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# RIEMANNIAN FORMULATION OF PONTRYGIN'S PRINCIPLE FOR ROBOTIC MANIPULATORS

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**Abstract.** In this work, we consider a mechanical system whose mass tensor implements a scalar product in a Riemannian manifold. This system is controlled with the help of forces and torques. A cost functional is minimized to achieve an optimal trajectory. In this contribution, this cost function is supposed to be an arbitrary regular function invariant under a change of coordinates. Optimal control evolution based on Pontryagin's principle induces a covariant second-order ordinary differential equation for an adjoint variable featuring the Riemann curvature tensor. This second order time evolution is derived in this contribution.

*Keywords:* optimal control, robotics, Riemannian geometry, Riemann curvature tensor, invariance, multibody dynamics.

*AMS classification:* 49S05, 51P05; 53A35, 70E60.

## §1. Introduction

This work is motivated by the controlled dynamics of articulated systems. The Euler-Lagrange equations are classically derived from the knowledge of kinetic and potential energies. Moreover, the control of the system can be modelled by the addition of external forces and torques. The search of an optimal dynamics depends on a given cost function. Then Pontryagin's approach [5] allows the emergence of a control law from the minimization of the cost function. After a remark of Brillouin [1], developed by Lazrak and Vallée [3] and Rojas-Quintero *et al.* [7, 9, 10]: a Riemannian structure is present in such a system. With a quadratic cost function, a remarkable result has been obtained in [7, 9, 10]: the Lagrange multiplier associated to Pontryagin's approach can be interpreted as the forces and torques submitted by the dynamical system. This property is revisited in this contribution where the cost function is not required to be a quadratic function anymore, but can be taken to be a general nonlinear function instead.

In the first section, we clarify the previous choice of a natural Riemann metric for robotics. Then in Section 2, we recall very classical results concerning differential operators on a regular Riemannian manifold. In the next section, the art of derivation suggested by Pontryagin is emphasized. In section 5, the essential of the work done by one of us [7] and published in [2, 8, 6] is briefly presented. A generalized approach is developped in Section 6: the cost function is no more quadratic as it was in our previous works. Comparing the results for quadratic and general cost functions is emphasized in the conclusion.

## §2. Riemannian metric for robotics

We consider a dynamical system parameterized by a finite number of functions of time $q^j(t)$. The manifold of states is denoted by $Q$: $q \equiv \{q^j\}$. In the case of an articulated system, the mass metric $M(q)$ depends on the general coordinates $q^j$. This mass tensor is symmetric and positive definite for each state. Then the kinetic energy

$$K(q, \dot{q}) \equiv \frac{1}{2} \sum_{k\,\ell} M_{k\ell}(q)\, \dot{q}^k\, \dot{q}^\ell \qquad (1)$$

is a positive definite quadratic form of the time derivatives $\dot{q}^j$. The coefficients $M_{k\ell}(q)$ are ideal candidates to define a Riemannian metric structure on the configuration space.

This property has been remarked many years ago by Brillouin [1]. It is also mentioned in the book of Spong and Vidyasagar [11]. In their contribution [3], Lazrak and Vallée emphasize the tensorial nature of this relation. From the positivity of the kinetic energy, the mass matrix naturally defines a Riemannian metric. This fundamental remark is the starting point of our contribution, incorporating Riemannian geometry in the field of poly-articulated systems, *id est* robotics.

## §3. Classical Riemannian geometry

We follow essentially the presentation of tensorial calculus presented in Lichnerowicz [4]. We use Einstein notation for implicit summation for repeted indices. We recall very briefly the main notions.

Inverse of the metric mass tensor $M^{-1}$: $M^{j\ell}$. We have the contraction $M_{ij}\, M^{j\ell} = \delta_i^\ell$ with $\delta_i^\ell$ the Kronecker symbol.

Covariant space differentiation along the manifold $\partial_j \equiv \frac{\partial}{\partial q^j}$. The associated contravariant basis of the tangent space $e_j$ is defined by $e_j \equiv \partial_j$. The covariant basis $e^j$ of the tangent space is defined by the relations $< e^j , e_k >= \delta_k^j$, where $< ., . >$ is the duality product between a vector space and its dual. A contravariant vector field $\varphi = \varphi^k\, e_k$ admits also covariant components $\varphi_j$. We have the relations $\varphi_j = M_{jk}\, \varphi^k$ and conversely $\varphi^k = M^{kj}\, \varphi_j$ between the contravariant components $\varphi^k$ and the covariant components.

Differentiation of a contravariant basis vector $\mathrm{d}e_j = \Gamma_{jk}^\ell\, \mathrm{d}q^k\, e_\ell$. It introduces the connection $\Gamma_{ik}^j = \frac{1}{2}\, M^{j\ell}\, (\partial_i M_{\ell k} + \partial_k M_{\ell i} - \partial_\ell M_{ik})$. These Riemann-Christofell coefficients $\Gamma_{ki}^j$ satisfy a symmetry property: $\Gamma_{ki}^j = \Gamma_{ik}^j$. Then the differentiation of the covariant basis vector satisfies the relation $\mathrm{d}e^j = -\Gamma_{k\ell}^j\, \mathrm{d}q^k\, e^\ell$.

Differentiation of a scalar field $V$: we have $\mathrm{d}V = \partial_j V\, e^j$. Then the gradient of the scalar field $V$ satisfies $\nabla V = \partial_\ell V\, e^\ell$; it is a covector field and we have $\mathrm{d}V = \partial_\ell V\, \mathrm{d}q^\ell = < \nabla V , \mathrm{d}q^j\, e_j >$. The covariant derivative of a vector field $\varphi \equiv \varphi^j\, e_j$ can be evaluated according to the relation $\mathrm{d}\varphi = (\partial_\ell \varphi^j + \Gamma_{\ell k}^j\, \varphi^k)\, \mathrm{d}q^\ell\, e_j$. Analogously, the covariant derivative of a covector field $\xi \equiv \xi_\ell\, e^\ell$ satisfies the condition $\mathrm{d}\xi = (\partial_k \xi_\ell - \Gamma_{k\ell}^j\, \xi_j)\, \mathrm{d}q^k\, e^\ell$. Then the gradient of a covector field satisfies the conditions $\nabla \xi = (\partial_k \xi_\ell - \Gamma_{k\ell}^j\, \xi_j)\, e^k\, e^\ell$. It is a two times covariant tensor and we have $\mathrm{d}\xi = < \nabla\xi \,,\, \mathrm{d}q^j \, e_j >$. Similarly, the second order gradient $\nabla^2 V$ of a scalar field $V$ is defined by the relation $\nabla^2 V = \nabla(\nabla V)$, *id est* $\nabla^2 V = (\partial_k \partial_\ell V - \Gamma^j_{k\ell} \, \partial_j V) \, e^k \, e^\ell$. It is also a two times covariant tensor.

Ricci identities for the differentiation of the metric: $\partial_j M_{k\ell} = \Gamma^p_{jk} \, M_{\ell p} + \Gamma^p_{j\ell} \, M_{kp}$. We have also $\partial_j M^{k\ell} = -\Gamma^k_{jp} \, M^{p\ell} - \Gamma^\ell_{jp} \, M^{pk}$.

The components $R^j_{ik\ell}$ of the Riemann tensor are defined by the relations

$$R^j_{ik\ell} \equiv \partial_\ell \Gamma^j_{ik} - \partial_k \Gamma^j_{i\ell} + \Gamma^p_{ik} \, \Gamma^j_{p\ell} - \Gamma^p_{i\ell} \, \Gamma^j_{pk} \,. \tag{2}$$

We observe the anti-symmetry of the Riemann tensor: $R^j_{ik\ell} = -R^j_{i\ell k}$. For a given vector field $\varphi$ and covector field $\xi$, we introduce the covector field $R_\varphi.\xi$ defined by

$$R_\varphi.\xi = R^i_{k\ell j} \, \varphi^k \, \varphi^\ell \, \xi_i \, e^j \tag{3}$$

and $(R_\varphi.\xi)_j = R^i_{k\ell j} \, \varphi^k \, \varphi^\ell \, \xi_i$.

The time derivative of a state $q(t)$ on the manifold defines a contravariant vector field $\zeta$ according to

$$\zeta = \frac{\mathrm{d}q}{\mathrm{d}t} = \Big( \frac{\mathrm{d}}{\mathrm{d}t} q^j \Big) \, e_j \equiv \dot{q}^j \, e_j \tag{4}$$

and $\zeta^j = \dot{q}^j$. In a similar way, the first order time derivative of a covector $\xi = \xi_j \, e^j$ along a trajectory $q(t)$ satisfies the conditions $\frac{\mathrm{d}\xi}{\mathrm{d}t} = (\dot{\xi}_j - \Gamma^k_{j\ell} \, \xi_k \, \zeta^\ell) \, e^j$.

**Proposition 1. - Variation of the first and second order time derivatives of a state on a Riemannian manifold**

*We consider a given trajectory position $q(t)$ on a Riemannian manifold $Q$. We denote the velocity tangent vector by $\zeta = \frac{\mathrm{d}q}{\mathrm{d}t}$. This trajectory position is supposed to vary in an infinitesimal way with the variation $\delta q = \delta q^j \, e_j$ of the state. We have the relations*

$$\delta \Big( \frac{\mathrm{d}q}{\mathrm{d}t} \Big) = \delta\zeta = \Big[ \delta(\zeta^j) + \Gamma^j_{k\ell} \, \zeta^\ell \, \delta q^k \Big] \, e_j \tag{5}$$

$$\delta \Big( \frac{\mathrm{d}^2 q}{\mathrm{d}t^2} \Big) = \delta \Big( \frac{\mathrm{d}\zeta}{\mathrm{d}t} \Big) = \Big[ \delta(\dot{\zeta}^j) + 2 \, \Gamma^j_{k\ell} \, \zeta^k \, \delta(\zeta^\ell) + \Big( \partial_k \Gamma^j_{\ell m} \, \zeta^\ell \, \zeta^m + \Gamma^j_{k\ell} \Big( \frac{\mathrm{d}\zeta}{\mathrm{d}t} \Big)^\ell \Big) \delta q^k \Big] \, e_j \tag{6}$$

- Proof of Proposition 1.

The relation (5) is an easy consequence of the variation $\delta e_\ell = \Gamma^j_{k\ell} \, \delta q^k \, e_j$ of a tangent vector in some infinitesimal variation. We have also $\frac{\mathrm{d}\zeta}{\mathrm{d}t} = [\dot{\zeta}^j + \Gamma^j_{k\ell} \, \zeta^\ell \, \zeta^k] \, e_j$. Then we have
$\delta(\frac{\mathrm{d}\zeta}{\mathrm{d}t}) = [\delta(\dot{\zeta}^j) + (\partial_k \Gamma^j_{m\ell}) \, \delta q^k \, \zeta^\ell \, \zeta^m + 2 \; \Gamma^j_{k\ell} \, \zeta^m \, \delta(\zeta^\ell)] \, e_j + (\frac{\mathrm{d}\zeta}{\mathrm{d}t})^\ell \, \Gamma^j_{k\ell} \; \delta q^k \, e_j$
and the relation (6) is established. □

**Proposition 2. - Second time derivative of a covariant vector**

*If $\xi = \xi_j \, e^j$ is a covector field on a manifold $Q$, we can explicit the components of the second time derivative $\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} = (\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2})_j \, e^j$ of this co-vector along a trajectory position $q(t)$:*

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} = \Big[ \ddot{\xi}_j - 2 \, \Gamma^k_{j\ell} \Big( \frac{\mathrm{d}\xi}{\mathrm{d}t} \Big)_k \zeta^l - \Gamma^k_{j\ell} \, \xi_k \Big( \frac{\mathrm{d}\zeta}{\mathrm{d}t} \Big)^\ell + \Big( R^k_{\ell m j} - \partial_j \Gamma^k_{\ell m} \Big) \xi_k \, \zeta^\ell \, \zeta^m \Big] \, e^j \,. \tag{7}$$

• Proof of Proposition 2.
We differentiate relatively to time the first order derivative $\frac{\mathrm{d}\xi}{\mathrm{d}t} = (\dot{\xi}_j - \Gamma^k_{j\ell}\, \xi_k\, \zeta^\ell)\, e^j$. Then

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\xi}_j - \Gamma^k_{j\ell}\, \xi_k\, \zeta^\ell)\, e^j + (\tfrac{\mathrm{d}\xi}{\mathrm{d}t})_k\, \frac{\mathrm{d}e^k}{\mathrm{d}t}$$
$$= \Big[\ddot{\xi}_j - (\partial_m \Gamma^k_{j\ell})\, \xi_k\, \zeta^\ell\, \zeta^m - \Gamma^k_{j\ell}\, \dot{\xi}_k\, \zeta^\ell - \Gamma^k_{j\ell}\, \xi_k\, \dot{\zeta}^\ell - (\tfrac{\mathrm{d}\xi}{\mathrm{d}t})_k\, \Gamma^k_{j\ell}\, \zeta^\ell\Big]\, e^j$$
$$= \Big[\ddot{\xi}_j - (\partial_m \Gamma^k_{j\ell})\, \xi_k\, \zeta^\ell\, \zeta^m - \Gamma^k_{j\ell}\Big((\tfrac{\mathrm{d}\xi}{\mathrm{d}t})_k + \Gamma^m_{kp}\, \xi_m\, \zeta^p\Big)\zeta^\ell - \Gamma^k_{j\ell}\, \xi_k\Big((\tfrac{\mathrm{d}\zeta}{\mathrm{d}t})^\ell - \Gamma^\ell_{pq}\, \zeta^p\, \zeta^q\Big)$$
$$-\Gamma^k_{j\ell}\, (\tfrac{\mathrm{d}\xi}{\mathrm{d}t})_k\, \zeta^\ell\Big]\, e^j$$
$$= \Big[\ddot{\xi}_j - 2\,\Gamma^k_{j\ell}\, (\tfrac{\mathrm{d}\xi}{\mathrm{d}t})_k\, \zeta^\ell - \Gamma^k_{j\ell}\, \xi_k\, (\tfrac{\mathrm{d}\zeta}{\mathrm{d}t})^\ell + (-\partial_m \Gamma^k_{j\ell} + \Gamma^s_{m\ell}\, \Gamma^k_{sj} - \Gamma^s_{j\ell}\, \Gamma^k_{sm})\, \xi_k\, \zeta^\ell\, \zeta^m\Big]\, e^j.$$

But thanks to (2), we have $R^k_{\ell m j} = \partial_j \Gamma^k_{\ell m} - \partial_m \Gamma^k_{j\ell} + \Gamma^s_{m\ell}\, \Gamma^k_{sj} - \Gamma^s_{j\ell}\, \Gamma^k_{sm}$
and we deduce that $-\partial_m \Gamma^k_{j\ell} + \Gamma^s_{m\ell}\, \Gamma^k_{sj} - \Gamma^s_{j\ell}\, \Gamma^k_{sm} = R^k_{\ell m j} - \partial_j \Gamma^k_{\ell m}$.
Then $\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} = [\ddot{\xi}_j - 2\,\Gamma^k_{j\ell}\, (\frac{\mathrm{d}\xi}{\mathrm{d}t})_k\, \zeta^l - \Gamma^k_{j\ell}\, \xi_k\, (\frac{\mathrm{d}\zeta}{\mathrm{d}t})^\ell + (R^k_{\ell m j} - \partial_j \Gamma^k_{\ell m})\, \xi_k\, \zeta^\ell\, \zeta^m]\, e^j$
and the property is established. □

## §4. Pontryagin framework for differential equations

We consider a dynamical system in a finite dimensional euclidian space. A state vector $y(t, \lambda) \in \mathbb{R}^d$ is submitted to a system of first order differential equations

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y(t), \lambda(t), t). \tag{8}$$

This system is controlled by a set of dynamical parameters $\lambda(t)$. The initial condition takes the form $y(0, \lambda) = x$. We search an optimal solution that minimizes the cost function

$$J(\lambda) \equiv \int_0^T g(y(t), \lambda(t), t)\, \mathrm{d}t \tag{9}$$

Pontryagin's main idea (see *e.g.* [5]) can be formulated as follows. Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y(t), \lambda(t), t)$ as a constraint satisfied by the variable $y$ and introduce a Lagrange multiplier $p = p(t)$ associated with this constraint. Then a Lagrangian functional

$$\mathcal{L}(y, \lambda, p) \equiv \int_0^T g(y, \lambda, t)\, \mathrm{d}t + \int_0^T p(t)\Big(\frac{\mathrm{d}y}{\mathrm{d}t} - f(y, \lambda, t)\Big)\, \mathrm{d}t$$

is naturally associated with the cost function and the differential equation viewed as a constraint. After a classical integration by parts of the variation $\delta\mathcal{L}$ of the Lagrangian (see *e.g.* [5]), it is well known that if the adjoint state $p(t)$ satisfies the following adjoint equation

$$\frac{\mathrm{d}p}{\mathrm{d}t} + p\,\frac{\partial f}{\partial y} - \frac{\partial g}{\partial y} = 0$$

and the final condition: $p\,(T) = 0$, then the variation $\delta J$ of the cost function is given by the relation

$$\delta J = \int_0^T \Big[\, \frac{\partial g}{\partial \lambda} - p\,\frac{\partial f}{\partial \lambda}\, \Big]\, \delta\lambda(t)\, \mathrm{d}t$$

for a given variation $\delta\lambda$ of the parameter. At the optimum this variation is identically null and this is expressed with the Pontryagin optimality condition $\frac{\partial g}{\partial \lambda} - p\,\frac{\partial f}{\partial \lambda} = 0$.

## §5. Optimal dynamics for a quadratic cost functional

We consider now a mechanical system described by a state $q(t)$ on a manifold $Q$ of finite dimension. We suppose given a mechanical Lagrangian

$$L(q,\,\zeta) \equiv K(q,\zeta) - V(q)$$

with $K(q,\,\zeta) \equiv M_{k\ell}(q)\,\zeta^k\,\zeta^\ell$ the kinetic energy of the system. It defines a metric through the mass matrix as observed previously in (1). The Euler-Lagrange equations of a free evolution take the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial \zeta^i}\Big) = \frac{\partial L}{\partial q^i}$$

for all degrees of freedom. These equations take a Riemannian form:

$$M_{k\ell}\,(\dot\zeta^\ell + \Gamma^\ell_{ij}\,\zeta^i\,\zeta^j) + \partial_k V = 0 \tag{10}$$

and the proof of this relation can be found in [2, 7]. After some index juggling, the relation (10) can be written $\dot\zeta^j + \Gamma^j_{k\ell}\,\zeta^k\,\zeta^\ell + M^{j\ell}\,\partial_\ell V = 0$.

The objective of an engineering process is the control of the state $q(t)$ along the time, adding forces and torques $u = u_k\,e^k$ to the natural evolution. Observe that the control source $u$ is a covariant vector field on the manifold. We obtain with this process (see *e.g.* [7]) the evolution equations

$$M_{k\ell}\,(\dot\zeta^\ell + \Gamma^\ell_{ij}\,\zeta^i\,\zeta^j) + \partial_k V = u_k.$$

We can introduce the contravariant components $u^j = M^{jk}\,u_k$ for the covector. Then the dynamical evolution equations can be written as

$$\dot\zeta^j + \Gamma^j_{k\ell}\,\zeta^k\,\zeta^\ell + M^{j\ell}\,\partial_\ell V = u^j\,. \tag{11}$$

A fundamental idea of our approach [3] is to enforce the coherence of the controlled mechanical system with a cost function $J(u)$ that respects the Riemannian structure of the free evolution. The choice of a quadratic functional is proposed in [7]:

$$J(u) = \frac{1}{2}\int_0^T M_{k\ell}(q)\,u^k\,u^\ell\,\mathrm{d}t \tag{12}$$

It is possible to make a link with the Pontryagin approach (8)(9) with the choice proposed in [2]:

$$y = \{q^j\,,\,\zeta^j\}\,,\ f = \{\zeta^j\,,\,-\Gamma^j_{k\ell}\,\zeta^k\,\zeta^\ell - M^{j\ell}\,\partial_\ell V + u^j\}\,,\ \lambda = \{u^k\}\,,\ g = \frac{1}{2}\,M_{k\ell}(q)\,u^k\,u^\ell.$$

Observe that the quadratic functional (12) has an intrinsic structure that respects the fundamental mechanical constraints. The Lagrange multiplers or adjoint states take the form

$p = \{\rho_j\,,\,\xi_j\}$ with $\rho = \rho_j\, e^j$ associated with the first equation $\frac{\mathrm{d}q}{\mathrm{d}t} = \zeta$ and $\xi = \xi_j\, e^j$ multiplying the dynamics $\dot{\zeta}^j + \Gamma^j_{k\ell}\, \zeta^k\, \zeta^\ell + M^{j\ell}\, \partial_\ell V - u^j = 0$. A very beautiful result established in [7] is the interpretation of the adjoint state $\xi$ as exactly equal to the forces and torques. We have

$$\xi = u,$$

*id est* $\xi_k = u_k$ for all the covariant components. Moreover, a precise evolution equation for the dual variable has been established.

**Theorem 3. - Covariant evolution equation of the optimal force**

*With the above notations and hypotheses, the forces and torques* $u$ *satisfy the following time evolution:*

$$\Big(\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}\Big)_j + R^i_{k\ell j}\, \dot{q}^k\, \dot{q}^\ell\, u_i + (\nabla^2_{jk} V)\, u^k = 0\,. \tag{13}$$

This relation has been derived in Rojas-Quintero's thesis [7], and is presented in [2].

One fundamental case is the double pendulum and it has been considered for an experimental confrontation. In this case, the manifold $Q$ is of dimension 2. The efficiency of the choice of a covariant quadradic functional is not *a priori* obvious. It is studied for the double pendulum and compared with experiments and simulations in the references [10] and [8].

## §6. General second order covariant adjoint equation

We consider in this contribution a general cost function

$$J(u) = \int_0^T \gamma(q,\, \zeta,\, u)\; \mathrm{d}t \tag{14}$$

instead of the quadratic functional (12). The Lagrangian of the problem introduces the adjoint states $\rho$ and $\xi$ relative to each equation of the dynamical system

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \zeta\,,\;\; \frac{\mathrm{d}\zeta}{\mathrm{d}t} - \psi(q) = u \tag{15}$$

and we have

$$\mathcal{L} = J(u) + \int_0^T \rho\Big(\frac{\mathrm{d}q}{\mathrm{d}t} - \zeta\Big)\; \mathrm{d}t + \int_0^T \xi\Big(\frac{\mathrm{d}\zeta}{\mathrm{d}t} - \psi(q) - u\Big)\; \mathrm{d}t. \tag{16}$$

**Proposition 4. - Variation of the Lagrangian**
*For arbitrary variations* $(\delta q,\, \delta\zeta)$ *of the state* $(q,\, \zeta)$*,* $(\delta p,\, \delta\xi)$ *of the Lagrange multipliers* $p$ *and* $\xi$*, and* $\delta u$ *of the control variable* $u$*, we have the following variation* $\delta\mathcal{L}$ *of the la-*

*grangian defined in (16):*

$$\delta\mathcal{L} = \begin{cases} \int_0^T \delta\rho\Big(\frac{\mathrm{d}q}{\mathrm{d}t} - \zeta\Big)\,\mathrm{d}t + \int_0^T \delta\xi\Big(\frac{\mathrm{d}\zeta}{\mathrm{d}t} - \psi(q) - u\Big)\,\mathrm{d}t \\ +\Big[\Big(\frac{\partial\gamma}{\partial\zeta} - \frac{\mathrm{d}\xi}{\mathrm{d}t}\Big)\delta q + \xi\,\delta\zeta\Big]_0^T + \int_0^T \Big(\frac{\partial\gamma}{\partial u} - \xi\Big)(\delta u)^j\,\mathrm{d}t \\ + \int_0^T \Big[\Big(\frac{\partial\gamma}{\partial q}\Big)_j - \Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial\gamma}{\partial\zeta}\Big)\Big)_j + \Big(\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2}\Big)_j \\ \qquad + R^k_{\ell jm}\,\xi_k\,\zeta^\ell\,\zeta^m - \xi_k\,(\partial_j\psi)^k\Big]\,\delta q^j\,\mathrm{d}t\,, \end{cases} \tag{17}$$

*where* $R^k_{\ell jm}$ *is the Riemann curvature tensor defined in (2).*

- Proof of Proposition 4.

The Lagrangien of this problem can be written $\mathcal{L} = J(u) + \mathcal{L}_1 + \mathcal{L}_2$ with

$$\mathcal{L}_1 = \int_0^T \rho\Big(\frac{\mathrm{d}q}{\mathrm{d}t} - \zeta\Big)\,\mathrm{d}t\,,\ \mathcal{L}_2 = \int_0^T \xi\Big(\frac{\mathrm{d}\zeta}{\mathrm{d}t} - \psi(q) - u\Big)\,\mathrm{d}t. \tag{18}$$

Recall that we have $\rho = \rho_j\,e^j$, $\xi = \xi_j\,e^j$, $(\delta q)^j = \delta(q^j)$, $(\delta\zeta)^j = \delta(\zeta^j) + \Gamma^j_{k\ell}\,\zeta^k\,\delta q^\ell$ and $(\delta u)^j = \delta(u^j) + \Gamma^j_{k\ell}\,u^k\,\delta q^\ell$. We take the variation of the three terms of the Lagrangian function. For the cost function defined in (14), we have

$\delta J = \int_0^T [\frac{\partial\gamma}{\partial q}\,\delta q + \frac{\partial\gamma}{\partial\zeta}\,\delta\zeta + \frac{\partial\gamma}{\partial u}\,\delta u]\,\mathrm{d}t$

$\quad = \int_0^T [(\frac{\partial\gamma}{\partial q})_j\,\delta q^j + (\frac{\partial\gamma}{\partial\zeta})_j\,(\delta\zeta)^j + (\frac{\partial\gamma}{\partial u})_j\,(\delta u)^j]\,\mathrm{d}t$

$\quad = \int_0^T [(\frac{\partial\gamma}{\partial q})_j\,\delta q^j + (\frac{\partial\gamma}{\partial\zeta})_j\,(\delta(\zeta^j) + \Gamma^j_{k\ell}\,\zeta^k\,\delta q^\ell) + (\frac{\partial\gamma}{\partial u})_j\,(\delta u)^j]\,\mathrm{d}t$

and

$$\delta J = \int_0^T \Big[\Big(\frac{\partial\gamma}{\partial q}\Big)_j + \Gamma^\ell_{kj}\,\Big(\frac{\partial\gamma}{\partial\zeta}\Big)_\ell\,\zeta^k\Big]\,\delta q^j\,\mathrm{d}t + \int_0^T \Big[\Big(\frac{\partial\gamma}{\partial\zeta}\Big)_j\,\delta(\zeta^j) + \Big(\frac{\partial\gamma}{\partial u}\Big)_j\,(\delta u)^j\Big]\,\mathrm{d}t. \tag{19}$$

From $\frac{\mathrm{d}q}{\mathrm{d}t} = \dot{q}^j\,e_j = \zeta^j\,e_j$, we have

$\delta(\frac{\mathrm{d}q}{\mathrm{d}t}) = \delta\dot{q}^j\,e_j + \dot{q}^j\,\delta e_j = (\delta\dot{q}^j + \Gamma^j_{k\ell}\,\zeta^k\,\delta q^\ell) = (\delta(\zeta^j) + \Gamma^j_{k\ell}\,\zeta^k\,\delta q^\ell) = (\delta\zeta)^j$

and by recalling (5) of Proposition 1,

$\delta(\rho\,(\frac{\mathrm{d}q}{\mathrm{d}t} - \zeta)) = \delta\rho\,(\frac{\mathrm{d}q}{\mathrm{d}t} - \zeta) + \rho_j\,(\delta\dot{q}^j + \Gamma^j_{k\ell}\,\zeta^k\,\delta q^\ell - (\delta\zeta)^j)$

$\qquad = \delta\rho\,(\frac{\mathrm{d}q}{\mathrm{d}t} - \zeta) + \frac{\mathrm{d}}{\mathrm{d}t}(\rho_j\,\delta q^j) - \dot{\rho}_j\,\delta q^j - \rho_j\,\delta(\zeta^j).$

Then integrating by parts

$\delta\mathcal{L}_1 = \int_0^T \delta\rho\,(\frac{\mathrm{d}q}{\mathrm{d}t} - \zeta)\,\mathrm{d}t + [\rho_j\,\delta q^j]_0^T - \int_0^T \dot{\rho}_j\,\delta q^j\,\mathrm{d}t - \int_0^T \rho^j\,\delta(\zeta^j)\,\mathrm{d}t$

and

$$\delta\mathcal{L}_1 = [\rho_j\,\delta q^j]_0^T + \int_0^T \delta\rho\Big(\frac{\mathrm{d}q}{\mathrm{d}t} - \zeta\Big)\,\mathrm{d}t - \int_0^T \Big(\dot{\rho}_j\,\delta q^j + \rho_j\,\delta(\zeta^j)\Big)\,\mathrm{d}t\,. \tag{20}$$

We observe now that we have for the contravariant vector field $\delta\psi = (\delta\psi^j + \Gamma^j_{k\ell}\,\psi^k\,\delta q^\ell)\,e_j$. We keep the compact expression $\delta u = (\delta u)^j\,e_j$. We can develop the third term:

$\delta\mathcal{L}_2 = \int_0^T \delta\xi \, (\frac{d\zeta}{dt} - \psi(q) - u) \, dt + \int_0^T \xi \, (\delta \frac{d\zeta}{dt} - \delta\psi(q) - \delta u) \, dt$

and from (6) and Lemma 2, we have

$$\delta\mathcal{L}_2 = \int_0^T \delta\xi \, (\tfrac{d\zeta}{dt} - \psi(q) - u) \, dt + \int_0^T \xi_j \, [\delta\dot{\zeta}^j + (\partial_k \Gamma^j_{\ell m} \, \zeta^\ell \, \zeta^m + \Gamma^j_{k\ell} \, (\tfrac{d\zeta}{dt})^\ell) \, \delta q^k + 2 \, \Gamma^j_{k\ell} \, \zeta^k \, \delta(\zeta^\ell)] \, dt$$
$$- \int_0^T \xi_j \, (\partial_\ell \psi^j + \Gamma^j_{k\ell} \, \psi^k) \, \delta q^\ell \, dt - \int_0^T \xi_j \, (\delta u)^j \, dt \quad \text{and}$$

$$\delta\mathcal{L}_2 = \begin{cases} \int_0^T \delta\xi \, (\frac{d\zeta}{dt} - \psi(q) - u) \, dt + [\xi_j \, \delta(\zeta^j)]_0^T + \int_0^T (-\dot{\xi}_j + 2 \, \Gamma^k_{j\ell} \, \xi_k \, \zeta^\ell) \, \delta(\zeta^\ell) \, dt \\ + \int_0^T [\xi_k \, (\partial_j \Gamma^k_{\ell m} \, \zeta^\ell \, \zeta^m + \Gamma^k_{j\ell} \, (\frac{d\zeta}{dt})^\ell - \partial_j \psi^k - \Gamma^k_{j\ell} \, \psi^\ell)] \, \delta q^j \, dt - \int_0^T \xi_j \, (\delta u)^j \, dt. \end{cases} \tag{21}$$

We can now add the three contributions detailed in the relations (19), (20) and (21):

$$\delta\mathcal{L} = [\rho_j \, \delta q^j + \xi_j \, \delta(\zeta^j)]_0^T + \int_0^T \delta\rho \, (\tfrac{dq}{dt} - \zeta) \, dt + \int_0^T \delta\xi \, (\tfrac{d\zeta}{dt} - \psi(q) - u) \, dt$$
$$+ \int_0^T [(\tfrac{\partial\gamma}{\partial q})_j + \Gamma^\ell_{kj} \, (\tfrac{\partial\gamma}{\partial\zeta})_\ell \, \zeta^k - \dot{\rho}_j + (\partial_j \Gamma^k_{\ell m}) \, \xi_k \, \zeta^\ell \, \zeta^m + \Gamma^k_{j\ell} \, \xi_k \, (\tfrac{d\zeta}{dt})^\ell - \xi_k \, (\partial_j \psi^k + \Gamma^k_{j\ell} \, \psi^\ell)] \, \delta q^j \, dt$$
$$+ \int_0^T [(\tfrac{\partial\gamma}{\partial u})_j - \xi_j] \, (\delta u)^j \, dt + \int_0^T [(\tfrac{\partial\gamma}{\partial\zeta})_j - \rho_j - \dot{\xi}_j + 2 \, \Gamma^k_{j\ell} \, \xi_k \, \zeta^\ell] \, \delta(\zeta^j) \, dt.$$

Because $\delta(\zeta^j) = \delta\dot{q}^j = \frac{d}{dt}(\delta q^j)$, we can integrate by parts the last term and we obtain

$$\delta\mathcal{L} = [\rho_j \, \delta q^j + \xi_j \, \delta(\zeta^j)]_0^T + \int_0^T \delta\rho \, (\tfrac{dq}{dt} - \zeta) \, dt + \int_0^T \delta\xi \, (\tfrac{d\zeta}{dt} - \psi(q) - u) \, dt$$
$$+ \int_0^T [(\tfrac{\partial\gamma}{\partial q})_j + \Gamma^\ell_{kj} \, (\tfrac{\partial\gamma}{\partial\zeta})_\ell \, \zeta^k - \dot{\rho}_j + (\partial_j \Gamma^k_{\ell m}) \, \xi_k \, \zeta^\ell \, \zeta^m + \Gamma^k_{j\ell} \, \xi_k \, (\tfrac{d\zeta}{dt})^\ell - \xi_k \, (\partial_j \psi^k + \Gamma^k_{j\ell} \, \psi^\ell)] \, \delta q^j \, dt$$
$$+ \int_0^T [(\tfrac{\partial\gamma}{\partial u})_j - \xi_j] \, (\delta u)^j \, dt + [((\tfrac{\partial\gamma}{\partial\zeta})_j - \rho_j - \dot{\xi}_j + 2 \, \Gamma^k_{j\ell} \, \xi_k \, \zeta^\ell) \, \delta q^j]_0^T$$
$$- \int_0^T \tfrac{d}{dt}[(\tfrac{\partial\gamma}{\partial\zeta})_j - \rho_j - \dot{\xi}_j + 2 \, \Gamma^k_{j\ell} \, \xi_k \, \zeta^\ell] \, \delta q^j \, dt$$
$$= [((\tfrac{\partial\gamma}{\partial\zeta})_j - \dot{\xi}_j + 2 \, \Gamma^k_{j\ell} \, \xi_k \, \zeta^\ell)) \, \delta q^j + \xi_j \, \delta(\zeta^j)]_0^T + \int_0^T \delta\rho \, (\tfrac{dq}{dt} - \zeta) \, dt + \int_0^T \delta\xi \, (\tfrac{d\zeta}{dt} - \psi(q) - u) \, dt$$
$$+ \int_0^T [(\tfrac{\partial\gamma}{\partial u})_j - \xi_j] \, (\delta u)^j \, dt + \int_0^T [(\tfrac{\partial\gamma}{\partial q})_j + \Gamma^\ell_{kj} \, (\tfrac{\partial\gamma}{\partial\zeta})_\ell \, \zeta^k - \tfrac{d}{dt}(\tfrac{\partial\gamma}{\partial\zeta})_j + \ddot{\xi}_j + (\partial_j \Gamma^k_{\ell m}) \, \xi_k \, \zeta^\ell \, \zeta^m$$
$$+ \Gamma^k_{j\ell} \, \xi_k \, (\tfrac{d\zeta}{dt})^\ell - 2 \, \tfrac{d}{dt}(\Gamma^k_{j\ell} \, \xi_k \, \zeta^\ell) - \xi_k \, (\partial_j \psi^k + \Gamma^k_{j\ell} \, \psi^\ell)] \, \delta q^j \, dt.$$

The boundary term can be simplified:

$$[((\tfrac{\partial\gamma}{\partial\zeta})_j - \dot{\xi}_j + 2 \, \Gamma^k_{j\ell} \, \xi_k \, \zeta^\ell)) \, \delta q^j + \xi_j \, \delta(\zeta^j)]_0^T = [((\tfrac{\partial\gamma}{\partial\zeta})_j - (\tfrac{d\xi}{dt})_j) \, \delta q^j + \xi_j \, (\delta(\zeta^j) + \Gamma^j_{k\ell} \, \zeta^k \, \delta q^\ell)]_0^T$$
$$= [((\tfrac{\partial\gamma}{\partial\zeta})_j - (\tfrac{d\xi}{dt})_j) \, \delta q^j + \xi_j \, (\delta\zeta)^j]_0^T = [(\tfrac{\partial\gamma}{\partial\zeta} - \tfrac{d\xi}{dt}) \, \delta q + \xi \, \delta\zeta]_0^T$$

and natural boundary conditions are put in evidence.

We focus now our attention on the term containing the variation $\delta q$ in factor. We have

$$\int_0^T [(\tfrac{\partial\gamma}{\partial q})_j + \Gamma^\ell_{kj} \, (\tfrac{\partial\gamma}{\partial\zeta})_\ell \, \zeta^k - \tfrac{d}{dt}(\tfrac{\partial\gamma}{\partial\zeta})_j + \ddot{\xi}_j + (\partial_j \Gamma^k_{\ell m}) \, \xi_k \, \zeta^\ell \, \zeta^m + \Gamma^k_{j\ell} \, \xi_k \, (\tfrac{d\zeta}{dt})^\ell$$
$$-2 \, \tfrac{d}{dt}(\Gamma^k_{j\ell} \, \xi_k \, \zeta^\ell) - \xi_k \, (\partial_j \psi^k + \Gamma^k_{j\ell} \, \psi^\ell)] \, \delta q^j \, dt$$
$$= \int_0^T [(\tfrac{\partial\gamma}{\partial q})_j - (\tfrac{d}{dt}(\tfrac{\partial\gamma}{\partial\zeta}))_j + \ddot{\xi}_j + (\partial_j \Gamma^k_{\ell m}) \, \xi_k \, \zeta^\ell \, \zeta^m + \Gamma^k_{j\ell} \, \xi_k \, (\tfrac{d\zeta}{dt})^\ell - 2 \, (\partial_m \Gamma^k_{j\ell}) \, \xi_k \, \zeta^\ell \, \zeta^m$$
$$-2 \, \Gamma^k_{j\ell} \, ((\tfrac{d\xi}{dt})_k + \Gamma^s_{kp} \, \xi_s \, \zeta^p) \, \zeta^\ell - 2 \, \Gamma^k_{j\ell} \, \xi_k \, ((\tfrac{d\zeta}{dt})^\ell - \Gamma^\ell_{sm} \, \zeta^s \, \zeta^m) - \xi_k \, (\partial_j \psi)^k] \, \delta q^j \, dt$$
$$= \int_0^T [(\tfrac{\partial\gamma}{\partial q})_j - (\tfrac{d}{dt}(\tfrac{\partial\gamma}{\partial\zeta}))_j + \ddot{\xi}_j - \Gamma^k_{j\ell} \, \xi_k \, (\tfrac{d\zeta}{dt})^\ell - 2 \, \Gamma^k_{j\ell} \, (\tfrac{d\xi}{dt})_k \, \zeta^\ell + (\partial_j \Gamma^k_{\ell m} - 2 \, \partial_m \Gamma^k_{j\ell}) \, \xi_k \, \zeta^\ell \, \zeta^m$$
$$+ 2 \, (\Gamma^s_{\ell m} \, \Gamma^k_{sj} - \Gamma^s_{j\ell} \, \Gamma^k_{sm}) \, \xi_k \, \zeta^\ell \, \zeta^m - \xi_k \, (\partial_j \psi)^k] \, \delta q^j \, dt$$
$$= \int_0^T [(\tfrac{\partial\gamma}{\partial q})_j - (\tfrac{d}{dt}(\tfrac{\partial\gamma}{\partial\zeta}))_j + \ddot{\xi}_j - \Gamma^k_{j\ell} \, \xi_k \, (\tfrac{d\zeta}{dt})^\ell - 2 \, \Gamma^k_{j\ell} \, (\tfrac{d\xi}{dt})_k \, \zeta^\ell + (\partial_j \Gamma^k_{\ell m}) \, \xi_k \, \zeta^\ell \, \zeta^m$$

$+ 2\,(R^k_{\ell m j} - \partial_j \Gamma^k_{\ell m})\,\xi_k\,\zeta^\ell\,\zeta^m - \xi_k\,(\partial_j\psi)^k]\,\delta q^j\ \mathrm{d}t$

because $R^k_{\ell m j} = \partial_j \Gamma^k_{\ell m} - \partial_m \Gamma^k_{j\ell} + \Gamma^s_{\ell m}\,\Gamma^k_{sj} - \Gamma^s_{j\ell}\,\Gamma^k_{sm}$

$= \int_0^T [(\frac{\partial\gamma}{\partial q})_j - (\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial\gamma}{\partial\zeta}))_j + \ddot{\xi}_j - \Gamma^k_{j\ell}\,\xi_k\,(\frac{\mathrm{d}\zeta}{\mathrm{d}t})^\ell - 2\,\Gamma^k_{j\ell}\,(\frac{\mathrm{d}\xi}{\mathrm{d}t})_k\,\zeta^\ell$

$+ 2\,R^k_{\ell m j}\,\xi_k\,\zeta^\ell\,\zeta^m - (\partial_j\Gamma^k_{\ell m})\,\xi_k\,\zeta^\ell\,\zeta^m - \xi_k\,(\partial_j\psi)^k]\,\delta q^j\ \mathrm{d}t$

$= \int_0^T [(\frac{\partial\gamma}{\partial q})_j - (\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial\gamma}{\partial\zeta}))_j + (\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2})_j + R^k_{\ell m j}\,\xi_k\,\zeta^\ell\,\zeta^m - \xi_k\,(\partial_j\psi)^k]\,\delta q^j\ \mathrm{d}t$

due to Lemma 2. We deduce a new expression for the variation of the Lagrangian:

$\delta\mathcal{L} = [(\frac{\partial\gamma}{\partial\zeta} - \frac{\mathrm{d}\xi}{\mathrm{d}t})\,\delta q + \xi\,\delta\zeta]_0^T + \int_0^T \delta\rho\,(\frac{\mathrm{d}q}{\mathrm{d}t} - \zeta)\,\mathrm{d}t + \int_0^T \delta\xi\,(\frac{\mathrm{d}\zeta}{\mathrm{d}t} - \psi(q) - u)\,\mathrm{d}t$

$+ \int_0^T (\frac{\partial\gamma}{\partial u} - \xi)\,\delta u\ \mathrm{d}t + \int_0^T [(\frac{\partial\gamma}{\partial q})_j - (\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial\gamma}{\partial\zeta}))_j + (\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2})_j + R^k_{\ell m j}\,\xi_k\,\zeta^\ell\,\zeta^m - \xi_k\,(\partial_j\psi)^k]\,\delta q^j\ \mathrm{d}t$

and the Proposition is established. □

We observe from (17) that the Pontryagin optimality condition is written

$$\frac{\partial\gamma}{\partial u} = \xi.$$

The adjoint variable $\xi$ is no more equal to the forces and torques $u$ but the relation between the two variables is completely explicited.

The boundary conditions take the quite unusual form

$$\Big[\Big(\frac{\partial\gamma}{\partial\zeta} - \frac{\mathrm{d}\xi}{\mathrm{d}t}\Big)\,\delta q + \xi\,\delta\zeta\Big]_0^T = 0 \tag{22}$$

because they can cover several cases. To fix the ideas, when the initial conditions take the usual form $q(0) = q_0$ and $\zeta(0) = \zeta_0$, with fixed given data $q_0$ and $\zeta_0$, we have in consequence $\delta q(0) = 0$ and $\delta\zeta(0) = 0$. Then the boundary conditions (22) express simply a null condition at the final time: $\xi(T) = 0$ and $(\frac{\mathrm{d}\xi}{\mathrm{d}t} - \frac{\partial\gamma}{\partial\zeta})(T) = 0$. We can also consider for other applications that initial and final states are imposed: $q(0) = q_0$ and $q(T) = q_T$. In this case, $\delta q(0) = \delta q(T) = 0$ and the expression (22) express conditions for the second Lagrange multiplier at the initial and final time: $\xi(0) = \xi(T) = 0$. Other cases can be naturally considered.

**Theorem 5. - Second order adjoint evolution equation**
*When the source term derives from a potential, id est* $\psi^k(q) = -\partial^k V = M^{k\ell}\,\partial_\ell V$*, then we have no constraint for the first adjoint state* $\rho$ *and we have a second order dynamics for the second Lagrange multiplier:*

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} - R_\zeta\,.\,\xi + \frac{\partial\gamma}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial\gamma}{\partial\zeta}\Big) + \nabla^2 V.\,\xi = 0. \tag{23}$$

- Proof of Theorem 5.

From the relation (17), the second order adjoint equation can be written as

$$\Big(\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2}\Big)_j + R^k_{\ell j m}\,\xi_k\,\zeta^\ell\,\zeta^m - \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial\gamma}{\partial\zeta}\Big)_j + \Big(\frac{\partial\gamma}{\partial q}\Big)_j - \xi_k\,(\partial_j\psi)^k = 0.$$

With $\psi = -\partial_\ell V \, e^\ell$, we have the following calculus:
$\partial_j \psi = -\partial_j \partial_\ell V \, e^\ell + \Gamma^s_{j\ell} \, \partial_s V \, e^\ell = -(\nabla^2 V)_{j\ell} \, e^\ell$ and
$\xi_k \, (\partial_j \psi)^k = -M^{k\ell} \, (\nabla^2 V)_{j\ell} \, \xi_k = -(\nabla^2 V)_{j\ell} \, \xi^\ell = -(\nabla^2 V . \xi)_j$. Additionally, we establish the contraction $R^k_{\ell jm} \, \xi_k \, \zeta^\ell \, \zeta^m = (R_\zeta \, . \, \xi)_j$. Then the evolution equation can be written

$$\Big( \frac{\mathrm{d}^2 \xi}{\mathrm{d}t^2} - R_\zeta \, . \, \xi + \frac{\partial \gamma}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \big( \frac{\partial \gamma}{\partial \zeta} \big) + \nabla^2 V . \xi \Big)_j = 0.$$

and the relation (23) is established. □

## §7. Conclusion

We first compared the result for the quadratic cost function (12) developed in paragraph 4 and the present result studied in the previous section. The cost function is now more general. It was written

$$J(u) = \frac{1}{2} \int_0^T M_{k\ell}(q) \, u^k \, u^\ell \, \mathrm{d}t$$

in [2] and we write it (14)

$$J(u) = \int_0^T \gamma(q, \, \zeta, \, u) \, \mathrm{d}t$$

in this contribution. Nevertheless, the equations of the dynamical system take the same form:

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \zeta \, , \;\; \frac{\mathrm{d}\zeta}{\mathrm{d}t} - \psi(q) = u$$

with the usual condition that the internal forces derive from a potential. With the particular cost function considered in [8], the optimality condition take the form

$$\xi = u.$$

The Lagrange multiplier associated to the dynamics equation is interpreted as a force. Then the adjoint equation derived in Vallée *et al.* [7, 6] is exactly a covariant evolution equation (13) for the optimal force. With the general cost function considered in this contribution, the optimality condition can be written

$$\frac{\partial \gamma}{\partial u} = \xi.$$

The dynamics of the adjoint variable $\xi$ differs *a priori* from the one of forces and torques $u$. We have explicited this condition in (23). We observe that in comparison with (13), two new terms are present: $\frac{\partial \gamma}{\partial q}$ and $-\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial \gamma}{\partial \zeta})$.
In this contribution, we have generalized the cost function used for the Pontryagin calculus in Riemannian geometry synthesized in [2]. The cost function is still chosen in coherence with the Riemannian geometry underlying the natural evolution of the mechanical system. The applications of this approach in robotics are into development and first results are proposed in [8]. The next step is the enrichement of the model with appropriate dissipation as fluid rubbing or dry Coulomb friction.

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## References

[1] Brillouin, L. *Les tenseurs en mécanique et en élasticité*. Masson, Paris, 1938.

[2] Dubois, F., Fortuné, D., Quintero, J. R., and Vallée, C. Pontryagin calculus in riemannian geometry. In *Geometric Science of Information; F. Nielsen, F. Barbaresco (eds.)*. Springer International Publishing, 2015, pp. 541–549.

[3] Lazrak, M., and Vallée, C. Commande de robots en temps minimal. *Revue d'Automatique et de Productique Appliquées (RAPA) 8*, 2-3 (1995), 217–222.

[4] Lichnerowicz, A. *Éléments de calcul tensoriel*. Armand Colin, Paris, 1946.

[5] Pontryagin, L., Boltyanskii, V., Gamkrelidze, R., and Mishchenko, E. John Wiley and Sons, New York, 1962.

[6] Quintero, J. R., Vallée, C., Gazeau, J., and Seguin, P. Covariant formulation of optimal control of jointed arm robots: an alternative to pontryagin's principle,, 2013. Available from: `https://arxiv.org/abs/1107.1153`, `doi:10.48550/ARXIV.1107.1153`.

[7] Rojas-Quintero, J. *Contribution à la manipulation dextre dynamique pour les aspects conceptuels et de commande en ligne optimale*. PhD thesis, Poitiers University, 2013.

[8] Rojas-Quintero, J., Dubois, F., and de Ávila, H. R. Riemannian formulation of pontryagin's maximum principle for the optimal control of robotic manipulators. *Mathematics 10* (2022), 1117.

[9] Rojas-Quintero, J., Vallée, C., Gazeau, J., Seguin, P., and Arsicault, M. An alternative to pontryagin's principle for the optimal control of jointed arm robots. In *Congrès Français de Mécanique, Bordeaux, 26-30 August 2013*.

[10] Rojas-Quintero, J., Villalobos-Chin, J., and Santibanez, V. Optimal control of robotic systems using finite elements for time integration of covariant control equations. *IEEE Access 9* (2021), 104980–105001.

[11] Spong, M. W., and Vidyasagar, M. *Robot Dynamics and Control*. John Wiley, New York, 1991.

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