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A JUSTIFICATION OF NONLINEAR TWO-DIMENSIONAL MODEL FOR FERROMAGNETIC PLATES WITH MAGNETOSTRICTION

Mouna Kassan, Gilles Carbou and Mustapha Jazar

Abstract. We consider a three-dimensional model of ferromagnetic material with magnetostriction, coupling the Landau-Lifschitz-Gilbert equation and the elasticity wave equation, with mixed boundary condition. We establish the existence of global in time weak solutions. By asymptotic method when the thickness of the sample tends to zero, we obtain and justify a new two-dimensional model for thin plates of ferromagnetic magnetostrictive material.

Keywords: Magnetostriction, weak solutions, thin plates..

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§1. Introduction

Ferromagnetic materials have promising applications in several domains, in particular for data storage (see [14]). In such materials, the magnetization induces deformations because of magnetostrictive effects (see [11], [13], [15] and references therein). Let us recall the three-dimensional model for ferromagnetic material with magnetostriction, in the magneto-static approximation. We denote by Ω the domain occupied by the ferromagnetic material, and by $m(t,x)$ the magnetic moment, satisfying the saturation constraint $|m(t,x)| = 1$ a.e. when the material is saturated. The variations of m are described by the Landau-Lifschitz-Gilbert equation (see [1], [4], [6] and [12]):

$$\begin{cases} \dfrac{\partial m}{\partial t} - m \times \dfrac{\partial m}{\partial t} = -m \times H_{eff} \ \text{ in } \mathbb{R}^+ \times \Omega, \\ \\ \partial_{\mathbf{n}} m = 0 \ \text{ in } \mathbb{R}^+ \times \partial\Omega, \end{cases} \tag{1}$$

where $\times$ is the usual cross product in $\mathbb{R}^3$ and H_{eff} is the effective field given by:

$$H_{eff} = \Delta m + h_d(m) + (\lambda^{\mathrm{m}} : \sigma) m.$$

The demagnetizing field $h_d(m)$ is calculated from m by solving the static Maxwell equations coupled with the law of Faraday:

$$\text{div } (h_d(m) + \overline{m}) = 0 \quad \text{and} \quad \text{curl } h_d(m) = 0, \tag{2}$$

where $\overline{m}$ is the extension of m by zero outside Ω.

In the magnetostriction field $(\lambda^{\mathrm{m}} : \sigma)m$, σ is the stress tensor and λ^{m} is a symmetric positive 4-tensor.

Remark 1. The usual notations and definitions of tensor calculus are detailed in [6], Section 1.2.1. Additionally, we say that a symmetric 4-tensor λ is positive, if there exists $\lambda^* > 0$ such that:

$$\forall \xi^{ij} \text{ symmetric}, \sum_{ijkl} \lambda_{ijkl}\xi^{ij}\xi^{kl} \geq \lambda^* \sum_{ij} (\xi^{ij})^2.$$

The Landau-Lifschitz equation (1) is coupled with the following elasticity wave equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \operatorname{div} \sigma = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u = 0 & \text{on } \mathbb{R}^+ \times \Gamma_1, \\ \sigma \cdot n = f & \text{on } \mathbb{R}^+ \times \Gamma_2, \end{cases} \tag{3}$$

where the stress tensor σ is given by:

$$\sigma = \lambda^{\mathrm{e}} : \varepsilon^e, \text{ with } \varepsilon(u) = \varepsilon^e + \varepsilon^m,$$

where:

- the deformation tensor $\varepsilon(u)$ is given by:

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

- the magnetic tensor ε^m satisfies :

$$\varepsilon^m = \lambda^{\mathrm{m}} : m \otimes m.$$

We assume that we are in the isotropic case, so that the 4-tensor λ^{e} is given by:

$$\lambda^{\mathrm{e}}_{ijkl} = \frac{E}{1+\nu}\left(\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}\right) \quad \text{for } i,\ j,\ k,\ l \in \{1, 2, 3\},$$

where $E > 0$ and $0 < \nu < \frac{1}{2}$.

Concerning the boundary conditions in (3), we assume that the sample is clamped on $\Gamma_1 \subset \partial\Omega$, where the surface measure of Γ_1 is non vanishing. We apply a surface force f on the other part $\Gamma_2 = \partial\Omega \setminus \Gamma_1$ of the boundary.

We define the space $V(\Omega)$ by:

$$V(\Omega) = \{v \in H^1(\Omega; \mathbb{R}^3);\ v = 0 \text{ on } \Gamma_1\}.$$

By Korn inequality (see [9], Theorem 6.3-4 page 292), there exists $c(\Omega) > 0$, such that for all $v \in V(\Omega)$, we have

$$\int_\Omega \varepsilon(v) : \varepsilon(v) \geq c(\Omega) \int_\Omega |v|^2. \tag{4}$$

This gives that $\|\cdot\|_V = \left(\int_\Omega \varepsilon(\cdot) : \varepsilon(\cdot)\right)^{\frac{1}{2}}$ is a norm on $V(\Omega)$ equivalent to the norm $\|\cdot\|_{H^1(\Omega)}$.

§2. Weak solutions for the Landau-Lifschitz-Gilbert equation with magnestostriction

First, we establish the existence of global-in-time weak solutions for System (1)-(3).

Theorem 1. *Let $m_0 \in H^1(\Omega; S^2)$, $u_0 \in V(\Omega)$, $u_1 \in L^2(\Omega)$ and $f \in H^{\frac{1}{2}}(\Gamma_2)$. Then, there exist m and u with $m \in L^\infty(\mathbb{R}^+; H^1(\Omega;\mathbb{R}^3))$, $|m(t,x)| = 1$ a.e., $\dfrac{\partial m}{\partial t} \in L^2(\mathbb{R}^+; L^2(\Omega;\mathbb{R}^3))$, $u \in L^\infty(\mathbb{R}^+; V(\Omega))$ and $\dfrac{\partial u}{\partial t} \in L^\infty(\mathbb{R}^+; L^2(\Omega;\mathbb{R}^3))$, satisfying:*

1. *$m(0,\cdot) = m_0$, $u(0,\cdot) = u_0$ and $\dfrac{\partial u}{\partial t}(0,\cdot) = u_1$ in the trace sense,*
2. *for all $\chi \in C_c^\infty(\mathbb{R}^+; H^1(\Omega;\mathbb{R}^3))$,*

$$\begin{aligned}\int_{\mathbb{R}^+\times\Omega}\left(\frac{\partial m}{\partial t} - m\times\frac{\partial m}{\partial t}\right)\cdot\chi(t,x)dt\,dx &= 2\int_{\mathbb{R}^+\times\Omega}\left[\sum_{i=1}^{3} m\times\frac{\partial m}{\partial x_i}\cdot\frac{\partial \chi}{\partial x_i} - m\times h_d(m)\cdot\chi\right]\\ -2\int_{\mathbb{R}^+\times\Omega} m\times\Big[(\lambda^m : (\lambda^e : \varepsilon(u)))m &- (\lambda^m : (\lambda^e : (\lambda^m : m\otimes m)))m\Big]\cdot\chi,\end{aligned} \tag{5}$$

3. *for all $\chi \in C_c^\infty(\mathbb{R}^+; V(\Omega))$,*

$$\begin{aligned}\int_{\mathbb{R}^+\times\Omega}\frac{\partial u}{\partial t}\cdot\frac{\partial \chi}{\partial t} - \int_{\mathbb{R}^+\times\Omega}(\lambda^e : \varepsilon(u)) : \varepsilon(\chi) + \int_{\mathbb{R}^+\times\Gamma_2} f\cdot\chi\\ -\int_\Omega u_1\cdot\chi(0,x) = -\int_{\mathbb{R}^+\times\Omega}(\lambda^e : (\lambda^m : m\otimes m)) : \varepsilon(\chi),\end{aligned} \tag{6}$$

4. *we have the following energy inequality: for all $t \geq 0$,*

$$\mathcal{E}(t) + \int_0^t\int_\Omega\left|\frac{\partial m}{\partial t}\right|^2 \leq \mathcal{E}(0), \tag{7}$$

where

$$\begin{aligned}\mathcal{E}(t) = \int_\Omega|\nabla m|^2 + \int_{\mathbb{R}^3}|h_d(m)|^2 + \frac{1}{2}\int_\Omega\Big[Q(m) - 2\varepsilon(u) : (\lambda^e : (\lambda^m : m\otimes m))\Big]\\ +\frac{1}{2}\int_\Omega\left[(\lambda^e : \varepsilon(u)) : \varepsilon(u) + \left|\frac{\partial u}{\partial t}\right|^2\right] - \int_{\Gamma_2} f\cdot u\,d\Gamma,\end{aligned}$$

with $Q(m) = (\lambda^e : (\lambda^m : m\otimes m)) : (\lambda^m : m\otimes m)$

Sketch of the proof: the proof of Theorem 1 follows the method due to Alouges and Soyeur [2] and generalized in [7]. First, we prove the existence of solution for a penalized system, in which the saturation constraint is relaxed. Then, we take the limit when the penalization constant tends toward zero. In [6], global existence for (1)-(3) is obtained in the case of a clamped sample, that is with $u = 0$ on $\partial\Omega$. We use essentially the same method to address mixed boundary condition.

§3. Asymptotic model for plates

We consider a thin plate Ω_η of the form:

$$\Omega_\eta = \omega \times]-\eta, \eta[,$$

where ω is a smooth bounded domain of $\mathbb{R}^2$. We assume that this plate is clamped on $\Gamma_1^\eta = C_1 \times]-\eta, \eta[$, (where $C_1 \subset \partial\omega$ such that the one-dimensional measure of C_1 is non vanishing). We denote by $C_2 = \partial\omega \setminus C_1$, $\Gamma_2^\eta = C_2 \times]-\eta, \eta[$ and $\Gamma_\pm^\eta = \bar{\omega} \times \{\pm\eta\}$.

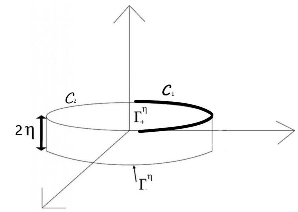

Figure 1: $\Omega_\eta = \omega \times]-\eta, \eta[$

For given initial data (precised below), we consider the weak solution (m^η, u^η) given by Theorem 1 on the domain Ω_η. In order to work on a fixed domain, we perform the following rescaling: for $t \geq 0$ and $x = (x_1, x_2, x_3) \in \Omega_1 = \omega \times]-1, 1[$, we define M^η, H^η and U^η by:

$$\begin{aligned} &M^\eta(t, x_1, x_2, x_3) = m^\eta(t, x_1, x_2, \eta x_3),\\ &H^\eta(t, x_1, x_2, x_3) = (h_d(m^\eta(t, .)))(x_1, x_2, \eta x_3),\\ &U_\alpha^\eta(t, x_1, x_2, x_3) = u_\alpha^\eta(t, x_1, x_2, \eta x_3) \text{ for } \alpha = 1, 2,\\ &U_3^\eta(t, x_1, x_2, x_3) = \eta\, u_3^\eta(t, x_1, x_2, \eta x_3). \end{aligned} \tag{8}$$

We also assume that the boundary conditions are of the form:

$$g_\alpha^{\eta\pm}(x_1, x_2, \pm\eta) = \eta\, g_\alpha^\pm(x_1, x_2) \quad \text{and} \quad g_3^{\eta\pm}(x_1, x_2, \pm\eta) = \eta^2 g_3^\pm(x_1, x_2) \quad \text{on } \Gamma_\pm^\eta,$$

$$h_\alpha^\eta(x_1, x_2, x_3) = h_\alpha(x_1, x_2) \quad \text{and} \quad h_3^\eta(x_1, x_2, x_3) = \eta\, h_3(x_1, x_2) \quad \text{on } \Gamma_2^\eta,$$

where $g^\pm \in H^{\frac{1}{2}}(\omega)$ and $h \in H^{\frac{1}{2}}(C_2)$. Then, the rescaled quantities satisfy:

- for all $\chi \in C_c^\infty(\mathbb{R}^+; H^1(\Omega_1; \mathbb{R}^3))$, we have

$$\begin{aligned} &\int_{\mathbb{R}^+\times\Omega_1} \left(\frac{\partial M^\eta}{\partial t} - M^\eta \times \frac{\partial M^\eta}{\partial t}\right) \cdot \chi = 2 \int_{\mathbb{R}^+\times\Omega_1} \sum_{\alpha=1}^{2} M^\eta \times \partial_\alpha M^\eta \cdot \partial_\alpha \chi \\ &+ \frac{2}{\eta^2} \int_{\mathbb{R}^+\times\Omega_1} M^\eta \times \partial_3 M^\eta \cdot \partial_3 \chi - 2 \int_{\mathbb{R}^+\times\Omega_1} M^\eta \times \left(H^\eta + (\lambda^{\mathrm{m}} : (\lambda^{\mathrm{e}} : \varepsilon(\eta, U^\eta)))M^\eta\right) \cdot \chi \\ &+ 2 \int_{\mathbb{R}^+\times\Omega_1} M^\eta \times ((\lambda^{\mathrm{m}} : (\lambda^{\mathrm{e}} : (\lambda^{\mathrm{m}} : M^\eta \otimes M^\eta)))M^\eta) \cdot \chi, \end{aligned} \tag{9}$$

- for all $\xi \in C_c^\infty(\mathbb{R}^+; V(\Omega_1))$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^+\times\Omega_1} \sum_{\alpha=1}^{2} \frac{\partial U_\alpha^\eta}{\partial t}\frac{\partial \xi_\alpha}{\partial t} + \frac{1}{\eta^2}\int_{\mathbb{R}^+\times\Omega_1} \frac{\partial U_3^\eta}{\partial t}\frac{\partial \xi_3}{\partial t} - \int_{\mathbb{R}^+\times\Omega_1} (\lambda^{\mathrm{e}} : \varepsilon(\eta, U^\eta)) : \varepsilon(\eta,\xi) \\
&+ \int_{\mathbb{R}^+\times\Gamma_-^1} g^+\cdot\xi - \int_{\mathbb{R}^+\times\Gamma_+^1} g^+\cdot\xi + \int_{\mathbb{R}^+\times\Gamma_2^1} h\cdot\xi - \int_{\Omega_1} u_1\cdot\xi(0,x) \\
&= \int_{\mathbb{R}^+\times\Omega_1} (\lambda^{\mathrm{e}} : (\lambda^{\mathrm{m}} : M^\eta\otimes M^\eta)) : \varepsilon(\eta,\xi),
\end{aligned} \tag{10}$$

- for all $t > 0$, we have

$$\mathcal{E}^\eta(t) + \int_0^t \int_{\Omega_1} \left|\frac{\partial M^\eta}{\partial t}\right|^2 \leq \mathcal{E}^\eta(0), \tag{11}$$

  with

$$\begin{aligned}
\mathcal{E}^\eta(t) := \frac{1}{\eta}\mathcal{E}(m^\eta, u^\eta) = &\int_{\Omega_1} \sum_{\alpha=1}^{2}\left[|\partial_\alpha M^\eta|^2 + \frac{1}{\eta^2}|\partial_3 M^\eta|^2\right] + \int_{\mathbb{R}^3} |H^\eta|^2 + \frac{1}{2}\int_{\Omega_1} Q(M^\eta) \\
&- \int_{\Omega_1}\left[\varepsilon(\eta, U^\eta) : (\lambda^{\mathrm{e}} : (\lambda^{\mathrm{m}} : M^\eta\otimes M^\eta)) + \frac{1}{2}(\lambda^{\mathrm{e}} : \varepsilon(\eta,U^\eta)) : \varepsilon(\eta, U^\eta)\right] \\
&+ \frac{1}{2}\int_{\Omega_1}\left[\sum_{\alpha=1}^{2}\left|\frac{\partial U_\alpha^\eta}{\partial t}\right| + \frac{1}{\eta^2}\left|\frac{\partial U_3^\eta}{\partial t}\right|^2\right] - \int_{\Gamma_-^1} g^-\cdot U^\eta - \int_{\Gamma_+^1} g^+\cdot U^\eta - \int_{\Gamma_2^1} h\cdot U^\eta,
\end{aligned}$$

where, for $\xi \in H(\Omega_1)$, the 2-tensor $\varepsilon(\eta,\xi)$ is given by

$$\begin{aligned}
&\varepsilon_{\alpha\beta}(\eta,\xi) = \varepsilon_{\alpha\beta}(\xi) \text{ for } \alpha,\beta\in\{1,2\}, \\
&\varepsilon_{\alpha3}(\eta,\xi) = \tfrac{1}{\eta}\varepsilon_{\alpha3}(\xi) \text{ for } \alpha\in\{1,2\}, \\
&\varepsilon_{33}(\eta,\xi) = \tfrac{1}{\eta^2}\varepsilon_{33}(\xi).
\end{aligned} \tag{12}$$

In order to ensure that the initial data are uniformly bounded, we assume that $\partial_3 m_0 = 0$, $\varepsilon_{i3}(u_0) = 0$ for $i \in \{1,2,3\}$ and $(u_1)_3 = 0$. Then, $\mathcal{E}^\eta(0)$ is independent of η and we can use inequality (11) to obtain that, for all $T > 0$ and η in a neighborhood of zero, there exists a constant C independent of η such that

- $\|\frac{\partial M^\eta}{\partial t}\|_{L^2(0,T;L^2(\Omega_1))} \leq C$,
- $\|\partial_1 M^\eta\|_{L^\infty(0,T;L^2(\Omega_1))} + \|\partial_2 M^\eta\|_{L^\infty(0,T;L^2(\Omega_1))} + \frac{1}{\eta}\|\partial_3 M^\eta\|_{L^\infty(0,T;L^2(\Omega_1))} \leq C$,
- $\|\frac{\partial U_1^\eta}{\partial t}\|_{L^\infty(0,T;L^2(\Omega_1))} + \|\frac{\partial U_2^\eta}{\partial t}\|_{L^\infty(0,T;L^2(\Omega_1))} + \frac{1}{\eta}\|\frac{\partial U_3^\eta}{\partial t}\|_{L^\infty(0,T;L^2(\Omega_1))} \leq C$,
- $\|\varepsilon(\eta, U^\eta)\|_{L^\infty(0,T;L^2(\Omega_1))} \leq C$,
- $\|H^\eta\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \leq C$.

So, we can extract subsequences, still denoted (M^η, U^η) and H^η, such that:

- $M^\eta \rightharpoonup M$ in $L^\infty(0,T;H^1(\Omega_1))$ weak-*,
- $\dfrac{\partial M^\eta}{\partial t} \rightharpoonup \dfrac{\partial M}{\partial t}$ in $L^2(0,T;L^2(\Omega_1))$ weak,
- $U^\eta \rightharpoonup U$ in $L^\infty(0,T;V(\Omega_1))$ weak-*,

- $\frac{\partial U^\eta}{\partial t} \rightharpoonup \frac{\partial U}{\partial t}$ in $L^\infty(0,T;L^2(\Omega_1))$ weak-*,
- $H^\eta \rightharpoonup H$ in $L^\infty(0,T;L^2(\mathbb{R}^3))$ weak-*.

Furthermore, we have $\partial_3 M^\eta \to 0$ in $L^\infty(0,T;L^2(\Omega_1))$ strong, so M only depends on $(t,x_1,x_2) \in \mathbb{R}^+ \times \omega$. In addition, $\frac{\partial U_3^\eta}{\partial t} \to 0$ in $L^\infty(0,T;L^2(\Omega_1))$ strong, so U_3 does not depend on t.

Since $(\varepsilon(\eta, U^\eta))_\eta$ is bounded, we obtain that $\varepsilon_{i3}(U^\eta) \to 0$ strongly in $L^\infty(0,T;L^2(\Omega))$, for all $T>0$, i.e. $\varepsilon_{i3}(U) = 0$ for $i \in \{1,2,3\}$. So, $U \in L^\infty(\mathbb{R}^+; V_{KL}(\Omega_1))$, where

$$V_{KL}(\Omega_1) = \left\{\xi \in H^1(\Omega_1)\,;\, \xi = 0 \text{ on } \Gamma_1^1 \text{ and } \varepsilon_{i3}(\xi) = 0 \text{ in } \Omega_1 \text{ for } i \in \{1,2,3\}\right\}.$$

Therefore, by Theorem 1.4.1 in [8], there exist $\tilde{u} \in L^\infty(\mathbb{R}^+; V(\omega))$ such that

$$\begin{aligned} U_\alpha(t,x_1,x_2,x_3) &= \tilde{u}_\alpha(t,x_1,x_2) - x_3\partial_\alpha \tilde{u}_3 \text{ for } \alpha \in \{1,2\}, \\ U_3(t,x_1,x_2,x_3) &= \tilde{u}_3(t,x_1,x_2), \end{aligned} \tag{13}$$

where

$$V(\omega) = \left\{v = (v_i) \in H^1(\omega)\times H^1(\omega)\times H^2(\omega)\,;\, v_i = 0 \text{ on } C_1 \text{ and } \partial_1 v_3 = \partial_2 v_3 = 0 \text{ on } C_1\right\}.$$

Hereafter, we denote $\tilde{u}_T = (\tilde{u}_1, \tilde{u}_2)$. Since $U^\eta \rightharpoonup U$ in $V(\Omega_1)$ weak, we have

$$\varepsilon(\eta, U^\eta) \rightharpoonup A \quad \text{weakly in } L^2(\Omega_1), \tag{14}$$

where A is the 2-tensor defined by $A_{\alpha\beta} = \varepsilon_{\alpha\beta}(U)$ for $\alpha = 1,2$.

Moreover, we prove that $M^\eta \to M$ in $L^\infty(0,T;L^r(\Omega_1)) \cap C^0([0,T];L^2(\Omega_1))$ strong for $r<6$ by using the Aubin-Simon lemma in [3], Theorem II.5.16. Thus, we can extract a subsequence, still denoted by $(M^\eta)_\eta$, such that $M^\eta \to M$ a.e. in $[0,T]\times\Omega_1$. Hence M satisfies the saturation constraint $|M(t,x_1,x_2)| = 1$ *a.e.* in $\mathbb{R}^+\times\omega$, and by continuity in time with values in $L^2(\Omega_1)$, $M(0,x) = m_0(x)$ in the trace sense.

Let $\tilde{\chi} \in C_c^\infty(\mathbb{R}^+; \mathcal{D}(\bar{\omega}))$, and define χ in $\mathbb{R}^+\times\Omega_1$ by $\chi(t,x_1,x_2,x_3) = \tilde{\chi}(t,x_1,x_2)$. Then, $\chi \in C_c^\infty(\mathbb{R}^+; H^1(\Omega))$ and $\partial_3\chi = 0$, so we can put χ as a test function in (9). Taking the limit as η tends to zero in (9), by using weak and strong convergences for M^η and U^η respectively, and by using (14) (see [6] for more details), we obtain that for all $\tilde{\chi} \in C_c^\infty(\mathbb{R}^+; H^1(\omega))$,

$$\begin{gathered} \int_{\mathbb{R}^+\times\omega}\left(\frac{\partial M}{\partial t} - M\times\frac{\partial M}{\partial t}\right)\cdot\tilde{\chi} = -2\int_{\mathbb{R}^+\times\omega} M\times\nabla M\cdot\nabla\tilde{\chi} - 2\int_{\mathbb{R}^+\times\omega} M\times H\cdot\tilde{\chi} \\ -2\int_{\mathbb{R}^+\times\omega} M\times(\lambda^{\mathrm{m}} : (\lambda^{\mathrm{e}} : \bar{A}))M\cdot\tilde{\chi} + 2\int_{\mathbb{R}^+\times\omega} M\times((\lambda^{\mathrm{m}} : (\lambda^{\mathrm{e}} : (\lambda^{\mathrm{m}} : M\otimes M)))M)\cdot\tilde{\chi}, \end{gathered} \tag{15}$$

where $\bar{A}$ is given by $\bar{A} = \dfrac{1}{2}\displaystyle\int_{-1}^{1} A\,dx_3$. Since $\displaystyle\int_{-1}^{1} x_3 dx_3 = 0$, using (13), $\bar{A}$ satisfies:

$$\bar{A}_{\alpha\beta} = \varepsilon_{\alpha\beta}(\tilde{u}) \text{ for } \alpha, \beta \in \{1,2\}. \tag{16}$$

We recall that λ^{e} is given by: for all symmetric 2-tensors S,

$$(\lambda^{\mathrm{e}} : S) = \frac{E}{1+\nu}\left(S + \frac{\nu}{1-2\nu}\mathrm{tr}(S)I_3\right), \tag{17}$$

where

$$\mathrm{tr}(S) = S_{11} + S_{22} + S_{33}.$$

Now, we take the limit as η tends to zero in (10). For this, we take a test function ξ in this equation of the form

$$\xi(t, x_1, x_2, x_3) = \begin{pmatrix} v_1(t, x_1, x_2) \\ v_2(t, x_1, x_2) \\ 0 \end{pmatrix},$$

where $v_\alpha \in H^1(\omega)$, such that $v_\alpha = 0$ on C_1. Then, $\varepsilon_{i3}(\xi) = 0$ for $i \in \{1, 2, 3\}$ and we obtain

$$\begin{aligned}\int_{\mathbb{R}^+\times\Omega_1} \sum_{\alpha=1}^{2} \frac{\partial U_\alpha^\eta}{\partial t} \cdot \frac{\partial \xi_\alpha}{\partial t} - \int_{\mathbb{R}^+\times\Omega_1} (\lambda^{\mathrm{e}} : \varepsilon(\eta, U^\eta)) : \varepsilon(\xi) + \int_{\mathbb{R}^+\times\Gamma_-^1} g^+ \cdot \xi - \int_{\mathbb{R}^+\times\Gamma_+^1} g^+ \cdot \xi \\ + \int_{\mathbb{R}^+\times\Gamma_2^1} h \cdot \xi - \int_{\Omega_1} u_1 \cdot \xi(0, x) = \int_{\mathbb{R}^+\times\Omega_1} (\lambda^{\mathrm{e}} : (\lambda^{\mathrm{m}} : M^\eta \otimes M^\eta)) : \varepsilon(\xi),\end{aligned} \tag{18}$$

By the weak convergence of $\frac{\partial U^\eta}{\partial t}$ and $\varepsilon(\eta, U^\eta)$ and by the strong limit of M^η, the limit of (18) as $\eta \to 0$ gives that for all $v_T \in H^1(\omega)$, with $v_T = 0$ on C_1, we have

$$\begin{aligned}\int_{\mathbb{R}^+\times\omega} \frac{\partial \tilde{u}_T}{\partial t} \cdot \frac{\partial v_T}{\partial t} - \int_{\mathbb{R}^+\times\omega} \sum_{\alpha\beta=1}^{2} (\lambda^{\mathrm{e}} : \bar{A})_{\alpha\beta} \varepsilon_{\alpha\beta}(v_T) + \frac{1}{2} \int_{\mathbb{R}^+\times\omega} (g_T^+ + g_T^-) \cdot v_T + \int_{\mathbb{R}^+\times C_2} h_T \cdot v_T \\ - \int_{\mathbb{R}^+\times\omega} u_{1,T} \cdot v_T(0, x) = - \int_{\mathbb{R}^+\times\omega} \sum_{\alpha\beta=1}^{2} (\lambda^{\mathrm{e}} : (\lambda^{\mathrm{m}} : M \otimes M))_{\alpha\beta} \varepsilon_{\alpha\beta}(v).\end{aligned} \tag{19}$$

By (16) and (17), we have:

$$(\lambda^{\mathrm{e}} : \bar{A})_{\alpha\beta} = \frac{E}{1+\nu} \left(\varepsilon_{\alpha\beta}(\tilde{u}_T) + \frac{\nu}{1-2\nu} \mathrm{tr}(\bar{A}) \delta_{\alpha\beta} \right),$$

where

$$\mathrm{tr}(\bar{A}) = \varepsilon_{11}(\tilde{u}_T) + \varepsilon_{22}(\tilde{u}_T) + \bar{A}_{33}.$$

To determine $\bar{A}_{33}$, we multiply (10) by η^2, and we use a test function ξ such that $\xi_\alpha = 0$ for $\alpha = 1, 2$ and $\xi_3 = x_3\,\varphi$, where $\varphi \in \mathcal{D}(\mathbb{R}^+ \times \omega)$. Taking the limit when $\eta \to 0$, we obtain:

$$2 \int_{\mathbb{R}^+\times\omega} (\lambda^{\mathrm{e}} : \bar{A})_{33} \varphi = \int_{\mathbb{R}^+\times\omega\times]-1,1[} (\lambda^{\mathrm{e}} : (\lambda^{\mathrm{m}} : M \otimes M))_{33} \varphi,$$

since M and φ are independent of x_3. Then

$$(\lambda^{\mathrm{e}} : \bar{A})_{33} = (\lambda^{\mathrm{e}} : (\lambda^{\mathrm{m}} : M \otimes M))_{33} \text{ in } L^2(\mathbb{R}^+ \times \omega). \tag{20}$$

Replacing λ^{e} by its value in (20), we obtain

$$\bar{A}_{33} = \left[\frac{1-2\nu}{1-\nu} (\lambda^{\mathrm{m}} : M \otimes M)_{33} + \frac{\nu}{1-\nu} \mathrm{tr}(\lambda^{\mathrm{m}} : M \otimes M) - \frac{\nu}{1-\nu} \sum_{\alpha=1}^{2} \varepsilon_{\alpha\alpha}(\tilde{u}_T) \right], \tag{21}$$

and we deduce that

$$\frac{1}{2}(\lambda^{\mathrm{e}} : \bar{A})_{\alpha\beta} - (\lambda^{\mathrm{e}} : (\lambda^{\mathrm{m}} : M \otimes M))_{\alpha\beta} = (\lambda^{\mathrm{et}} : \varepsilon(\tilde{u}_T))_{\alpha\beta} - (\lambda^{\mathrm{et}} : (\lambda^{\mathrm{m}} : M \otimes M))_{\alpha\beta}, \qquad (22)$$

where λ^{et} is a 4-tensor given by: for a symmetric 2-tensor S,

$$\begin{aligned} (\lambda^{\mathrm{et}} : S)_{\alpha\beta} &= \frac{E}{1+\nu}\left[S_{\alpha\beta} + \frac{\nu}{1-\nu}\,\tilde{\mathrm{tr}}(S)\delta_{\alpha\beta}\right] \qquad \text{for } \alpha, \beta \in \{1,2\}, \\ (\lambda^{\mathrm{et}} : S)_{i3} &= 0 \qquad \text{for } i \in \{1,2,3\}, \end{aligned} \qquad (23)$$

with

$$\tilde{\mathrm{tr}}(S) = S_{11} + S_{22}.$$

Remark 2. This 4-tensor is also obtained in [10].

To describe H, the limit for the demagnetizing field H^η, we use Lemma 2.1 in [5], to obtain that

$$H(t, x_1, x_2, x_3) = \begin{cases} 0 & \text{for } x \notin \mathbb{R}^3 \setminus \Omega_1, \\ -(M(t, x_1, x_2) \cdot \vec{e_3})\vec{e_3} & \text{for } x \in \Omega_1. \end{cases} \qquad (24)$$

Since $(M, \bar{A})$ is a solution for (15) and (19), and by using (22) and (24), $(M, \tilde{u}_T)$ is a weak solution for

$$\begin{cases} \dfrac{\partial M}{\partial t} - M \times \dfrac{\partial M}{\partial t} = -2M \times H_{eff}(M) \quad \text{in } \mathbb{R}^+ \times \omega \\ H_{eff}(M) = \Delta M - (M, \vec{e_3})\vec{e_3} + (\lambda^{\mathrm{m}} : \tilde{\sigma})M, \end{cases} \qquad (25)$$

coupled with

$$\begin{cases} \dfrac{\partial^2 \tilde{u}_T}{\partial t^2} - \tilde{\mathrm{div}}\,\tilde{\sigma} = \dfrac{1}{2}(g_T^+ + g_T^-) \quad \text{in } \omega \\ \tilde{u}_T = u_{0T} \quad \text{on } \omega, \\ \dfrac{\partial \tilde{u}_T}{\partial t}(t=0) = u_{1T} \quad \text{on } \omega, \\ \tilde{u}_T = 0 \quad \text{on } \mathbb{R}^+ \times C_1, \\ \tilde{\sigma} n = h_T \quad \text{on } \mathbb{R}^+ \times C_2, \end{cases} \qquad (26)$$

where

$$\tilde{\sigma} = ((\lambda^{\mathrm{et}} : \varepsilon(\tilde{u})) - (\lambda^{\mathrm{et}} : (\lambda^{\mathrm{m}} : M \otimes M))) \;\text{ and }\; \left(\tilde{\mathrm{div}}\,(\tilde{\sigma})\right)_\alpha = \sum_{\beta=1}^{2} \partial_\beta \tilde{\sigma}_{\alpha\beta}.$$

§4. Conclusion

We have obtained an equivalent 2d-model for thin ferromagnetic plates with magnetostriction. This model will be easier to study, to discretize and to solve numerically than the 3d-model. As observed in [5], the 2d demagnetizing field is local while it is not the case in 3d.

The obtained model does not take into account the third component of the deformation, since in the studied regime, the magnetization does not influence the normal deformations.

We also remark that the forces applied on the two faces of the plate appear as a source term in the 2d-model whereas they are modeled by boundary conditions in the 3d-model.

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Mouna Kassan and Gilles Carbou
Universite de Pau et des Pays de l'Adour,
E2S UPPA, CNRS, LMAP,
Pau, France
`mouna.kassan@univ-pau.fr` and
`gilles.carbou@univ-pau.fr`

Mouna Kassan and Mustapha Jazar
LaMA, Laboratoire de Mathématiques
Applications,
Lebanese University,
Tripoli, Lebanon.
`mouna.kassan@univ-pau.fr` and
`mjazar@laser-lb.org`