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DISCRETE EMBEDDING OF LAGRANGIAN/HAMILTONIAN SYSTEMS AND THE MARSDEN-WEST APPROACH TO VARIATIONAL INTEGRATORS-THE ORDER ONE CASE

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Abstract. We give a self-contained introduction to the discrete embedding of Lagrangian and Hamiltonian systems using a discrete differential and integral calculus of order one. This theory is compared with the seminal work of J-E. Marsden and M. West on variational integrators.

§1. Introduction

In recent years, many efforts have been devoted to the construction of numerical algorithms for the simulation of Lagrangian and Hamiltonian systems respecting the variational structure underlying these systems. These algorithms are called **variational integrators** and belong to the more general class of **geometric numerical integrators** (see [8] for a review). The most well known and systematic approach to the construction of variational integrators is due to J.E. Marsden and M. West and a review of this approach can be found in [11]. Briefly, a Lagrangian system is determined by critical points of a functional

$$\mathscr{L}(q) = \int_a^b L(q(s), \dot{q}(s))\, ds, \tag{1}$$

where $L(q, v)$ is called a Lagrangian. The critical points of $\mathscr{L}$ are solutions to the Euler-Lagrange equation $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial v}(q(t), \dot{q}(t))\right) = \dfrac{\partial L}{\partial q}(q(t), \dot{q}(t))$, where $q : \mathbb{R} \mapsto \mathbb{R}^d$. The Marsden-West approach to variational integrators consists in the most simple case in replacing the functional (1) by an approximation of order one which depends on (q_{i+1}, q_i) by introducing a discrete functional denoted by $\mathbb{L}$ and defined by

$$\mathbb{L}(q_{i+1}, q_i) \approx \int_{t_i}^{t_{i+1}} L(q(t), \dot{q}(t))\, dt = hL\left(q_i, \frac{q_{i+1} - q_i}{h}\right), \tag{2}$$

where $q_i = q(t_i)$, $t_i \in \mathbb{T}$, and $\mathbb{T}$ is a discrete time scale on $[a, b]$ with a uniform time step h. The discrete Euler-Lagrange equation is then characterized by extremising the following quantity

$$\mathbb{S}(q_0, \ldots, q_N) = \sum_{i=0}^{N-1} \mathbb{L}(q_{i+1}, q_i). \tag{3}$$

Regarding $\mathbb{L}$ as a function of (y, x), the resulting **discrete Euler-Lagrange equation** is given by $\frac{\partial \mathbb{L}}{\partial x}(q_{i+1}, q_i) + \frac{\partial \mathbb{L}}{\partial y}(q_i, q_{i-1}) = 0$, $i = 1, \dots, N-1$. The Marsden-West approach has the following drawbacks:

- The algebraic structure of the classical Euler-Lagrange equation is lost.
- The dichotomy between position and speed is not preserved because the discrete Lagrangian L depends on (q_{i+1}, q_i) which means that they replace the tangent space by doubling the configuration space.
- The functional framework underlying the definition of the discrete functional is not explicit. In particular, the integral nature of the discrete functional is not clear.

In order to solve these difficulties, we introduce, following the **discrete embedding formalism** [3, 2, 4, 5, 6], the functional space of discrete functions on which we define a **discrete extension of the differential and integral calculus**. This framework allows us to reformulate the construction of variational integrators and to obtain a complete correspondence with the continuous setting. Moreover, it gives new insight into the definition of discrete momentum and the property of symplecticity of variational integrators.

§2. Discrete embedding of order 1

In this section, we remind how to define a discrete differential and integral calculus on discrete functions following [3, 2, 4, 5, 6].

2.1. Discrete functional space and functional

Let $N \in \mathbb{N}^\star$, we consider an interval $[a, b] \subset \mathbb{R}$ and a discrete finite subset denoted by $\mathbb{T}$ and defined by $\mathbb{T} = \{t_i\}_{i=0,\dots,N}$. $\mathbb{T}$ is called a **discrete time scale** in the literature (see [1]). For simplicity, in all that follows we consider a **uniform time scale** $\mathbb{T}$, meaning that points $t_i \in \mathbb{T}$ are uniformly distributed with constant time step $h = (b-a)/N > 0$, i.e. $\mathbb{T} = \{t_i = a + ih, i = 0, \dots, N\}$. All the computations and arguments can be extended without difficulties to an arbitrary discrete time scale. We denote by $\mathbb{T}^+ = [a, b[\cap \mathbb{T} = \mathbb{T} \setminus \{t_N\}$, $\mathbb{T}^- =]a, b] \cap T = \mathbb{T} \setminus \{t_0\}$ and $C(\mathbb{T}, \mathbb{R}^d)$ the set of functions defined on $\mathbb{T}$ with values in $\mathbb{R}^d$. A discrete functional on $C(\mathbb{T}, \mathbb{R}^d)$ is a mapping from $C(\mathbb{T}, \mathbb{R}^d)$ with values in $\mathbb{R}$.

2.2. Discrete differential and integral calculus- General strategy

The strategy in order to construct a discrete version of the classical differential and integral calculus can be summarized as follows:

- Embed the set of discrete functions $C(\mathbb{T}, \mathbb{R}^d)$ into piecewise continuous or differentiable functions.
- Define the derivative of a discrete function q as the restriction of the action of the classical derivative on the appropriate embedded version of q.
- Construct a discrete integral theory using embedding such that a discrete version of the fundamental theorem of the differential calculus is satisfied.

Of course, one can reverse the previous construction starting from a discrete integral calculus and constructing of the corresponding discrete differential calculus.

2.3. Continuous/differentiable embedding of discrete functions

Let $\mathbb{T}$ be a given discrete time scale. For a given operator A acting in a continuous setting (for example integral or derivative) the construction of its discrete analogue is done by introducing a mapping e_A from $C(\mathbb{T}, \mathbb{R}^d)$ into $D(A)$ where $D(A)$ is the domain of definition of A. As we are interested in the construction of discrete analogues of integral and derivative, we then are lead to introduce the following sets and mappings:

- $P^{0,+}_{\mathbb{T}}([a,b[,\mathbb{R}^d)$ (resp. $P^{0,-}_{\mathbb{T}}(]a,b],\mathbb{R}^d)$) the set of piecewise left (resp. right) continuous constant functions on $[a,b]$ with discontinuities on $\mathbb{T}$.
- $P^1_{\mathbb{T}}([a,b],\mathbb{R}^d)$ the set of piecewise continuous polynomial of order 1 on $[a,b]$ non differentiable on $\mathbb{T}$.

We denote by $e_{0,+}$ (resp. $e_{0,-}$) and e_1 the mappings defined for all $q \in C(\mathbb{T}, \mathbb{R}^d)$ by

$$e_{0,+}(q) = \sum_{k=0}^{N-1} q(t_k)\mathbf{1}_{[t_k,t_{k+1}[}, \quad \left(\text{resp.} e_{0,-}(q) = \sum_{k=1}^{N} q(t_k)\mathbf{1}_{]t_{k-1},t_k]}\right) \tag{4}$$

and

$$e_1(q) = \sum_{k=0}^{N-1}\left[q(t_k) + \frac{q(t_{k+1}) - q(t_k)}{h}(t - t_k)\right]\mathbf{1}_{[t_k,t_{k+1}]}, \tag{5}$$

where $\mathbf{1}_I$ is the indicator function of the set I. A natural way to recover a discrete function from a function q defined on an interval $I \subset [a,b]$ is to take its restriction on $\mathbb{T}$. This mapping is denoted by π. It must be noted that for $q \in C(I,\mathbb{R}^d)$, its image $\pi(q)$ belongs to $C(\mathbb{T}_I,\mathbb{R}^d)$ where $\mathbb{T}_I = \mathbb{T} \cap I$.

2.4. Discrete derivatives

In this section, we define discrete analogue of the classical right and left derivative denoted by d^+/dt and d^-/dt respectively using the mapping e_1. Precisely, we denote by Δ_+ and Δ_- the discrete operators defined over $C(\mathbb{T},\mathbb{R}^d)$ by $\Delta_+ = \pi \circ \dfrac{d^+}{dt} \circ e_1$ and $\Delta_- = \pi \circ \dfrac{d^-}{dt} \circ e_1$ respectively. The operator Δ_+ (resp. Δ_-) goes from $C(\mathbb{T},\mathbb{R}^d)$ in $C(\mathbb{T}^+,\mathbb{R}^d)$ (resp. $C(\mathbb{T}^-,\mathbb{R}^d)$). This corresponds to the following commutative diagrams:

$$\begin{array}{ccc} P^1([a,b],\mathbb{R}^d) & \xrightarrow{d^+/dt} & P^{0,+}([a,b[,\mathbb{R}^d) \\ e_1 \uparrow & & \downarrow \pi \\ C(\mathbb{T},\mathbb{R}^d) & \xrightarrow{\Delta_+} & C(\mathbb{T}^+,\mathbb{R}^d) \end{array}, \quad \begin{array}{ccc} P^1([a,b],\mathbb{R}^d) & \xrightarrow{d^-/dt} & P^{0,-}(]a,b],\mathbb{R}^d) \\ e_1 \uparrow & & \downarrow \pi \\ C(\mathbb{T},\mathbb{R}^d) & \xrightarrow{\Delta_-} & C(\mathbb{T}^-,\mathbb{R}^d). \end{array}$$

A simple computation leads to

$$\Delta_+[q](t_i) = \frac{q(t_{i+1}) - q(t_i)}{h}, t_i \in \mathbb{T}^+ \quad \textit{and} \quad \Delta_-[q](t_i) = \frac{q(t_i) - q(t_{i-1})}{h}, t_i \in \mathbb{T}^-.$$

2.5. Discrete integral

Following the general strategy, we want to define a discrete integral such that the fundamental theorem of differential calculus is preserved. Using $e_{0,+}$ one can obtain a discrete analogue of the classical integral $\displaystyle\int_a^t ds$ denoted by $\displaystyle\int_a^t \Delta_+ s$ and defined by $\displaystyle\int_a^t \Delta_+ s = \pi \circ \int_a^t ds \circ e_{0,+}$, corresponding to the following diagram:

$$\begin{array}{ccc} P^{0,+}([a,b[,\mathbb{R}^d) & \xrightarrow{\int_a^t ds} & P^1([a,b],\mathbb{R}^d) \\ {\scriptstyle e_{0,+}}\Big\uparrow & & \Big\downarrow{\scriptstyle \pi} \\ C(\mathbb{T},\mathbb{R}^d) & \xrightarrow{\int_a^t \Delta_+ s} & C(\mathbb{T},\mathbb{R}^d). \end{array}$$

An explicit computation gives for all $q \in C(\mathbb{T},\mathbb{R}^d)$ and all $t_i, t_j \in \mathbb{T}$, $j > i$ that

$$\int_{t_i}^{t_j} q(s)\Delta_+ s = \sum_{k=i}^{j-1} q(t_k)h.$$

One can verify that we have a **fundamental theorem of the discrete differential calculus**, precisely that $\displaystyle\int_a^b \Delta_+[q](s)\Delta_+ s = q(b) - q(a)$, $\displaystyle\Delta_+\left[\int_a^t q(s)\Delta_+ s\right] = q(t)$, $\forall\, t \in \mathbb{T}$, and an **integration by parts formula** $\displaystyle\int_a^b q(t)\Delta_+[g](t)\Delta_+ t = -\int_a^b \Delta_-[q](t)g(t)\Delta_+ t + q(b)g(b) - q(a)g(a)$. We refer to [5, 6] for more details.

2.6. Why order 1 ?

Let q be a continuous function on $[a,b]$ and $\mathbb{T}$ be a discrete time scale. The order of the discrete embedding is the order of approximation in the parameter h of the classical integral $\displaystyle\int_a^b q(s)\,ds$ by $\displaystyle\int_a^b q(s)\Delta_+ s$. In the same way, if q is of class C^1 then $\Delta_+[q]$ is an approximation of order one of dq/dt at each point $t \in \mathbb{T}$. As a consequence, a discrete embedding theory can be seen as a reformulation of the classical **theory of approximation** in a functional point of view.

§3. Discrete Lagrangian formalism

We follow the embedding formalism approach to define a discrete analogue of Lagrangian systems. Our definition is compared with the notion of discrete functional introduced by J-E. Marsden and M. West in [11].

3.1. Discrete Lagrangian functional - Embedding case

Using the previous discrete differential and integral calculus, we define the **discrete Lagrangian functional** denoted by $\mathscr{L}_h$ over $C(\mathbb{T},\mathbb{R}^d)$ as follows

Definition 1. Let $\mathbb{T}$ a discrete time scale with uniform time step h. The discrete Lagrangian functional associated to L given in (1) is defined for all $q \in C(\mathbb{T}, \mathbb{R}^d)$ by

$$\mathscr{L}_h(q) = \int_a^b L(q(s), \Delta_+[q](s))\, \Delta_+ s. \tag{6}$$

As we see using this approach, we preserve the algebraic structure of the classical integral.

3.2. Discrete Lagrangian functional - Marsden-West case

The discrete Lagrangian functional $\mathscr{L}_h(q)$ given in (6) coincides with the one defined by Marsden-West $\mathbb{S}(q_0, \dots, q_N)$ given in (3) where $(q_0, \dots, q_N) = (q(t_0), \dots, q(t_N)) \in (\mathbb{R}^d)^{N+1}$. However, it must be noted that the Lagrangian $\mathbb{L}$ defined in (2) has an integral nature. Indeed, denoting by Ψ the mapping defined by

$$\Psi(q_i, v_i) = (hv_i + q_i, q_i), \tag{7}$$

we deduce that $\mathbb{L}(q(t_{i+1}), q(t_i)) = \int_{t_i}^{t_{i+1}} L(\Psi^{-1}(q(t+h), q(t)))\Delta_+ t$. This integral nature of $\mathbb{L}$ makes the computations more cumbersome in the Marsden-West approach than using the function $L(q(t), \Delta_+[q](t)) = L(\Psi^{-1}(q(t+h), q(t)))$ in the discrete embedding framework.

§4. Discrete calculus of variations

A classical tool of studying Lagrangian functional is the calculus of variations. As we will see, the use of a discrete version of the calculus of variations is precisely the place where the functional setting of our discrete formulation will be the most efficient and will alight some classical computations used in [11].

4.1. Discrete Euler-Lagrange equation - Embedding case

Let us consider a discrete Lagrangian functional $\mathscr{L}_h(q)$ of the form (6). We denote by $\mathscr{V}$ the set of variations defined by $\mathscr{V} = \{v \in C(\mathbb{T}, \mathbb{R}^d)\, , v(a) = v(b) = 0\}$. The Frechet derivative $D\mathscr{L}_h(q)$ of $\mathscr{L}_h$ at point $q \in C(\mathbb{T}, \mathbb{R}^d)$ in the direction $w \in \mathscr{V}$ is then given by $D\mathscr{L}_h(q)(w) = \lim_{\epsilon \to 0} \dfrac{\mathscr{L}_h(q + \epsilon w) - \mathscr{L}_h(q)}{\epsilon}$. Assuming that L is sufficiently smooth, a simple Taylor expansion leads to

$$D\mathscr{L}_h(q)(w) = \int_a^b \left[w(s) \frac{\partial L}{\partial q}(q(s), \Delta_+[q](s)) + \Delta_+[w](s) \frac{\partial L}{\partial v}(q(s), \Delta_+[q](s)) \right] \Delta_+ s. \tag{8}$$

As in the continuous case, we perform a **discrete integration by parts** which gives

$$D\mathscr{L}_h(q)(w) = \int_a^b \left(\frac{\partial L}{\partial q}(q(s), \Delta_+[q](s)) - \Delta_- \left[\frac{\partial L}{\partial v}(q(s), \Delta_+[q](s)) \right] \right) w(s)\, \Delta_+ s. \tag{9}$$

The critical points of $\mathscr{L}_h$ satisfy $D\mathscr{L}_h(q)(w) = 0$ for all $w \in \mathscr{V}$. It can be proved that if $\int_a^b f(t) g(t) \Delta_+ t = 0$ for all $g \in \mathscr{V}$ then $f(t) = 0$ for $t \in \mathbb{T}^\pm = \mathbb{T}^+ \cap \mathbb{T}^-$ (the **Discrete Dubois**

Raymond lemma). As a consequence, we deduce that if q is a critical point of $\mathscr{L}_h$ then it satisfies the **discrete Euler-Lagrange equation** given by

$$\frac{\partial L}{\partial q}(q(s), \Delta_+[q](s)) - \Delta_- \left[\frac{\partial L}{\partial v}(q(s), \Delta_+[q](s))\right] = 0, \ s \in \mathbb{T}^{\pm}. \tag{10}$$

The previous formula keeps the classical algebraic form of the Euler-Lagrange equation, moreover it shows that a mixing between the backward and forward derivative is unavoidable due to the duality between these operators with respect to the discrete integration.

4.2. Discrete Euler-Lagrange equation - Marsden-West case

A discrete Euler-Lagrange equation was derived by J-E. Marsden and M. West in [11] and must of course coincide with our equation (10). However, as we will see, this is not transparent due to the introduction of the function $\mathbb{L}$ and the fact that they do not use discrete operators to formulate the equation.

For all $x, y \in \mathbb{R}^d$, let $\mathbb{L}(y, x) = hL\left(x, \dfrac{y-x}{h} \right)$. J-E. Marsden and M. West define a variation of $\mathbb{S}$ as a family v_i, $i = 0, \dots, N$ such that $v_0 = v_N = 0$ corresponding to the choice of a function $v \in \mathscr{V}$. They consider the quantity denoted $\delta\mathbb{S}(q_0, \dots, q_N, h)$ corresponding to the limit when ϵ goes to 0 of $\dfrac{\mathbb{S}(q_0 + \epsilon v_0, \dots, q_N + \epsilon v_N) - \mathbb{S}(q_0, \dots, q_N)}{\epsilon}$.
By a Taylor expansion, they obtain

$$\delta\mathbb{S}(q_0, \dots, q_N, h) = \sum_{i=0}^{N-1} \left(\frac{\partial \mathbb{L}}{\partial x}(q_{k+1}, q_k)v_k + \frac{\partial \mathbb{L}}{\partial y}(q_{k+1}, q_k)v_{k+1} \right). \tag{11}$$

Then, they use a rearrangement of the sum that they call discrete integration by part (see [11],p.363):

$$\delta\mathbb{S}(q_0, \dots, q_N, h) = \sum_{i=1}^{N-1} \left(\frac{\partial \mathbb{L}}{\partial x}(q_{k+1}, q_k) + \frac{\partial \mathbb{L}}{\partial y}(q_k, q_{k-1}) \right) v_k. \tag{12}$$

Using the fact that $v_0 = v_N = 0$. As the v_i for $i = 1, \dots, N-1$ are arbitrary, they deduce that the equation $\delta\mathbb{S}(q_0, \dots, q_N, h) = 0$ is equivalent to the discrete Euler-Lagrange equation

$$\frac{\partial \mathbb{L}}{\partial x}(q_{i+1}, q_i) + \frac{\partial \mathbb{L}}{\partial y}(q_i, q_{i-1}) = 0. \tag{13}$$

Computing explicitly each part of the previous quantities, we obtain

$$\partial_x \mathbb{L}(q_{i+1}, q_i) = h\left[\frac{\partial L}{\partial q}(q_i, \frac{q_{i+1} - q_i}{h}) - \frac{1}{h}\frac{\partial L}{\partial v}(q_i, \frac{q_{i+1} - q_i}{h})\right].$$
$$\partial_y \mathbb{L}(q_i, q_{i-1}) = h\left[\frac{1}{h}\frac{\partial L}{\partial v}(q_{i-1}, \frac{q_i - q_{i-1}}{h})\right].$$

Replacing $\partial_x \mathbb{L}(q_{i+1}, q_i)$ and $\partial_y \mathbb{L}(q_i, q_{i-1})$ by their quantities in (13), we recover our Euler-Lagrange equation (10) evaluated on a time $t_i \in \mathbb{T}^{\pm}$.

§5. Discrete Hamiltonian systems

5.1. Discrete Hamiltonian systems - Embedding case

The classical definition of a Hamiltonian system in $\mathbb{R}^{2d}$ with the canonical symplectic structure is the following:

Definition 2. Let $d \in \mathbb{N}^\star$, $p \in C([a,b],\mathbb{R}^d)$ and $q \in C([a,b],\mathbb{R}^d)$. Let $H\colon \mathbb{R}^{2d} \to \mathbb{R}$. A $2d$-dimensional differential system of the form $\begin{pmatrix}\dot p\\ \dot q\end{pmatrix} = J.\nabla H(p,q)$ where $J = \begin{pmatrix}0 & -Id\\ Id & 0\end{pmatrix}$, $\nabla H = \begin{pmatrix}\partial_p H\\ \partial_q H\end{pmatrix}$, is called a Hamiltonian system with Hamiltonian H.

When the mapping $v \mapsto \dfrac{\partial L}{\partial v}$ is invertible for all $q \in \mathbb{R}^d$, one can associate to a given Lagrangian system a natural Hamiltonian system which is a convenient reformulation of the Euler-Lagrange equation using a change of variables called the **Legendre transform** where $p \in \mathbb{R}^d$, called the **momentum**, is defined by $p(t) = \dfrac{\partial L}{\partial v}(q(t),\dot q(t))$. Following the discrete embedding formalism, a natural choice for the discrete momentum is given by:

Definition 3. (Discrete momentum) Let L a given Lagrangian function. Assuming that the Lagrangian is *admissible*.The discrete momentum is defined for all $t \in \mathbb{T}^+$ by

$$p(t) = \frac{\partial L}{\partial v}(q(t),\Delta_+ q(t)). \tag{14}$$

For each $q \in \mathbb{R}^d$, we denote by $g(p,q)$ the inverse of the invertible mapping $v \mapsto \dfrac{\partial L}{\partial v}$. We then have $\Delta_+[q] = g(p,q)$. Then, the discrete Euler-Lagrange equation can be reformulate using the following discrete system

$$\begin{cases} \Delta_-[p](t) &= \dfrac{\partial L}{\partial q}(p(t),q(t)), \qquad \text{for all } t \in \mathbb{T}^\pm.\\ \Delta_+[q](t) &= g(p(t),q(t)), \qquad \text{for all } t \in \mathbb{T}^\pm.\end{cases} \tag{15}$$

Using the classical Hamiltonian function defined in the continuous case by $H(p,q) = pg(p,q) - L(q,g(p,q))$, we obtain the discrete Hamiltonian form of the discrete Euler-Lagrange equation

$$\begin{cases} \Delta_-[p](t) &= -\dfrac{\partial H}{\partial q}(p(t),q(t)), \qquad \text{for all } t \in \mathbb{T}^\pm,\\ \Delta_+[q](t) &= \dfrac{\partial H}{\partial p}(p(t),q(t)), \qquad \text{for all } t \in \mathbb{T}^\pm.\end{cases} \tag{16}$$

A natural demand in order to justify the terminology of discrete Hamiltonian system is to show that solutions to (16) correspond to critical points of a functional. Following our strategy, we introduce the discrete Hamiltonian functional

$$\mathscr{L}_{H,h}(p,q) = \int_a^b \left(p\Delta_+[q] - H(p,q)\right)\, \Delta_+ t. \tag{17}$$

Theorem 1. *The critical points of the discrete Hamiltonian functional* (17) *correspond to the solutions to the discrete system* (15).

As a consequence, all the relations and structures of the continuous case are preserved in the discrete case thanks to the discrete embedding procedure.

5.2. Discrete Hamiltonian systems - Marsden-West case

The discrete analogue of the Legendre transform is defined inductively by J-E. Marsden and M. West as follows: we denote by $P_{M,0}$ the quantity defined by $P_{M,0} = -\frac{\partial \mathbb{L}}{\partial x}(q_1, q_0)$. The connection between our definition of the discrete momentum p in (14) and the one defined by Marsden-West is given by $P_{M,0} = -h\frac{\partial L}{\partial q}(q_0, v_0) + p_0$, where $v_0 = \Delta_+[q](t_0) = (q_1 - q_0)/h$, which can be summarized by the diagram

$$\begin{array}{ccc} (q_0, v_0) & \xrightarrow{P} & (q_0, p_0) \\ {\scriptstyle \Psi^{-1}}\uparrow & & \downarrow{\scriptstyle \Theta} \\ (q_1, q_0) & \xrightarrow{P_M} & (q_0, P_{M,0}), \end{array}$$

where $\Theta\colon (q_0, p_0) \to (q_0, p_0 - h\frac{\partial L}{\partial q}(q_0, v_0))$. We then observe that the Marsden-West definition of the discrete momentum introduces a distortion between the definition in the continuous case and the discrete one encoded by the mapping Θ which is corrected in the discrete embedding formalism.

§6. Variational integrators and symplecticity

An important property of variational integrators is that they are symplectic, meaning that the corresponding mapping preserves the symplectic structure. We show how these results are related in the two formalisms.

6.1. Discrete flows: embedding and Marsden-West case

Lagrangian systems induce an algorithm which can be initialized by the data of q_0 and v_0 in the discrete embedding case and q_0 and q_1 in the Marsden-West case, the two representations are connected by the mapping Ψ defined in (7). Denoting by Φ_M the induced flow in the Marsden-West case defined by $\Phi_M(q_1, q_0) = (q_2, q_1)$ and by Φ the one induced by the discrete embedding approach and defined by $\Phi(q_0, v_0) = (q_1, v_1)$, we easily prove that these two maps are conjugated, meaning that we have the following diagram:

$$\begin{array}{ccc} (q_0, v_0) & \xrightarrow{\Phi} & (q_1, v_1) \\ {\scriptstyle \Psi^{-1}}\uparrow & & \downarrow{\scriptstyle \Psi} \\ (q_1, q_0) & \xrightarrow[\Phi_M]{} & (q_2, q_1). \end{array}$$

We deduce that $\Phi_M = \Psi \circ \Phi \circ \Psi^{-1}$. It is well known that the mapping Φ_M is symplectic. By conjugacy, we deduce that the mapping Φ is also symplectic. However, one can go further and try to reproduce the variational proof of the symplecticity given by J-E. Marsden and T. Ratiu in [12] for the discrete flow Φ directly.

6.2. Symplecticity

We denote by S the functional defined on $C(\mathbb{T}, \mathbb{R}^d)$ by $S(q) = \int_{t_0}^{t_1} L(q(t), \Delta_+ q(t))\Delta_+ t$. Let $\mathscr{C}_L$ denote the set of solutions to the discrete Euler-Lagrange equation (10). For each $q_0 \in \mathbb{R}^d$ and $v_0 \in \mathbb{R}^d$, there exists a unique solution over $\{t_0, t_1\}$ of the discrete Euler-Lagrange equation denoted by $\psi_t(q_0, v_0)$ such that $q(0) = q_0$ and $\Delta_+[q](0) = v_0$. We denote by $\mathbb{S}$ the action integral defined on $\mathbb{R}^d \times \mathbb{R}^d$ by $\mathbb{S}(q_0, v_0) = S(\psi_t(q_0, v_0))$. Considering a variation such that $q + u$ is also in $\mathscr{C}_L$, a simple computation leads to

$$d\mathbb{S}(q_0, v_0)(u_0, w_0) = \theta_L(\psi(q_0, v_0))w_1 - \theta_L(q_0, v_0)w_0, \quad (18)$$

where $(u_1, w_1) = \psi(u_0, w_0)$ and θ_L is the classical **Lagrange 1-form** defined for all $q_0 \in \mathbb{R}^d$ and $w_0 \in \mathbb{R}^d$ by $\theta_L(q_0, v_0).w_0 = \dfrac{\partial L}{\partial v}(q_0, v_0)w_0$. The quantity (18) can be rewritten as

$$d\mathbb{S}(q_0, v_0)(u_0, w_0) = \psi^*(\theta_L)(q_0, v_0)w_1 - \theta_L(q_0, v_0)w_0. \quad (19)$$

Taking the exterior derivative of this quantity, we obtain

$$0 = d^2\mathbb{S} = \psi^*(d\theta_L) - d\theta_L = -\psi^*(\omega_L) + \omega_L. \quad (20)$$

As a consequence, the mapping ψ preserves the two form $\omega_L = -d\theta_L$. It is then a symplectic map.

§7. Conclusion and perspectives

We have shown that the classical approach derived by J-E. Marsden and M. West in [11] for the construction of variational integrators can be made more transparent from the point of view of the connection with the continuous Lagrangian/Hamiltonian formalism as well as from the point of view of the calculus of variations by introducing a suitable discrete differential and integral calculus following the discrete embedding formalism. Moreover, the definition of the geometric structure is also simplified in the discrete case by cancelling the distortions with respect to the continuous geometrical framework induced by the encoding of the discrete Lagrangian on a doubled configuration space instead of a tangent bundle.

Although the present presentation is limited to variational integrators of order one, all the constructions can be extended to variational integrators of an arbitrary order. We refer to [7] for a complete presentation of the Marsden-Wendlandt approach [14] to an order two variational integrator using a mid-point quadrature formula and to [9] for a general presentation of Galerkin variational integrators [10, 13] in the discrete embedding framework.

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