}^2 = r^2 + z^2 + l_j^2 - (-1)^j 2 l_j \left[ z \cos \theta + r \sin \theta \sin(\phi - \lambda) \right]$$

and

$$\psi = \phi - \lambda, \quad P_\psi = P_\phi,$$
$$\omega = \lambda, \quad P_\omega = P_\theta + P_\lambda,$$

and $m$ and $A$ are the following constants:

$$m = \frac{m_1(m_2 + m_3)}{m_1 + m_2 + m_3} \quad \text{and} \quad A = \frac{m_2 m_3}{m_2 + m_3} l^2.$$
With this election of variables the problem is reduced to four degrees of freedom. Since angle $\omega$ is cyclic, its conjugate moment $P_\omega$ is an integral of the motion. The Hamiltonian itself is another integral.

Then the equations of motion are

\[
\dot{r} = \frac{P_r}{m}, \quad \dot{P}_r = \frac{(P_\omega - P_\psi)^2}{mr^3} - \frac{\partial U}{\partial r},
\]

\[
\dot{z} = \frac{P_z}{m}, \quad \dot{P}_z = -\frac{\partial U}{\partial z},
\]

\[
\dot{\theta} = \frac{P_\theta}{A}, \quad \dot{P}_\theta = \frac{P_\psi^2 \cos \theta}{A \sin^3 \theta} - \frac{\partial U}{\partial \theta},
\]

\[
\dot{\psi} = -\frac{P_\omega - P_\psi}{mr^2} + \frac{P_\psi}{A \sin^3 \theta}, \quad \dot{P}_\psi = -\frac{\partial U}{\partial \psi},
\]

and equilibria are found by zeroing this system. Thus, there results that

\[
P_r = P_z = P_\theta = 0, \quad P_\omega - P_\psi = \frac{mr^2}{A \sin^2 \theta} P_\psi,
\]

and

\[
\frac{\partial U}{\partial r} = \frac{mr}{A^2 \sin^4 \theta} P_\psi^2, \quad \frac{\partial U}{\partial z} = 0,
\]

\[
\frac{\partial U}{\partial \theta} = \frac{A \sin \theta \cos \theta}{mr} \frac{\partial U}{\partial r}, \quad \frac{\partial U}{\partial \psi} = 0.
\]

Defining the shortcuts

\[
F = \mathcal{G} \left( \frac{m_3}{r_{13}^3} + \frac{m_2}{r_{12}^3} \right), \quad G = \mathcal{G} \left( \frac{m_3 l_3}{r_{13}^3} - \frac{m_2 l_2}{r_{12}^3} \right),
\]

the partial derivatives of the potential $U$ may be put as

\[
\frac{\partial U}{\partial r} = Fr + G \sin \theta \sin \psi, \quad \frac{\partial U}{\partial z} = Fz + G \cos \theta, \quad \frac{\partial U}{\partial \theta} = G(-z \sin \theta + r \cos \theta \sin \psi), \quad \frac{\partial U}{\partial \psi} = G \sin \theta \cos \psi,
\]

and equations for equilibria reduce to

\[
P_\psi = \frac{A \sin^2 \theta}{mr^2 + A \sin^2 \theta} P_\omega
\]

\[
Fr + G \sin \theta \sin \psi = \frac{mr}{A^2 \sin^4 \theta} P_\psi^2
\]

\[
Fz + G \cos \theta = 0,
\]

\[
A \sin \theta \cos \theta (Fr + G \sin \theta \sin \psi) - mr G(-z \sin \theta + r \cos \theta \sin \psi) = 0,
\]

\[
Gr \sin \theta \cos \psi = 0.
\]
The finding of general solution of this system is rather complicated, hence, we will only look for conic solutions, i.e., verifying $z \cos \theta \neq 0$. Cases $r = 0$ and $\theta = 0$ will be excluded since they correspond to singularities of the problem. Other solutions as linear and isosceles cases have already been studied by the authors [4].

§3. Conic stationary motion

3.1. Existence of conic motions

We will consider here not plane stationary solutions of equations (3)–(7) that satisfy $z \cos \theta \neq 0$.

One particular solution to the equation (7) corresponds to $\psi = \pi/2$ or $3\pi/2$. In this case (see figure 2), the three bodies $M_1$, $M_2$ and $M_3$ lay on the same plane $M_1M_2s_3$, and the axis of the dumb-bell describes a conic surface around the axis $M_1s_3$ with semiangle at the apex $\theta$; the line passing through $M_1$ and $C$ also describes a conic surface of semi-angle $\beta$, $\cos \beta = z/r$. Hence, we can call it conic solution.

All the bodies must move around $G_3$ axis ($G$ being the center of masses of the whole system) along circles in planes orthogonal to it with frequency of rotation $\dot{\phi}$ and radius given, respectively, by

\[
\rho_1 = \frac{m_2 + m_3}{m_1 + m_2 + m_3} r, \quad \rho_2 = \frac{m_1}{m_1 + m_2 + m_3} r + l_2 \sin \theta, \quad \rho_3 = \frac{m_1}{m_1 + m_2 + m_3} r - l_3 \sin \theta.
\]

Let us note that for $z = 0$ and $\theta \neq \pi/2$, from (5), it must be $G = 0$, it is to say, $r_{12} = r_{13}$, and again (from (6)) $\theta = \pi/2$, in contradiction with the hypothesis. So, we will study conic stationary solutions with $z \neq 0$ and $\theta \neq \pi/2$. Let us deduce the existence of these equilibria studying the rest of equations. Writing $\varepsilon = \sin \psi = \pm 1$, equations (5) and (6) become:

\[
F z + G \cos \theta = 0, \quad (8)
\]

\[
A \sin \theta \cos \theta F r + (\varepsilon A \sin^2 \theta \cos \theta - m r (-z \sin \theta + \varepsilon r \cos \theta)) G = 0, \quad (9)
\]
These equations compose an indeterminate compatible linear system in the variables $F, G$ (defined by (2)) if its determinant vanishes, it is to say, if

$$(m r z + \varepsilon A \sin \theta \cos \theta) (\varepsilon r \cos \theta - z \sin \theta) = 0,$$

what gives us two interesting particular solutions:

$$r_1 = \varepsilon z \tan \theta,$$

$$r_2 = -\frac{\varepsilon A}{2mz} \sin 2\theta. \quad (11)$$

### 3.2. Case I: $r = \varepsilon z \tan \theta$

Conditions of equilibria are now written as

$$P_{\psi} = \frac{A \sin^2 \theta}{m r^2 + A \sin^2 \theta} P_{\omega}, \quad (12)$$

$$F r + \varepsilon G \sin \theta = \frac{m r}{A^2 \sin^4 \theta} P_{\psi}^2 = \frac{m r}{(m r^2 + A \sin^2 \theta)^2} P_{\omega}^2, \quad (13)$$

$$z F + G \cos \theta = 0, \quad (14)$$

$$\varepsilon r \cos \theta - z \sin \theta = 0. \quad (15)$$

The frequency of the motion is given by

$$\dot{\omega}^2 = \frac{P_{\omega}^2}{(m r^2 + A \sin^2 \theta)^2} = \frac{F r + \varepsilon G \sin \theta}{m r} = \varepsilon \tan \theta \frac{F z + G \cos \theta}{m r} = 0.$$

It means that the three bodies are situated at fixed positions on a straight line. It must be $P_{\omega} = 0$, hence, condition (13) reduces to condition (14).
Now, taking $\eta$ as the distance $M_1$ to $C$ and $\nu = m_3/m_2$, condition (14) is transformed into
\[(\eta + I_3)(l_2 - \eta) = \nu(l_2 - \eta)^3(I_3 + \eta), \quad \text{or} \quad (\eta + \frac{l}{1+\nu})^2 = \nu\left(\frac{\nu l}{1+\nu} - \eta\right)^2,
\]
that provides the following solutions:
\[\eta_1 = l \frac{-1 + \nu \sqrt{\nu}}{(1+\nu)(1 + \sqrt{\nu})}, \quad (16)
\]
\[\eta_2 = l \frac{1 + \nu \sqrt{\nu}}{(1+\nu)(-1 + \sqrt{\nu})}.
\]

3.3. Case II: $r = \frac{\varepsilon A \sin 2\theta}{2mz}$

Conditions of equilibria are now written as
\[P\psi = \frac{A \sin^2 \theta}{m r^2 + A \sin^2 \theta} P\omega, \quad (18)
\]
\[F r + \varepsilon G \sin \theta = \frac{m r}{A^2 \sin^4 \theta} P^2 = \frac{m r}{(m r^2 + A \sin^2 \theta)^2} P^2\omega, \quad (19)
\]
\[z F + G \cos \theta = 0, \quad (20)
\]
\[m r z + \varepsilon A \sin \theta \cos \theta = 0. \quad (21)
\]
The frequency of the motion is given by
\[\omega^2 = \frac{P^2}{(m r^2 + A \sin^2 \theta)^2} = \frac{F r + \varepsilon G \sin \theta}{m r} \geq 0,
\]
hence, taking into account (20),
\[0 \leq F r + \varepsilon G \sin \theta = F (r - \varepsilon z \tan \theta) \implies r \geq \varepsilon z \tan \theta \iff \sin \theta \leq \varepsilon.
\]

Introducing the variable $\zeta = z/(l \cos \theta)$, condition (20) can be written in the following form:
\[r_{13}^3 [v - \zeta (1 + \nu)] = r_{12}^3 [v + \nu (1 + \nu) \zeta],
\]
where $r_{12}$ and $r_{13}$ are now written as
\[r_{12}^2 = l^2 \left[\frac{v^2}{(1 + \nu)^2} - \frac{2 v \zeta \cos^2 \theta}{1 + \nu} + \zeta^2 \cos^2 \theta\right] + \frac{A^2 \sin^2 \theta}{l^2 m^2 \zeta^2} - \frac{2 A \nu \sin^2 \theta}{m(1 + \nu) \zeta},
\]
\[r_{13}^2 = l^2 \left[\frac{1}{(1 + \nu)^2} + \frac{2 \zeta \cos^2 \theta}{1 + \nu} + \zeta^2 \cos^2 \theta\right] + \frac{A^2 \sin^2 \theta}{l^2 m^2 \zeta^2} + \frac{2 A \sin^2 \theta}{m(1 + \nu) \zeta}.
\]

In this way, the equation (20) is equivalent to a polynomal one of degree thirteen in the variable $\zeta$ with coefficients known function of constants of the problem.
§4. Sufficient conditions for stability of the conic solutions.

The stationary solutions are defined by the following found values:

\[ P_x^{(0)} = P_y^{(0)} = P_{z}^{(0)} = 0, \quad r^{(0)}, \quad z^{(0)}, \quad \psi^{(0)}, \quad \theta^{(0)}. \]

Introducing the vector \( \mathbf{v} = (y_1, y_2, y_3, y_4, x_1, x_2, x_3, x_4) \) of variations of the coordinates and momenta

\[ y_1 = P_r, \quad y_2 = P_z, \quad y_3 = P_\psi - P_\psi^{(0)}, \quad y_4 = P_\theta, \]
\[ x_1 = r - r^{(0)}, \quad x_2 = z - z^{(0)}, \quad x_3 = \psi - \psi^{(0)}, \quad x_4 = \theta - \theta^{(0)}, \]

the Hamiltonian of the linearized perturbed problem \[2\] is, formally, the same as the non-linearized, but with coefficients evaluated at the equilibrium solution. Consequently, the quadratic part of the Hamiltonian of the linearized perturbed problem is the sum of a positive defined part, the kinetic energy, and the Hessian of the potential energy. This last part is

\[ V_2 = \frac{1}{2} \sum_{i,j=1}^{4} V_{ij} x_i x_j, \quad (22) \]

where \( V_{ij} \) are the following second derivatives of the potential evaluated at the equilibrium solution. We will use this function as a Lyapunov function for our analysis of the stability. In this way, the Lyapunov’s stability of the stationary solutions follows (taking into account the Dirichlet theorem) from the fact that the quadratic form \( (22) \) be positively defined, i.e., in agreement with the Jacobi’s criterium, if all the principal minors of the matrix which elements are \( (V_{ij}) \) have positive value. In the case of conic solutions, the matrix \( (V_{ij}) \) has \( V_{13} = V_{23} = V_{34} = 0 \), hence, the conditions of Jacobi’s criterium become:

\[ V_{11} > 0, \quad V_{11} V_{22} - V_{12}^2 > 0, \quad V_{33} > 0, \quad \det \begin{bmatrix} V_{11} & V_{12} & V_{14} \\ V_{12} & V_{22} & V_{24} \\ V_{14} & V_{24} & V_{44} \end{bmatrix} > 0. \quad (24) \]

These conditions, in the case I, reduce to

\[ V_{11} = F - 3G m_1 \left[ \frac{m_2}{r_{12}^2} (\eta - l_2)^2 + \frac{m_3}{r_{13}^2} (\eta^2 - l_3^2) \right] \sin^2 \theta \geq 0, \]
\[ V_{11} V_{22} - V_{12}^2 = F \left( F - 3G m_1 \left[ \frac{m_2}{r_{12}^2} (\eta - l_2)^2 + \frac{m_3}{r_{13}^2} (\eta^2 - l_3^2) \right] \right) \geq 0, \]
\[ V_{33} = \varepsilon \eta G \sin^2 \theta \geq 0. \]

Analogously, we should proceed in the case II.

§5. Conclusions

The equations of motion of one three-body problem composed of a dumb-bell (two masses at fixed distance) moving around a central mass have been established. Conic stationary solutions
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of these equations have studied and sufficient conditions for stability has been found in term of Lyapunov’s stability functions. It seems that this could be a good approximation for the study of the motion of a body around to a massic segment and, one more general situation, the motion of two solid bodies under a gravitational field.

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