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CORNER CUTTING ALGORITHMS FOR Q-BÉZIER CURVES AND SURFACES

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Abstract. In this manuscript, we present a new evaluation algorithm for q-Bernstein polynomials that is corner cutting, a desired property for its good stability properties and nice geometric interpretation. Moreover, this algorithm is generalized and applied to the evaluation of rational q-Bézier curves as well as to the evaluation of rational q-Bézier surfaces.

Keywords: q-Bernstein, rational q-Bernstein, corner cutting, De Casteljau.

AMS classification: 65D15, 65D17.

§1. Introduction

For curve design purposes, the basis $(u_0(t), \dots, u_n(t))$ has to be normalized (i.e., it forms a partition of the unity: $\sum_{i=0}^n u_i(t) = 1$ for all $t \in I$) and nonnegative (i.e., $u_i(t) \geq 0$ for all $t \in I$ and $i = 0, \dots, n$). It is well known in Computer Aided Geometric Design (CAGD) that a curve representation presents nice shape preserving properties when the used normalized basis is *totally positive*, that is, when all its collocation matrices have nonnegative minors (see [1], [2])

The *Bernstein polynomials* $b_i^n(x)$, $i = 0, 1, \dots, n$, of degree n are defined as

$$b_i^n(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad x \in [0,1].$$

The Bernstein polynomials $(b_0^n, \dots, b_n^n)$ form a normalized totally positive basis of the space of polynomials of degree at most n, Π_n. Using the Bernstein polynomials, we can construct a *Bézier curve* as

$$\gamma(x) = \sum_{i=0}^n P_i b_i^n(x), \quad x \in [0,1], \tag{1}$$

where $P_i \in \mathbb{R}^k$ $(k = 2, 3)$ are the *control points* of the curve. Bézier curves and surfaces are frequently used in CAGD. In order to get greater flexibility, the q-Bernstein bases of polynomial spaces, for values $0 < q \leq 1$, have been also used to design q-Bézier curves and surfaces (see [4, 9, 10]). They belong to the field of Quantum Calculus (see [8]), which deals with q-integers, q-binomial coefficients, and other q-analogues of classical calculus that will be introduced in Section 2. The most desired algorithms in CAGD are the algorithms called corner cutting algorithms, in which all steps are linear convex combinations. In addition to their nice geometric interpretation, they satisfy nice stability properties. To the authors' knowledge, the existing evaluation algorithms in the literature for curve and surface evaluation of q-Bézier and rational q-Bézier curves and surfaces are not corner cutting algorithms. This paper fills this gap.

In Section 2 we present the corner cutting evaluation algorithm for q-Bézier curves, deriving an explicit expression for the intermediate control points defined by the algorithm. Section 3 presents the corner cutting evaluation algorithm for rational q-Bézier curves. Finally, Section 4 introduces the corner cutting evaluation algorithm for rational q-Bézier surfaces. As a particular case, it can be derived a corner cutting evaluation algorithm for tensor product q-Bézier surfaces.

§2. q-Bernstein polynomials and q-Bézier curves

In [10], Phillips introduced a generalization of the Bernstein polynomials based on the q-integers. Given a positive real number q, we define a *q-integer* $[r]$ as

$$[r] = \begin{cases} 1 + q + \cdots + q^{r-1} = \frac{1-q^r}{1-q}, & \text{if } q \neq 1 \\ r, & \text{if } q = 1. \end{cases}$$

Then we can define the following q-analogues in terms of the q-integers. The *q-factorial* $[r]!$ is defined as

$$[r]! = \begin{cases} [r] \cdot [r-1] \cdots [1], & \text{if } r \in \mathbb{N}, \\ 1, & \text{if } r = 0 \end{cases}$$

and the *q-binomial* coefficients are defined as

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n][n-1]\cdots[n-r+1]}{[r]!} = \frac{[n]!}{[r]![n-r]!}$$

for integers $n \geq r \geq 0$ and as zero otherwise. The q-binomial coefficients satisfy the following recurrence relationships (see Proposition 6.1 of [8]):

$$\begin{bmatrix} i \\ j \end{bmatrix} = \begin{bmatrix} i-1 \\ j-1 \end{bmatrix} + q^j \begin{bmatrix} i-1 \\ j \end{bmatrix}, \tag{2}$$

$$\begin{bmatrix} i \\ j \end{bmatrix} = q^{i-j} \begin{bmatrix} i-1 \\ j-1 \end{bmatrix} + \begin{bmatrix} i-1 \\ j \end{bmatrix}. \tag{3}$$

The *q-Bernstein polynomials* of degree n are defined as

$$b^n_{i,q}(x) = \begin{bmatrix} n \\ i \end{bmatrix} x^i \prod_{s=0}^{n-i-1} (1 - q^s x), \quad x \in [0,1], \quad i = 0, 1, \ldots, n, \tag{4}$$

for $0 < q \leq 1$. The q-Bernstein polynomials of degree n, $\mathcal{B}^q = (b^n_{0,q}, b^n_{1,q}, \ldots, b^n_{n,q})$, also form a normalized totally positive basis of Π_n. Let us observe that for the case $q = 1$ the q-Bernstein polynomials coincide with the classical Bernstein polynomials. The q-Bernstein polynomials can be defined in terms of the following recurrence relationship (see section 2 of [9]).

Proposition 1. *The q-Bernstein polynomials* (4) *are given by the following recurrence relationship*

$$b^n_{i,q}(x) = q^{n-i}xb^{n-1}_{i-1,q}(x) + (1 - q^{n-i-1}x)b^{n-1}_{i,q}(x), \tag{5}$$

where we consider $b^0_{0,q}(x) = 1$ and $b^k_{-1,q}(x) = b^k_{k+1,q}(x) = 0$ for any $k = 0, \ldots, n$.

Following the definition of Bézier curves, we define a *q-Bézier curve* as

$$\gamma(x) = \sum_{i=0}^{n} P_i b^n_{i,q}(x), \quad x \in [0, 1], \tag{6}$$

where the q-Bernstein polynomials are given by (4). Observe that for $q = 1$ we have again the classical Bézier curves.

The well-known de Casteljau algorithm for the evaluation of Bézier curves is an example of a corner cutting algorithm. However, the evaluation algorithms known for q-Bernstein polynomials and q-Bézier curves in general do not satisfy this property (see [4, 5]). We now introduce a corner cutting evaluation algorithm for a q-Bézier curve.

Theorem 2. *Given the control points $\{P_0, \ldots, P_n\}$, let us define the intermediate points $f^{(r)}_i$, for $r = 0, \ldots, n$ and $i = 0, \ldots, n - r$, by the following recurrence relationship:*

$$\begin{aligned} f^{(0)}_i(x) &= P_i, \\ f^{(r)}_i(x) &= q^{n-r-i}xf^{(r-1)}_{i+1}(x) + (1 - q^{n-r-i}x)f^{(r-1)}_i(x). \end{aligned} \tag{7}$$

Then we have that $\gamma(x) = f^{(n)}_0(x)$.

Proof. Let us consider the q-Bézier curve $\gamma(x) = \sum_{i=0}^{n} P_i b^n_i(x)$, $x \in [0, 1]$. We can deduce an expression for $\gamma(x)$ in terms of the q-Bernstein polynomials of degree $n - 1$ thanks to (5):

$$\gamma(x) = \sum_{i=0}^{n} P_i b^n_i(x) = \sum_{i=0}^{n} P_i \left[q^{n-i}xb^{n-1}_{i-1}(x) + (1 - q^{n-i-1})xb^{n-1}_i(x)\right]. \tag{8}$$

Then we rewrite (8) by rearranging the summands and we deduce that

$$\begin{aligned} \gamma(x) &= \sum_{i=0}^{n-1} \left[P_{i+1}q^{n-i-1}x + P_i(1 - q^{n-i-1}x)\right] b^{n-1}_i(x) \\ &= \sum_{i=0}^{n-1} \left[f^{(0)}_{i+1}(x)q^{n-i-1}x + f^{(0)}_i(x)(1 - q^{n-i-1}x)\right] b^{n-1}_i(x) \\ &= \sum_{i=0}^{n-1} f^{(1)}_i(x)b^{n-1}_i(x), \end{aligned}$$

where $f^{(1)}_i(x)$ are given by (7). Applying this argumentation r times, we deduce the following formula for the evaluation of the curve $\gamma(x)$ in terms of the q-Bernstein polynomials of degree $n - r$,

$$\gamma(x) = \sum_{i=0}^{n-r} f_i^{(r)}(x) b_i^{n-r}(x). \tag{9}$$

In particular, for $r = n$ we have that $\gamma(x) = f_0^{(n)}(x)$ and the result holds. □

The previous theorem introduces a new evaluation algorithm for q-Bézier curves. Its main advantage with respect to other known evaluation algorithms is presented in the following corollary.

Corollary 3. *All the steps performed in the evaluation algorithm given by Theorem 2 are convex combinations of the previously computed quantities.*

Proof. Since $x \in [0, 1]$ and $q \in (0, 1]$, we have that the coefficients appearing in every step of (7) are nonnegative and sum up to one. □

We have seen the relationship between the intermediate control points and the q-Bernstein bases. Now, we introduce the explicit expression of the intermediate control points, which are key for the extension of the evaluation algorithm to rational curves.

Proposition 4. *Given the control points $\{P_0, \dots, P_n\}$, the intermediate control points defined in Theorem 2 have the following expression:*

$$f_i^{(r)}(x) = \sum_{k=0}^{r} P_{i+k} \begin{bmatrix} r \\ k \end{bmatrix} (q^{n-r-i}x)^k \prod_{t=k+1}^{r} (1 - q^{n-i-t}x). \tag{10}$$

Proof. We proceed by induction over r. For $r = 0$, we see that (10) gives $f_i^{(0)} = P_i$ for any $i = 0, \dots, n$. Now let us suppose that (10) holds for $r - 1$. Then we apply the recurrence formula (7) to compute $f_i^{(r)}$:

$$\begin{aligned} f_i^{(r)}(x) =& (1 - q^{n-r-i}x) f_i^{(r-1)} + q^{n-r-i}x f_{i+1}^{(r-1)} \\ =& (1 - q^{n-r-i}x) \sum_{k=0}^{r-1} P_{i+k} \begin{bmatrix} r-1 \\ k \end{bmatrix} (q^{n-r-i+1}x)^k \prod_{t=k+1}^{r-1} (1 - q^{n-i-t}x) \\ &+ q^{n-r-i}x \sum_{k=0}^{r-1} P_{i+1+k} \begin{bmatrix} r-1 \\ k \end{bmatrix} (q^{n-r-i}x)^k \prod_{t=k+1}^{r-1} (1 - q^{n-i-1-t}x). \end{aligned} \tag{11}$$

In (11), we can compute the coefficient of every point P_i using formula (2)

$$\begin{aligned} f_i^{(r)}(x) &= \sum_{k=0}^{r} P_{i+k} \left(\begin{bmatrix} r-1 \\ k \end{bmatrix} q^k + \begin{bmatrix} r-1 \\ k-1 \end{bmatrix} \right) (q^{n-r-i}x)^k \prod_{t=k+1}^{r} (1 - q^{n-i-t}x) \\ &= \sum_{k=0}^{r} P_{i+k} \begin{bmatrix} r \\ k \end{bmatrix} (q^{n-r-i}x)^k \prod_{t=k+1}^{r} (1 - q^{n-i-t}x). \end{aligned}$$

And so, formula (10) holds. □

Finally, we provide Algorithm 1 with the pseudocode for the evaluation algorithm defined by Theorem 2.

Algorithm 1 eval.qBezier

Require: $(P_i)_{0\le i\le n},\ q\in(0,1],\ x\in[0,1]$
Ensure: $f_0^{(n)}(x)=\gamma(x)$
  for $i=0:n$ **do**
    $f_i^{(0)}(x)=P_i$
  end for
  for $r=1:n$ **do**
    for $i=0:n-r$ **do**
      $f_i^{(r)}(x)=(1-q^{n-r-i}x)f_i^{(r-1)}(x)+q^{n-r-i}xf_{i+1}^{(r-1)}(x)$
    end for
  end for

§3. Rational q-Bézier curves

In [5], rational q-Bézier curves were presented as a generalization of rational Bézier curves. Given a sequence $(w_i)_{i=0}^n$ of strictly positive weights, a *rational q-Bernstein basis* $(r_{0,q}^n,\dots,r_{n,q}^n)$ was defined as

$$r_{i,q}^n(x)=\frac{w_i\,b_{i,q}^n(x)}{\sum_{i=0}^n w_i\,b_{i,q}^n(x)},\quad x\in[0,1],\quad \text{for } i=0,1,\dots,n, \tag{12}$$

and a *rational q-Bézier curve* as

$$\gamma(x)=\sum_{i=0}^n P_i\,r_{i,q}^n(x)=\sum_{i=0}^n P_i\,\frac{w_i\,b_{i,q}^n(x)}{\sum_{i=0}^n w_i\,b_{i,q}^n(x)},\quad x\in[0,1], \tag{13}$$

where $P_i\in\mathbb{R}^k$ $(k=2,3)$ are the control points of the curve. Since the numerator and denominator of formula (13) are given by q-Bézier polynomials, we could evaluate this expression just by computing the numerator, the denominator and finally the quotient. However, this might lead to numerical instabilities depending on the sequence of weights used (see [3, 7]).

In order to avoid this phenomenon, the idea used for rational Bézier curves and rational q-Bézier curves has been the normalization of the weights on every step of the De Casteljau algorithm (see [4, 6]). Following the same strategy, our proposed evaluation algorithm for rational q-Bézier curves is presented in Algorithm 2.

The following Proposition presents the formula for the intermediate control points computed at every step of Algorithm 2 and shows that the output is in fact the evaluation of the rational curve.

Proposition 5. *Given the control points $\{P_0,\dots,P_n\}$ and the positive weights $w_0,\dots,w_n$, the intermediate control points defined in Algorithm 2 satisfy the following expressions:*

$$w_i^{(r)}(x)=\sum_{k=0}^r w_{i+k}\begin{bmatrix}r\\k\end{bmatrix}(q^{n-r-i}x)^k\prod_{t=k+1}^r(1-q^{n-i-t}x), \tag{14}$$

$$f_i^{(r)}(x) = \frac{\sum_{k=0}^{r} w_{i+k} P_{i+k} \begin{bmatrix} r \\ k \end{bmatrix} (q^{n-r-i}x)^k \prod_{t=k+1}^{r} (1 - q^{n-i-t}x)}{w_i^{(r)}(x)}. \tag{15}$$

In particular, we have that $f_0^{(n)}(x) = \gamma(x)$.

Proof. We have that formula (14) comes from the application of Proposition 4 to $\sum_{i=0}^{n} w_i\, b_{i,q}^n(x)$. The argumentation for the numbers defined by (15) is analogous to the one given in Proposition 4. Let us see that these numbers satisfy the recurrence relationship defined in Algorithm 2:

$$\begin{aligned}
f_i^{(r)}(x) =&(1 - q^{n-r-i}x)\frac{w_i^{(r-1)}(x)}{w_i^{(r)}(x)} f_i^{(r-1)}(x) + q^{n-r-i}x\frac{w_{i+1}^{(r-1)}(x)}{w_{i+1}^{(r)}(x)} f_{i+1}^{(r-1)}(x)\\
=&(1 - q^{n-r-i}x)\frac{w_i^{(r-1)}(x)}{w_i^{(r)}(x)} \frac{\sum_{k=0}^{r-1} w_{i+k} P_{i+k} \begin{bmatrix} r-1 \\ k \end{bmatrix} (q^{n-r-i+1}x)^k \prod_{t=k+1}^{r-1} (1 - q^{n-i-t}x)}{w_i^{(r-1)}(x)}\\
&+ q^{n-r-i}x\frac{w_{i+1}^{(r-1)}(x)}{w_{i+1}^{(r)}(x)} \frac{\sum_{k=0}^{r-1} w_{i+1+k} P_{i+1+k} \begin{bmatrix} r-1 \\ k \end{bmatrix} (q^{n-r-i}x)^k \prod_{t=k+1}^{r-1} (1 - q^{n-i-1-t}x)}{w_{i+1}^{(r-1)}(x)}.
\end{aligned} \tag{16}$$

Finally, we can apply Proposition 4 to the numerator of the fractions appearing in (16) using $w_i P_i$ as the control points and deduce (15). □

We know present Algorithm 2 for the evaluation of a rational q-Bézier curve.

Algorithm 2 eval.rational

Require: $(P_i)_{0\le i\le n},\ (w_i)_{0\le i\le n},\ q \in (0,1],\ x \in [0,1]$
Ensure: $f_0^{(n)}(x) = \gamma(x)$
  for $i = 0 : n$ **do**
    $f_i^{(0)}(x) = P_i$
    $w_i^{(0)}(x) = w_i$
  end for
  for $r = 1 : n$ **do**
    for $i = 0 : n - r$ **do**
      $w_i^{(r)}(x) = (1 - q^{n-r-i}x)w_i^{(r-1)}(x) + q^{n-r-i}xw_{i+1}^{(r-1)}(x)$
      $f_i^{(r)}(x) = (1 - q^{n-r-i}x)\frac{w_i^{(r-1)}(x)}{w_i^{(r)}(x)} f_i^{(r-1)}(x) + q^{n-r-i}x\frac{w_{i+1}^{(r-1)}(x)}{w_{i+1}^{(r)}(x)} f_{i+1}^{(r-1)}(x)$
    end for
  end for

Moreover, we have that the nice properties of the De Casteljau algorithm deduced for q-Bézier curves are also derived for the rational case.

Corollary 6. *All the steps performed in Algorithm 2 for the computation of the intermediate points* $f_i^{(r)}(x)$ *are convex combinations of the previously computed points.*

§4. Rational q-Bézier surfaces

Given a matrix of positive weights $(w_{ij})_{0\le i\le m;0\le j\le n}$, a control net $(P_{ij})_{0\le i\le m;0\le j\le n}$ in $\mathbb{R}^3$ and $q_1, q_2 \in (0,1]$, we define the rational q-Bézier surface

$$F(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} P_{ij}\frac{w_{ij}b^m_{i,q_1}(x)b^n_{j,q_2}(y)}{\sum_{i=0}^m\sum_{j=0}^n w_{ij}b^m_{i,q_1}(x)b^n_{j,q_2}(y)}, \quad (x,y)\in[0,1]\times[0,1]. \tag{17}$$

Let us now see how we can evaluate the surface combining the previous algorithms. The rational surface defined by (17) can be written as

$$F(x,y) = \frac{\sum_{i=0}^m \left[\left(\sum_{j=0}^n \frac{P_{ij}w_{ij}b^n_{j,q_2}(y)}{\sum_{j=0}^n w_{ij}b^n_{j,q_2}(y)}\right)\left(\sum_{j=0}^n w_{ij}b^n_{j,q_2}(y)\right)b^m_{i,q_1}(x)\right]}{\sum_{i=0}^m\left(\sum_{j=0}^n w_{ij}b^n_{j,q_2}(y)\right)b^m_{i,q_1}(x)}, \quad (x,y)\in[0,1]\times[0,1].$$

In this expression, we see that we can evaluate the surface using the evaluation algorithms for q-Bézier curves and for rational q-Bézier curves. Let us define the following control points and weights

$$P_i(y) := \sum_{j=0}^n \frac{P_{ij}w_{ij}b^n_{j,q_2}(y)}{\sum_{j=0}^n w_{ij}b^n_{j,q_2}(y)}, \quad w_i(y) := \sum_{j=0}^n w_{ij}b^n_{j,q_2}(y),$$

for $i = 0,\dots,n$. After computing these quantities, we can use them to compute the following rational curve

$$F(x,y) = \sum_{i=0}^m P_i(y)\,\frac{w_i(y)\,b^m_{i,q_1}(x)}{\sum_{i=0}^m w_i(y)\,b^m_{i,q_1}(x)}.$$

Hence, we provide Algorithm 3 for the evaluation of the rational surface $F(x,y)$. Let us notice that Algorithm 3 also provides an evaluation algorithm for q-Bézier surfaces whenever $w_{ij} = 1$ for all $i = 0,\dots,m$ and $j = 0,\dots,n$. This is the case of tensor product q-Bézier surfaces.

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Algorithm 3 eval.surface

Require: $(w_{ij})_{0\le i\le m;0\le j\le n}, (P_{ij})_{0\le i\le m;0\le j\le n},\ q_1, q_2 \in (0,1],\ x, y \in [0,1]$
Ensure: $F(x,y)$
  for $i = 0 : m$ **do**
    for $j = 0 : n$ **do**
      $f_{ij}^{(00)} = P_{ij}$
      $w_{ij}^{(00)} = w_{ij}$
    end for
  end for
  for $r = 1 : n$ **do**
    for $i = 0 : m$ **do**
      for $j = 0 : n - r$ **do**
        $w_{ij}^{(0r)} = (1 - q_2^{n-r-j} y) w_{ij}^{(0,r-1)} + q_2^{n-r-j} y w_{i,j+1}^{(0,r-1)}$
        $f_{ij}^{(0r)} = (1 - q_2^{n-r-j} y) \frac{w_{ij}^{(0,r-1)}}{w_{ij}^{(0,r)}} f_{ij}^{(0,r-1)} + q_2^{n-r-j} y \frac{w_{i,j+1}^{(0,r-1)}}{w_{i,j}^{(0,r)}} f_{i,j+1}^{(0,r-1)}$
      end for
    end for
  end for
  for $r = 1 : m$ **do**
    for $i = 0 : m - r$ **do**
      $w_{i0}^{(rn)} = (1 - q_1^{n-r-i} x) w_{i0}^{(r-1,n)} + q_1^{n-r-i} x w_{i+1,0}^{(r-1,n)}$
      $f_{i0}^{(rn)} = (1 - q_1^{n-r-j} x) \frac{w_{i0}^{(r-1,n)}}{w_{i0}^{(rn)}} f_{i0}^{(r-1,n)} + q_1^{n-r-j} x \frac{w_{i+1,0}^{(r-1,n)}}{w_{i,0}^{(rn)}} f_{i+1,0}^{(r-1,n)}$
    end for
  end for

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