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ON THE STABILITY CONDITIONS FOR A HEAVY GYROSTAT

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Abstract. We consider the stability of some permanent rotations of a heavy gyrostat with a fixed point. Provided the system can be regarded as a Lie-Poison one, by means of the Energy-Casimir method, we give sufficient stability conditions in terms of the parameters defining the geometry of the spinning body and also in terms of the gyrostatic moments. We prove that two rotors are enough to get stable permanent rotations along any space direction. Besides, for some configurations of the gyrostat, sufficient conditions are also necessary.

Keywords: gyrostat rotation, stability, Energy–Casimir method.

AMS classification: 70E55; 37J25; 37N05.

§1. Introduction

A gyrostat $\mathcal{G}$ is a mechanical system composed of a rigid body $\mathcal{P}$, the platform, and other bodies $\mathcal{R}$, the rotors, connected to $\mathcal{P}$ in such a way that their motion does not modify the mass distribution of the system. The first one to introduce this model seems to be Zhukovskii [5], a little before Volterra used it to describe the Earth's rotational motion [25]. Noadays, gyrostats serve, in the field of Astrodynamics, to model a spacecraft, where the rotors are used to stabilize its rotations, see e.g. [6, 12, 14, 15, 20].

When there are no external torques, the motion of a gyrostat is an extension of the classical problem of a rigid body in torque free motion. Despite this problem is integrable, there is a lot of literature related to it, due to the number of parameters involved, as the principal moments of inertia and the gyrostatic moments. As a consequence, a great variety of bifurcations appear and stable rotations can turn into unstable ones [8, 9, 10, 23, 24]. However, this is a simplified model of a more complex one, when internal or external torques act on the gyrostat. Here, we focus on the motion of a gyrostat, when it is subject to a uniform gravity field. In particular, on the stability of permanent rotations around the vertical axis, along which the gravity force acts. Different methods allow to derive stability conditions for permanent rotations. The classical way uses Lyapunov functions [1, 2, 18, 23], but it is also possible to get insight by analyzing the invariant manifolds and their bifurcations [11]. However, these results can also be obtained and, in some cases, improved by using the Energy-Casimir method [21, 22], provided the problem is a Lie-Poison system. Indeed, the Energy-Casimir method has been successfully used in rigid body dynamics [4, 7] and more recently in the study the stability of permanent rotations of a heavy gyrostat [16, 17, 19].

Our aim is to extend the previous results to the case of two spinning rotors aligned along two of the principal axes of inertia, when the center of mass is on one of them. In this way,

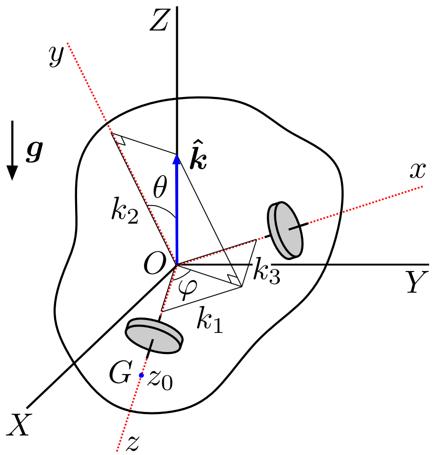

Figure 1: Asymmetric gyrostat and reference frames.

we address two families of permanent rotations, we name E_0 and E_1 for which we obtain sufficient conditions of stability. We find that the family E_1 is stable if I_2 is the largest moment of inertia or if $I_3 > I_2 > I_1$ and one of the gyrostatic moments is great enough. In the other cases not covered by the above conditions, we also obtain stable rotations by turning off one of the rotors and turning on the the rotor along the other principal axis. Regarding the other family of equilibria, E_0, we find that it is a limit case of the other one and, it does not matter the geometry of the body, an appropriate combination of the spinning rotors produce stable permanent rotations. Hence, in the case here considered, given a rotation axis, the action of the rotors leads to stable permanent rotations provided that the center of mass is lying on one of the principal axis, which is the most frequent practical case.

§2. Equations of motion and equilibrium solutions

Let us consider an asymmetric gyrostat with two rotors in a uniform gravity field. We assume that their axes are aligned with the principal axes of the platform and that the whole gyrostat rotates with a fixed point O, which may be different of the center of mass G.

To describe the problem, we introduce two orthonormal reference frames centered at the fixed point O (see Fig. 1). On the one hand, the space or inertial reference frame $\mathcal{F}\{O, X, Y, Z\}$ fixed in the space, with the direction of the Z axis opposite to the acceleration $\mathbf{g}$ of the gravity field. On the other hand, the body frame $\mathcal{B}\{O, x, y, z\}$ fixed with the gyrostat, whose axes coincide with the principal axes of inertia of the gyrostat. The relative motion of the reference frames is described by three consecutive rotations involving three angles, for instance the Euler angles. However, only two of them are needed to define the orientation of the rotating gyrostat in the inertial frame $\mathcal{F}$. Let $\mathbb{I} = (I_1, I_2, I_3)$ and $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ be the inertia tensor and the angular velocity of the gyrostat, respectively, expressed in the body frame $\mathcal{B}$. Thus, when considered as a rigid body, the angular momentum of the gyrostat is given by $\boldsymbol{\pi} = \mathbb{I}\boldsymbol{\omega}$. Now, let $\mathbf{l} = (l_1, l_2, l_3)$ be the angular momentum of the rotors in the body frame and assume

$l_2 = 0$. Moreover, $\hat{\mathbf{k}} = (k_1, k_2, k_3)$ is the unitary vector in the direction of the fixed Z axis, which can expressed in the body frame $\mathcal{B}$ as

$$\hat{\mathbf{k}} = (\sin\varphi\sin\theta, \cos\theta, \cos\varphi\sin\theta), \tag{1}$$

where the angles $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$ give us the orientation of the gyrostat with respect to the inertial reference frame $\mathcal{F}$ (see Fig. 1). If $(0, 0, z_0)$ are the coordinates of the center of mass G in the body frame, the equations of the motion result to be (see e.g. [5])

$$\begin{aligned}
\frac{d\pi_1}{dt} &= \left(\frac{I_2 - I_3}{I_2 I_3}\right)\pi_2\pi_3 - \frac{l_3\pi_2}{I_2} + mgz_0k_2,\\
\frac{d\pi_2}{dt} &= \left(\frac{I_3 - I_1}{I_1 I_3}\right)\pi_1\pi_3 + \frac{l_3\pi_1}{I_1} - \frac{l_1\pi_3}{I_3} - mgz_0k_1,\\
\frac{d\pi_3}{dt} &= \left(\frac{I_1 - I_2}{I_1 I_2}\right)\pi_1\pi_2 + \frac{l_1\pi_2}{I_2},\\
\frac{dk_1}{dt} &= \frac{k_2\pi_3}{I_3} - \frac{k_3\pi_2}{I_2},\\
\frac{dk_2}{dt} &= \frac{k_3\pi_1}{I_1} - \frac{k_1\pi_3}{I_3},\\
\frac{dk_3}{dt} &= \frac{k_1\pi_2}{I_2} - \frac{k_2\pi_1}{I_1}.
\end{aligned} \tag{2}$$

Permanent rotations are the equilibrium solutions of Eqs. (2). In this way, we have the following result.

Theorem 1. *There are two families of equilibrium points. The first one is given by those points of the form*

$$E_0 \equiv (I_1\omega\sin\varphi, 0, I_3\omega\cos\varphi, \sin\varphi, 0, \cos\varphi),$$

where $\varphi \in [0, 2\pi)$ *and* $\omega \in \mathbb{R}$ *such that*

$$(l_3\omega - gmz_0)\sin\varphi - \omega\cos\varphi(l_1 + (I_1 - I_3)\omega\sin\varphi) = 0. \tag{3}$$

The second one is defined by points of the form

$$E_1 \equiv (I_1\omega\sin\varphi\sin\theta, I_2\omega\cos\theta, I_3\omega\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta, \cos\varphi\sin\theta),$$

where $\varphi \in [0, 2\pi)$*,* $\theta \in [0, \pi/2) \cup (\pi/2, \pi]$ *and* $\omega \in \mathbb{R}$ *such that*

$$l_1 + (I_1 - I_2)\omega\sin\varphi\sin\theta = 0, \qquad (I_2 - I_3)\omega^2\cos\varphi\sin\theta + gmz_0 - l_3\omega = 0. \tag{4}$$

Proof. The proof is strightforward, taking into account that we are looking for equilibrium solutions of the form

$$\pi_1 = \omega I_1\sin\varphi\sin\theta, \quad \pi_2 = \omega I_2\cos\theta, \quad \pi_3 = \omega I_3\cos\varphi\sin\theta, \tag{5}$$

where ω is the modulus of the angular velocity and the components of the unit vector $\hat{\mathbf{k}}$ are those in equation (1). □

Remark 1. It is worth noting that the family E_0 is a limit case of the family E_1, when $\theta = \pi/2$. However, the two conditions (4) do not need to be satisfied at the same time, but only the linear combination (3). In addition, we remark that E_0 and E_1 cover any possible orientation of the two reference frames, if the corresponding gyrostatic moments verify appropriate conditions.

§3. Stability analysis

Let us analyze the stability of the equlibrium solutions E_0 and E_1. To this end, we take into account that (2) is a Lie-Poisson system (see [5, 7]), whose associated Hamiltonian function is given by

$$\mathcal{H} = \frac{1}{2}\left(\frac{\pi_1^2}{I_1} + \frac{\pi_2^2}{I_2} + \frac{\pi_3^2}{I_3}\right) + mgz_0k_3, \tag{6}$$

and the corresponding Poisson bracket defined as

$$\{\mathcal{F},\mathcal{G}\}(\boldsymbol{\pi},\hat{\mathbf{k}}) = -(\boldsymbol{\pi}+\mathbf{l})\cdot(\nabla_\pi\mathcal{F}\times\nabla_\pi\mathcal{G}) - \hat{\mathbf{k}}\cdot(\nabla_\pi\mathcal{F}\times\nabla_k\mathcal{G} + \nabla_k\mathcal{F}\times\nabla_\pi\mathcal{G}). \tag{7}$$

Moreover, there are two Casimir functions:

$$C_1 \equiv k_1^2 + k_2^2 + k_3^2 = 1, \tag{8}$$

$$C_2 \equiv (\pi_1 + l_1)k_1 + \pi_2k_2 + (\pi_3 + l_3)k_3 = p_\psi, \tag{9}$$

where p_ψ is the component of the total angular momentum $\boldsymbol{\pi} + \mathbf{l}$ along the fixed Z axis. Now, by using the two Casimir functions, we can define the augmented Hamiltonian given by

$$\mathcal{H}_A = \frac{1}{2}\left(\frac{\pi_1^2}{I_1} + \frac{\pi_2^2}{I_2} + \frac{\pi_3^2}{I_3}\right) + mgz_0k_3 + ((\pi_1+l_1)k_1 + \pi_2k_2 + (\pi_3+l_3)k_3)\lambda + (k_1^2+k_2^2+k_3^2)\mu, \tag{10}$$

where λ and μ are suitable parameters, in such a way that E_0 and E_1 are critical points of $\mathcal{H}_A$.

Under these considerations, we can give sufficient stability conditions by using the classical Energy-Casimir method [3, 13]. In particular, a generalized result given by Ortega and Ratiu [22], which reads as

Theorem 2 (Generalized energy-Casimir method)**.** *Let $(M, \{.,.\}, h)$ be a Poisson system, and $m \in M$ be an equilibrium of the Hamiltonian vector field X_h. If there is a set of conserved quantities $C_1, \ldots, C_n \in C^\infty(M)$ for which*

$$\mathbf{d}(h + C_1 + \cdots + C_n)(m) = 0,$$

and

$$\mathbf{d}^2(h + C_1 + \cdots + C_n)(m)\Big|_{W\times W}$$

is definite for $W = \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m)$, then m is stable. If $W = \{0\}$, m is always stable.

In order to apply Theorem 2, the first step is to identify the space $W = \ker \mathbf{d}C_1 \cap \ker \mathbf{d}C_2$, where C_1 and C_2 are given by Eqs. (8) and (9). In this way, using (1) and (5), we obtain $W = \operatorname{span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\})$, where

$$\begin{cases} \mathbf{v}_1 = \hat{\mathbf{e}}_1 \cos\varphi \sin\theta - \hat{\mathbf{e}}_3 \cos\theta, \\ \mathbf{v}_2 = \hat{\mathbf{e}}_2 \cos\varphi \sin\theta - \hat{\mathbf{e}}_3 \sin\varphi \sin\theta, \\ \mathbf{v}_3 = \hat{\mathbf{e}}_3(l_3 \sec\varphi \cot\theta + (I_3 - I_2)\omega \cos\theta) + \hat{\mathbf{e}}_4 \cos\varphi \sin\theta - \hat{\mathbf{e}}_6 \cos\theta, \\ \mathbf{v}_4 = \hat{\mathbf{e}}_3(l_3 \tan\varphi - l_1 - (I_1 - I_3)\omega \sin\varphi \sin\theta) + \hat{\mathbf{e}}_5 \cos\varphi \sin\theta - \hat{\mathbf{e}}_6 \sin\varphi \sin\theta, \end{cases}$$

provided $\cos\varphi \sin\theta \neq 0$. Now, let us consider a vector $\mathbf{v}$ in W, expressed as

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4,$$

where $x_i \in \mathbb{R}$, $i = 1, \ldots, 4$. Thus, the quadratic form

$$\mathbf{d}^2(h + C_1 + \cdots + C_n)(m)\Big|_{W\times W}$$

in the variables x_i is obtained from $\mathbf{v}^T \cdot \operatorname{Hess}(\mathcal{H}_A) \cdot \mathbf{v}$, the coefficients of the Hessian matrix given by

$$\begin{aligned}
h_{11} &= \frac{1}{I_2 I_3}(I_2 \cos^2\theta + I_3 \cos^2\varphi \sin^2\theta), \\
h_{12} &= \frac{1}{I_3} \sin\varphi \sin\theta \cos\theta, \\
h_{13} &= \frac{1}{I_3}\left[(I_3\lambda + (I_2 - I_3)\omega)\cos^2\theta - l_3 \sec\varphi \cos\theta \cot\theta + I_3\lambda \cos^2\varphi \sin^2\theta\right], \\
h_{14} &= \frac{1}{I_3}[l_1 + (I_3\lambda + (I_1 - I_3)\omega)\sin\varphi \sin\theta - l_3 \tan\varphi]\cos\theta, \\
h_{22} &= \frac{1}{I_1 I_3}(I_3 \cos^2\varphi + I_1 \sin^2\varphi)\sin^2\theta, \\
h_{23} &= \frac{1}{I_3}\left[(I_3\lambda + (I_2 - I_3)\omega)\sin\varphi \sin\theta - l_3 \tan\varphi\right]\cos\theta, \\
h_{24} &= \frac{1}{I_3}\left[(I_3\lambda + (I_1 - I_3)\omega \sin^2\varphi)\sin\theta + (l_1 - l_3 \tan\varphi)\sin\varphi\right]\sin\theta, \\
h_{33} &= \frac{1}{I_3}\Big[(2I_3\mu + 2(I_2 - I_3)I_3\lambda\omega + (I_2 - I_3)^2\omega^2)\cos^2\theta + l_3^2 \sec^2\varphi \cot^2\theta - \\
&\qquad 2(I_3\lambda + (I_2 - I_3)\omega)l_3 \sec\varphi \cos\theta \cot\theta + 2I_3\mu \cos^2\varphi \sin^2\theta\Big], \\
h_{34} &= \frac{1}{I_3}\Big[(l_3 \tan\varphi - l_1)l_3 \sec\varphi \cot\theta + (l_1(I_3\lambda + (I_2 - I_3)\omega)\cos\theta + \\
&\qquad ((I_1 - I_3)(I_2 - I_3)\omega^2 + (I_1 + I_2 - 2I_3)I_3\omega\lambda + 2I_3\mu)\sin\varphi \sin\theta \cos\theta - \\
&\qquad (2I_3(\lambda - \omega) + (I_1 + I_2)\omega)l_3 \tan\varphi \cos\theta\Big] \\
h_{44} &= \frac{1}{I_3}\Big[2I_3\mu \cos^2\varphi \sin^2\theta + (2I_3\mu + 2(I_1 - I_3)I_3\lambda\omega + (I_1 - I_3)^2\omega^2)\sin^2\varphi \sin^2\theta + \\
&\qquad 2(I_3\lambda + (I_1 - I_3)\omega)(l_1 - l_3 \tan\varphi)\sin\varphi \sin\theta + (l_1 - l_3 \tan\varphi)^2\Big].
\end{aligned} \tag{11}$$

Once we have the quadratic form $\mathbf{d}^2(h + C_1 + \cdots + C_n)(m)\Big|_{W\times W}$, the next step is to check whether it is positive definite for each of the equilibrium solutions.

3.1. Stability of the equilibrium E_1

We first focus on the equilibrium E_1 in order to give both sufficient and necessary conditions of stability. In this way, for the sufficient conditions we obtain the following result.

Theorem 3. *If* $\cos\varphi\sin\theta \neq 0$*, the equilibrium* E_1 *is stable if* I_2 *is the biggest moment of inertia, or if* $I_3 > I_2 > I_1$ *and*

$$|l_3| > \max\left|2(I_2 - I_3)\omega\cos\varphi\sin\theta \pm w\sqrt{(I_3 - I_2)(I_2 - (I_2 - I_3)\cos^2\varphi\sin^2\theta)}\right| \qquad (12)$$

Proof. The proof follows from Sylvester's criterion to determine whether the matrix, with the coefficients (11) evaluated for λ and μ corresponding to E_1, is positive-definite. A deteiled proof is given in [19]. □

Remark 2. In the limiting cases $\sin\theta = 0$, $\cos\varphi = 0$ the stability conditions of Theorem 3 are still valid. However, in these cases, E_1 exits if $l_3\omega - gmz_0 = 0$ and (12) can be written as

$$l_3^4 > I_2(I_3 - I_2)m^2g^2z_0^2. \qquad (13)$$

We also note that, when $\sin\theta = 0$, $l_2 = 0$ and only one rotor acts upon the gyrostat, recovering the results given in [17, 18].

Remark 3. For the other situations not covered by Theorem 3, namely I_1 is the biggest moment of inertia, and $I_3 > I_1 > I_2$, we can also get stable rotations by acting the l_2 gyrostatic moment. Indeed, we get a result analogous to Theorem 3 by considering the parameterization

$$\begin{aligned} k_2 &= \sin\varphi\sin\theta, & k_1 &= \cos\theta, & k_3 &= \cos\varphi\sin\theta \\ \pi_2 &= \omega I_2\sin\varphi\sin\theta, & \pi_1 &= \omega I_1\cos\theta, & \pi_3 &= \omega I_3\cos\varphi\sin\theta, \end{aligned}$$

replacing l_1 by l_2 and switching I_1 and I_2.

Remark 4. When $I_3 > I_2 > I_1$, if we introduce the relation between l_3 and ω given in (4) into the stability condition (12), it follows that there exist stable permanent rotations if $|\omega|$ is small enough, which is equivalent to say that l_3 is great enough.

Remark 5. It is worth noting that necessary conditions of stability are the same as the sufficient ones, given in Theorem 3, if I_2 is the biggest moment of inertia or $I_3 > I_2 > I_1$. This fact is readily deduced from the analysis of the linearized system around E_1.

3.2. Stability of the equilibrium E_0

From the results in the previous subsection, we conclude that, regardless the values of the moments of inertia, if the center of mass lies on one of the principal axis and one of the rotors is aligned with the same axis, it is possible to have stable permanent rotations with the gyrostat oriented in any space direction. The only possible exception is $\theta = \pi/2$, which corresponds to the orientation of the permanent rotations of the family E_0. Unfortunately, Theorem 3 cannot be extended to this case and it must be considered separately. By doing so, we arrive at the following stability result.

Theorem 4. *The equilibrium E_0 is stable if*

1. $\cos\varphi > 0$ *and* $l_3\omega > K_1$,
2. $\cos\varphi < 0$ *and* $l_3\omega < K_2$,
3. $\varphi = \pm\pi/2$ *and* $\pm l_1\omega > K_3$,

where K_1 and K_2 are the maximun and minimun, respectively, of

$$gmz_0 + (I_2 - I_3)\omega^2\cos\varphi,$$

$$\frac{I_1\omega^2 mgz_0 + (I_1 - I_3)\omega^2\cos^2\varphi(I_3\omega^2\cos\varphi - mgz_0\cos 2\varphi) - m^2g^2z_0^2\cos\varphi\sin^2\varphi}{\omega^2(I_1\sin^2\varphi + I_3\cos^2\varphi)},$$

and K_3 is the maximum of

$$(I_2 - I_1)\omega^2, \qquad \frac{(I_3 - I_1)I_1\omega^4 - g^2m^2z_0^2}{I_1\omega^2}.$$

Proof. The proof mimics that of Theorem 3. However, the space W is different and must be obtained for this specific case. For a detailed proof, see [19]. □

Remark 6. It is worth noting that, for any value of the angle φ, it is possibe to obtain stable rotations of the family E_0 by an appropriate selection of the gyrostatic moments. Even more, for this family, the range of values of the gyrostatic moments is greater because the two constraints (4) get reduced to (3).

Remark 7. We note that, if the following inequality holds

$$(I_1\sin^2\varphi + I_3\cos^2\varphi)(l_3\omega - gmz_0 + (I_3 - I_2)\omega^2\cos\varphi)\cos^3\varphi > 0,$$

then sufficient and necessary stability conditions are the same. However, as it is pointed out in [19], it seems that necessary conditions are also sufficient.

§4. Conclusions

The main conclusion of this work is about the existence of permanent stable rotations around an axis oriented in any direction of the space by the action of two rotors, one of them aligned along the principal axis where the center of mass lies. Indeed, given a particular gyrostat and a concrete orientation, it is possible to find appropriate gyrostatic moments and angular velocities in such a way that the gyrostat maintains its orientation along the time, even in the case of small perturbations. Moreover, necessary and sufficient stability conditions match in many cases and there is evidence that necessary conditions are also sufficient ones. However, this result cannot be proved using the Energy-Casimir method.

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