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BILINEAR CONTROL PROBLEMS ASSOCIATED TO CHEMO-REPULSION MODELS

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Abstract. Chemotaxis models try to reproduce the spatial transport of a living organism with respect to a chemical substance, which can be attractive or repulsive. Other interactions between both variables are considered such as production and/or consumption of chemical by cells, degradation of chemical or logistic reaction for living organisms. In this work, we consider a suitable bilinear control over the system acting on the chemical substance, by considering the case of chemo-repulsion and production effects. Then, we analyze the existence of global optimal solution, and the obtention of first-order optimality conditions for local optimal solutions by using a Lagrange multipliers theorem.

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§1. What is chemotaxis

Chemotaxis is the biological process in which living organisms move spatially in response to a chemical stimulus which can be given towards a higher (attractive) or lower (repulsive) concentration of a chemical substance. At the same time, the presence of living organisms can produce or consume chemical substance. The classical chemotaxis system was introduced by Keller and Segel in 1970-1971 [14], relating $u = u(t, x) \geq 0$ the cell density and $v = v(t, x)$ the chemical concentration, in the time $t \geq 0$ and the space $x \in \Omega$, and can be written as follows:

$$\begin{cases} \partial_t u - \Delta u \pm \nabla \cdot (u\nabla v) = 0 \quad \text{in } \Omega,\ t > 0, \\ \partial_t v - \Delta v + v = g(u) \quad \text{in } \Omega,\ t > 0, \\ \frac{\partial v}{\partial \mathbf{n}} = \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega,\ t > 0, \\ v(\mathbf{x}, 0) = v_0(\mathbf{x}) \geq 0,\ u(\mathbf{x}, 0) = u_0(\mathbf{x}) \geq 0 \ \text{ in } \Omega, \end{cases} \tag{1}$$

where the term $g(u)$ models the production of chemical substance and the term $\pm\nabla\cdot(u\nabla v)$ models the transport of cells by either chemo-attraction (if + is considered) or chemo-repulsion (if − is taken). Finally isolated boundary conditions and initial conditions for both variables are considered.

From the biological point of view, some (general) properties must be satisfied:

- **positivity:** $u \geq 0$ and $v \geq 0$.

- u **mass-conservation:** This can be obtained by integrating $(1)_1$ in Ω,

$$\frac{d}{dt}\left(\int_\Omega u\right) = 0, \quad \text{i.e.} \quad \int_\Omega u(t) = \int_\Omega u_0, \quad \forall t > 0.$$

For a detailed analysis of this kind of models the reader can consult the review of Bellomo et al. [3] and, in particular the following result in which some production terms are considered for the equation for both density and chemical substance:

Theorem 1 (Local classical solution and extensibility criteria). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and consider $f, g \in C^1([0,+\infty) \times \overline{\Omega} \times \mathbb{R}^2_+)$, $f(t,x,0,v) \geq 0$, $g(t,x,u,0) \geq 0$ (if $u, v \geq 0$), and the initial conditions $(u_0, v_0) \in C^0(\overline{\Omega}) \times W^{1,q}(\Omega)$ $(q > N)$ for the system:*

$$\begin{cases} \partial_t u - \Delta u \pm \nabla \cdot (u \nabla v) = f(t,x,u,v) \\ \partial_t v - \Delta v + v = g(t,x,u,v) \end{cases}$$

Then, there exists a local in time solution, that is, there exists a $T_{max} \in (0,+\infty]$ such that the exists unique $u, v \geq 0$ being a classical solution $(u,v) \in C^{1,2}((0,T_{max}) \times \overline{\Omega})$. Moreover, the following extensibility criterium holds:

If $T_{max} < \infty$, then $\lim \sup_{t \to T_{max}} \left(\|u(t,\cdot)\|_{L^\infty} + \|v(t,\cdot)\|_{W^{1,q}} \right) \to +\infty.$

§2. The case of chemo-repulsion with potential production

From now on, we focus on chemo-repulsion models and, in particular, with a potential production term. Then, (1) reads:

$$\begin{cases} \partial_t u - \Delta u - \nabla \cdot (u \nabla v) = 0, \\ \partial_t v - \Delta v + v = u^p, \end{cases} \tag{2}$$

endowed with the boundary and initial conditions. Several values of $p \geq 1$ can be analyzed:

- $p = 1$: linear term
- $1 < p \leq 2$: superlinear term ($p = 2$ quadratic).
- $p > 2$: superquadratic term (this case remains as an open problem).

In all cases, it is also posible to prove the positivity ($u \geq 0$ and $v \geq 0$) and the mass-conservation for u, that is, $\int_\Omega u(t) = \int_\Omega u_0$. Other properties are analyzed onwards.

2.1. Energy equality

By considering as test functions $F'(u) = \log(u)$ if $p = 1$ or u^{p-1} if $1 < p \leq 2$ in the u-equation and $-\frac{1}{p}\Delta v$ in the v-equation of (2), then the attraction and production effects cancel, obtaining the energy law:

$$\frac{d}{dt}\mathcal{E}(u,v) + \mathcal{D}(u,v) = 0, \tag{3}$$

where the energy functional is

$$\mathcal{E}(u,v) = \int_\Omega F(u) + \frac{1}{2p}\int_\Omega |\nabla v|^2$$

and the dissipation functional is

$$\mathcal{D}(u,v) = \int_\Omega F''(u)|\nabla u|^2 + \frac{1}{p}\int_\Omega (|\nabla v|^2 + |\Delta v|^2).$$

The energy potential is $F(u) = u\log(u) - u$ (in the linear case), $F(u) = \frac{1}{p}u^p$ (in the superlinear case). The cancellation of the chemotaxis term (in the equation for u) with the production term (in the equation for v) is crucial for obtaining the energy equality (3). Such energy equality allows to deduce the weak regularity estimates leading to the existence of global in time weak solution of problem (2). Concretely, this weak solution has the regularity

$$(\sqrt{u\log(u)} \text{ or } u^{p/2},\ \nabla v) \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)). \tag{4}$$

The mass-conservation for u together with (4) helps to the obtention of a global in time estimate for $\left(\int_\Omega v\right)$, namely:

$$\frac{d}{dt}\left(\int_\Omega v\right) + \int_\Omega v = \int_\Omega u^p \le C \quad \Rightarrow \int_\Omega v \le C$$

In the proof, we use interpolation regularity results. For instance, from (4) for $p \neq 1$, it can be proved that $u^{p/2} \in L^{10/3}(0,T;L^{10/3}(\Omega))$, which implies that $u^p \in L^{5/3}(0,T;L^{5/3}(\Omega))$ and thus $u \in L^{5p/3}(0,T;L^{5p/3}(\Omega))$. Similar results can be deduced for ∇v, for instance $\nabla v \in L^{10/3}(0,T;L^{10/3}(\Omega))$, and thus $u\nabla v \in L^q(0,T;L^q(\Omega))$ for $q = 10p/(3p+6)$. However, when looking for estimates for ∇u, we must restrict to the case of $p \le 2$, because the following equality only works well in that case:

$$\nabla u = u^{1-p/2}\nabla(u^{p/2}) \in L^{5p/(3+p)}(0,T;L^{5p/(3+p)}(\Omega)).$$

2.2. Space framework

We define the spaces where the definition of solution and the results of existence will be studied. Concretely, the “weak (L^p) space"

$$W_p := \{w : \ w^{p/2} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))\}, \quad p > 1$$

and the “strong (L^q) space":

$$X_q := \{w \in C([0,T];W^{2-2/q,q}(\Omega)) \cap L^q(0,T;W^{2,q}(\Omega)) : \ \partial_t w \in L^q(0,T;L^q(\Omega))\}, \quad q > 1.$$

Space W_p is chosen due to the energy estimates, and space X_q follow from the L^q regularity for parabolic equations with homogeneous Neumann boundary conditions using a bootstrap argument once the regularity for u^p and $\nabla\cdot(u\nabla v)$ is known.

2.3. Existence results

The analysis of the existence of solution for (2), for the different values of $p \in [1,2]$ is summarized as follows:

- Global in time weak solution $(u, v) \in W_p \times X_2$ and convergence to constant states $(\oint u_0, \oint (u_0)^p)$ as $t \to +\infty$
- Unique global in time classical solution (as in Theorem 1) for $1D$ and $2D$ domains.

Results for $p = 1$ for the continuous model were analyzed by Cieslak et al in [4]. The case $p = 2$ for the continuous model and also the analysis of several numerical schemes can be found in the authors' works [8, 9, 10]; and similar results for $p \in (1, 2)$ can be found in [11]. In fact, in [3] one can see that the regularity $u \in L^\infty(0, T; L^r(\Omega))$ for $r \geq 1$ and $r > Np/2$ implies the extensibility criterium given in Theorem 1. Note that this regularity only holds for $1D$ domains. However, for $2D$ domains, it is proved that the regularity $\Delta v \in L^2(0, T; L^2(\Omega))$ implies $u \in L^\infty(0, T; L^q(\Omega))$ for any $q < \infty$. Therefore, there exists global classical solution in $1D$ and $2D$ domains. Finally, questions for 3-dimensional domains, as blow-up versus global regularity of the solutions, remain as open problems.

§3. The case of chemo-attraction with linear production

The "classical" Keller-Segel model is a chemo-attractive and linear production system:

$$\begin{cases} \partial_t u - \Delta u + \nabla \cdot (u \nabla v) = 0 \\ \partial_t v - \Delta v + v = u. \end{cases} \tag{5}$$

Compared with the chemo-repusive models, as (2), the analysis of existence of (5) is more difficult because the energy laws that can be obtained by using adequate test functions contains terms difficult to control. For example, using $(F'(u) - v = \log(u) - v, \partial_t v)$ as test functions for (5), we obtain:

$$\frac{d}{dt} E(u, v) + D(u, v) = 0$$

where the energy associated is $E(u, v) = \int_\Omega F(u) - \int_\Omega uv + \frac{1}{2} \int_\Omega (|\nabla v|^2 + |v|^2)$ and the dissipation rate is $D(u, v) = \int_\Omega u |\nabla (F'(u) - v)|^2 + \int_\Omega |\partial_t v|^2$. Observe that the negative term $-\int_\Omega uv$ must be controlled in order to bound from below the energy, which lead to deduce regularity estimates for (u, v). Moreover, blow-up is expected when $\left(-\int_\Omega uv\right) \to -\infty$. In the last 20 years, many works have been written trying to do an answer to this and other related questions (see [16, 3, 13, 12]), obtaining (briefly) that there is no blow-up in $1D$, and the blow-up in $2D$ and $3D$ depends mainly on $\left(\int_\Omega u_0\right)$ and chemotactic coefficients.

§4. Optimal control problems for chemo-repulsion models

Acting over a system in order to force the solution to behave in a convenient manner is something interesting specially if the system reproduces a biological situation. The analysis of such behaviour can be made through an optimal control problem. In this case, we want to minimize the functional (for simplicity, we write the 2D version):

$$\min J(u, v, f) = \frac{\gamma_u}{2} \int_0^T \!\!\int_\Omega |u - u_d|^2 + \frac{\gamma_v}{2} \int_0^T \!\!\int_\Omega |v - v_d|^2 + \frac{\gamma_f}{2^+} \int_0^T \!\!\int_{\Omega_c} |f|^{2^+} \tag{6}$$

subject to the control $f \in \mathcal{F}$ being a closed convex of $L^{2^+}((0,T)\times\Omega_c)$ (2^+ means $2+\varepsilon$ for $\varepsilon > 0$ small enough), where Ω_c is the control subdomain of Ω and being the state (u, v) the solution of

$$\begin{cases} \partial_t u - \Delta u \pm \nabla\cdot(u\nabla v) = h(u) \\ \partial_t v - \Delta v = u^p + f\, v\, 1_{\Omega_c} \end{cases} \tag{7}$$

endowed with boundary and initial conditions. That means that we want to drive the state (u, v) near of desirable states (u_d, v_d) by using a control f as small as possible. Observe that we have introduced the control term as a bilinear one, $fv1_{\Omega_c}$, because we do not want to impose positivity on f and we want to use a control f with the less regularity possible.

Note that the existence of classical in time solutions for (2) deduced for instance from Theorem 1 cannot be applied to (7) because of the presence of the bilinear control term $f(t, x)v$ implies that the generic function $g = g(t, xu, v)$ given in Theorem 1 has not the classical regularity $g \in C^1([0,+\infty)\times\overline{\Omega}\times\mathbb{R}^2_+)$.

In a serie of papers, we have studied the existence of global optimal solution and first order optimality conditions for (6)-(7). When the reaction term $h(u) = 0$, the chemo-repulsion system is analyzed in [5] for $2D$ domains, in [6] for $3D$ domains, while the $2D$ repulsion system replacing the production term u by the nonlinear one u^p ($p \in (1, 2)$) is analyzed in [7].

On the other hand, the $2D$-chemo-attraction case with logistic term $h(u) = u(1-u)$ is the subject of [15], where the existence of optimal solution of (7) and the first order optimality condition for any local optimal solution including the existence of Lagrange multipliers can be deduced only in the weak regularity framework.

4.1. Existence of solution for the chemo-repulsion control problem

For simplicity, we reduce to the repulsion case with linear production ($p = 1$) and without reaction in the u-equation ($h(u) = 0$), remaining the problem

$$\begin{cases} \partial_t u - \Delta u - \nabla\cdot(u\nabla v) = 0 \\ \partial_t v - \Delta v = u + f\, v\, 1_{\Omega_c} \end{cases} \tag{8}$$

The first step in the analysis of the optimal control problem is to guarantee the existence of solution of (8) and a priori estimates of the possible states (u, v) depending on the control f.

The $2D$ case is studied in [5]. In this case, if $f \in L^{2^+}(0,T; L^{2^+}(\Omega_c))$, there exists a unique strong solution $(u, v) \in X_2 \times X_{2^+}$. Moreover, there exists $C > 0$ such that

$$\|(u,v)\|_{X_2\times X_{2^+}} \le C\|f\|_{L^{2^+}(0,T;L^{2^+}(\Omega_c))} \tag{9}$$

The proof of this existence result is based on the Leray-Schauder Theorem, that we apply to the operator $R : (\overline{u}, \overline{v}) \to (u, v)$ solving

1. v : $\quad \partial_t v - \Delta v = \overline{u}_+ + f\,\overline{v}_+\, 1_{\Omega_c}$
2. u : $\quad \partial_t u - \Delta u = \nabla\cdot(\overline{u}_+\nabla v)$

where u_+ and v_+ denotes the positive part of u and v, respectively (by the way, the solution (u, v) obtained is non-negative). The main key of the proof is the obtention of the energy estimates of (possible) fixed-points

$$(u, v) = \lambda R(u, v), \quad \lambda \in [0, 1]$$

A bootstrapping argument is also used via L^p regularity of the Heat-Neumann problem.

The 3D case is studied in [6]. Now, higher regularity for f is needed. In fact, for each $f \in L^4(Q_c)$, there exists $(u, v) \in W_2 \times X_2$ a weak solution of (8). However, in order to have sufficient regularity to study the adjoint system associated to the optimal control problem, we need to obtain a more regular solution. For this aim, it is necessary to assume for instance the regularity criterium $u \in L^{20/7}(Q)$. If such criterium is satisfied, then $(u, v) \in X_2 \times X_4$ is the unique strong solution of (8) and

$$\|(u, v)\|_{X_2 \times X_4} \leq C(\|f\|_{L^4(0,T;L^4(\Omega_c))})$$

Even the proof of the existence of weak solution in $W_2 \times X_2$ is not trivial (and not similar to the 2D case). This time, we use a regularization procedure: for all $\varepsilon > 0$, let $(u^\varepsilon, z^\varepsilon)$ the solution of

$$\begin{cases} \partial_t u^\varepsilon - \Delta u^\varepsilon - \nabla \cdot (u^\varepsilon \nabla v(z^\varepsilon)) = 0 \\ \partial_t z^\varepsilon - \Delta z^\varepsilon = u^\varepsilon + f\, v(z^\varepsilon)_+\, 1_{\Omega_c} \end{cases}$$

with $v(z^\varepsilon)$ the solution of the elliptic problem $v - \varepsilon\Delta v = z^\varepsilon$ with homogeneous Neumann boundary conditions. Then, the existence of solution $(u^\varepsilon, z^\varepsilon)$ for the regularized system is obtained via Leray-Schauder Theorem, together with ε-independent energy estimates that allow us to pass to the limit as $\varepsilon \to 0$ ($z^\varepsilon - v(z^\varepsilon) \to 0$), and finally obtain the existence of weak solution of (8).

In a second step, the regularity criterium $u \in L^{20/7}(Q)$ provides the strong solution $(u, v) \in X_2 \times X_4$ by using a bootstrapping argument.

4.2. Existence of global minimum and optimality conditions

For simplicity, we reduce to the case of $2D$ chemo-repulsive bilinear optimal control problem, studied in [5]. We define the admisible set:

$$\mathcal{S}_{ad} = \{s = (u, v, f) \in W_2 \times X_{2+} \times \mathcal{F} \;:\; s \text{ is a weak solution of (8)}\}.$$

The existence of a global optimal solution is based on minimizing sequences and the a priori estimate (9).

To deduce necessary optimality conditions, we use the following general procedure appearing in Zowe and Kurcyusz (see [17]), where the notions we describe onwards are used.

Definition 1. We consider the following abstract optimization problem

$$\min_{s \in \mathbb{M}} J(s) \;\text{ subject to } G(s) = 0, \tag{10}$$

where $J : \mathbb{X} \to \mathbb{R}$ is a functional, $G : \mathbb{X} \to \mathbb{Y}$ is an operator, $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces, and $\mathbb{M}$ is a nonempty closed and convex subset of $\mathbb{X}$. In fact, the admissible set for problem (10) is $\mathcal{S} = \{s \in \mathbb{M} \;:\; G(s) = 0\}$.

Now, we define the so-called Lagrangian functional related to problem (10) as $\mathcal{L} : \mathbb{X} \times \mathbb{Y}' \to \mathbb{R}$, given by

$$\mathcal{L}(s, \xi) = J(s) - \langle \xi, G(s) \rangle_{\mathbb{Y}'} \tag{11}$$

Definition 2 (Lagrange multipliers)**.** Let $\tilde{s} \in \mathcal{S}$ be a local optimal solution for problem (10). Suppose that J and G are Fréchet differentiable in $\tilde{x}$, with derivatives $J'(\tilde{s})$ and $G'(\tilde{s})$, respectively. Then, $\xi \in \mathbb{Y}'$ is called a Lagrange multiplier for (10) at the point $\tilde{s}$ if

$$\langle \xi, G(\tilde{s})\rangle_{\mathbb{Y}'} = 0, \quad \mathcal{L}'(\tilde{s},\xi)[r] = J'(\tilde{s})[r] - \langle \xi, G'(\tilde{s})[r]\rangle_{\mathbb{Y}'} \geq 0 \quad \forall r \in C(\tilde{s}), \tag{12}$$

where $C(\tilde{s}) = \{\theta(s - \tilde{s}) \ : \ s \in \mathbb{M}, \ \theta \geq 0\}$ is the conical hull of $\tilde{s}$ in $\mathbb{M}$.

Definition 3 (Regular points)**.** Let $\tilde{s} \in \mathcal{S}$. It will be said that $\tilde{s}$ is a regular point if

$$G'(\tilde{s})[C(\tilde{s})] = \mathbb{Y}.$$

Theorem 2 (Existence of Lagrange multipliers)**.** *Let $\tilde{x} \in \mathcal{S}$ be a local optimal solution for problem (10). If $\tilde{x}$ is a regular point, then the set of Lagrange multipliers for (10) at $\tilde{x}$ is nonempty.*

To apply the previous setting, we consider the operator $G = (G_1, G_2) : \mathbb{X} \to \mathbb{Y}$, where

$$\begin{array}{l} \mathbb{X} := \widetilde{X}_2 \times \widetilde{X}_4 \times L^4((0,T)\times\Omega_c), \ \ \mathbb{Y} := L^2((0,T)\times\Omega)\times L^4((0,T)\times\Omega), \\ \widetilde{X}_p = \{v \in X_p \ : \ \partial_n v|_{\partial\Omega} = 0\} \\ G_1 : \mathbb{X} \to L^2((0,T)\times\Omega), \quad G_2 : \mathbb{X} \to L^4((0,T)\times\Omega), \end{array}$$

and for each point $s = (u, v, f) \in \mathbb{X}$:

$$\left\{ \begin{array}{ll} G_1(s) = \partial_t u - \Delta u - \nabla\cdot(u\nabla v) & \text{in } L^2((0,T)\times\Omega), \\ G_2(s) = \partial_t v - \Delta v + v - u - fv & \text{in } L^4((0,T)\times\Omega). \end{array} \right. \tag{13}$$

Note that the boundary conditions for u and v have been directly considered in the spaces $\widetilde{X}_p$. By defining

$$\mathbb{M} := (\hat{u}, \hat{v}, 0) + \widehat{X}_2 \times \widehat{X}_4 \times \mathcal{F},$$

with $(\hat{u}, \hat{v}, \hat{f})$ a global weak solution of (8), $\widehat{X}_p = \{v \in X_p \ : \ v(0) = 0, \ \partial_n v|_{\partial\Omega} = 0\}$ and $\mathcal{F}$ is defined by

$$\mathcal{F} \subset= L^4(0,T;L^4(\Omega_c)) \ \text{ is a nonempty, closed and convex set,}$$

then, the optimal control problem (6) and (8) is reformulated as (10).

4.2.1. Step 1.

Any $(\widehat{u},\widehat{v},\widehat{f}) \in \mathcal{S}_{ad}$ is a **"regular point"**, i.e. for any data $(g_1, g_2) \in L^2((0,T)\times\Omega)\times L^4((0,T)\times\Omega)$, the linearized problem around $(\widehat{u},\widehat{v},\widehat{f})$:

$$\left\{ \begin{array}{ll} \partial_t U - \Delta U - \nabla\cdot(U\nabla\widehat{v} + \widehat{u}\nabla V) = g_u, & \text{in } L^2((0,T)\times\Omega), \\ \partial_t V - \Delta V + V - U - \widehat{f}\, V\, 1_{\Omega_c} = g_v & \text{in } L^4((0,T)\times\Omega), \end{array} \right.$$

has a solution $(U, V, F) \in \widehat{X}_2 \times \widehat{X}_4 \times C(\mathcal{F})$, with $C(\mathcal{F}) = \{\theta(f - \widehat{f}) : \ f \in \mathcal{F}, \ \theta \geq 0\}$.

4.2.2. Step 2.

By applying Theorem 2, for any $(\tilde{u}, \tilde{v}, \tilde{f})$ local optimal solution, then there exists Lagrange multipliers

$$(\lambda, \eta) \in L^2((0,T)\times\Omega) \times L^{4/3}((0,T)\times\Omega),$$

which is a (very-weak) solution of the variational problem:

$$\int_0^T \Big(\partial_t U - \Delta U - \nabla\cdot(U\nabla\widehat{v} + \widehat{u}\nabla V)\Big)\lambda - \int_0^T \int_\Omega U\eta = \gamma_u \int_0^T \int_\Omega (\tilde{u} - u_d)U,$$
$$\int_0^T \int_\Omega \Big(\partial_t V - \Delta V + V\Big)\eta - \int_0^T \int_{\Omega_c} \tilde{f} V\eta - \int_0^T \int_\Omega \nabla\cdot(\tilde{u}\nabla V)\lambda = \gamma_v \int_0^T \int_\Omega (\tilde{v} - v_d)V, \quad (14)$$

for any $U \in \widehat{X}_2$ and $V \in \widehat{X}_4$.

4.2.3. Step 3.

Let $\tilde{s} = (\tilde{u}, \tilde{v}, \tilde{f}) \in \mathcal{S}_{ad}$ be a local optimal solution for the control problem (10). Then, by using existence of regular solution of the multiplier problem (14) and uniqueness between a very-weak solution and a regular solution of this multiplier problem ([5]), then the Lagrange multiplier (λ, η) is regular and, jointly to the local optimal solution $(\tilde{u}, \tilde{v}, \tilde{f})$, the following optimality system holds

$$\left\{\begin{array}{rcl} -\partial_t\lambda - \Delta\lambda + \nabla\lambda\cdot\nabla\tilde{v} - \eta &=& \gamma_u(\tilde{u} - u_d) \ \text{ in } Q,\\ -\partial_t\eta - \Delta\eta + \eta - \nabla\cdot(\tilde{u}\nabla\lambda) - \tilde{f}\,\eta\,1_{\Omega_c} &=& \gamma_v(\tilde{v} - v_d) \ \text{ in } Q,\\ \lambda(T) = 0, \ \ \eta(T) &=& 0 \quad \text{in } \Omega,\\ \dfrac{\partial\lambda}{\partial\mathbf{n}} = 0, \ \dfrac{\partial\eta}{\partial\mathbf{n}} &=& 0 \quad \text{on } (0,T)\times\partial\Omega.\\ \gamma_f \displaystyle\int_0^T \int_{\Omega_c} (\tilde{f})^3 F + \int_0^T \int_{\Omega_c} \tilde{v}\,\eta\, F &\geq& 0, \quad \forall\, F \in C(\tilde{f}). \end{array}\right. \quad (15)$$

§5. Conclusions in Chemorepulsion

The bilinear control introduces a non-regular term in the chemotaxis PDE problem. In particular, classical solutions (see, for instance, Amann's arguments [1, 2]) cannot be used. Thanks to regularization and compactness arguments, the existence of weak or strong solutions for optimal bilinear control problem (7), with $h(u) = 0$, can be proved. In 1D or 2D domains, any weak solution is the unique strong solution, but in 3D-domains the existence of a more regular solution needs to impose a regularity criterium. In all cases, the existence of global optimal solution is made via a minimizing sequence argument and convexity. Finally, for any local optimal solution, it is possible to prove optimality conditions together to the existence of regular Lagrange multipliers.

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