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Induced potential in Stochastic Newtonian dynamics

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Abstract. We derive a stochastic virial theorem and a stochastic Hamilton-Jacobi equation for Newtonian dynamics over diffusion processes with non constant diffusion term extending our previous results (J. Cresson and al., J. Math. Phys. 62, 072702 (2021)).

§1. Introduction

The rotation curve of a spiral galaxy is a plot of the orbital speed of visible stars versus their radial distance from the galaxy's centre. Based on Newton's gravitational law, one expects a decreasing of the orbital velocity with distance. However, experimental curves show that stars revolve with constant or increasing speed over a large range of distances. Proposals to capture this effect include an invisible source of mass called "dark matter" (see [10]) or a modification of Newton's law of gravity see [8].

In [7], we propose an alternative based on the fact that stochastic processes can be used to model long term dynamics of stars in a galaxy following a previous work of S. Chandrasekhar (see [4, 3]). We then assume that the resulting dynamics is obtained through the stochastic embedding theory developed in [6], leading to a stochastic Newton's equation. It is proved that a new potential term called the induced potential appears in the stochastic analogue of the virial theorem for this equation. This potential is explicitly given as a function of the density of the process. The density can be compute through the connexion existing between the stochastic Newton's equation and the linear Schrödinger equation. For a Kepler's potential, using as a solution of the Schrödinger equation the ground state solution representing the contribution of the central bubble of a spiral galaxy, we compute the induced potential which corresponds to the ad-hoc "dark potential" used in astrophysics (see [7]).

The "dark potential" is not sufficient in order to reproduce the full complexity of the rotation curve for a given galaxy. Two strategies can be used to obtain a better agreement:

- first, to use more complex solutions of the Schrödinger equation reproducing more closely the visible shape of a galaxy for example arms of spiral galaxies and not only the central bubble following S. Albeverio et al. [1] work on the morphology of galaxies.
- Second, to take into account possible fluctuations of the diffusion coefficient.

In this article, we explore the second possibility by considering stochastic processes which are isotropic but dependent on space. A stochastic virial theorem is obtained as well as a stochastic Hamilton-Jacobi equation. In particular, we give the expression of the induced potential in this case which depends again on the density of the stochastic process. However, the connection to a (nonlinear) Schrodinger equation is lost and a more general partial differential equation is obtained. A derivation of the induced potential seems more delicate in this case.

§2. Nelson's stochastic derivatives and the stochastic derivative

We refer to [9, 6] for more details. We denote vectors in $\mathbb{R}^3$ with bold letters $\mathbf{v} \in \mathbb{R}^3$ and components of $\mathbf{v}$ with classical letters v_i, $\mathbf{v} = (v_1, v_2, v_3)$. The usual scalar product in $\mathbb{R}^3$ is denoted by $\langle \cdot, \cdot \rangle$. Let X be a stochastic process in $\mathbb{R}^3$ of the form

$$dX_{i,t} = b_i(t, X_t)\, dt + \sigma(X_t)\, dW_{i,t},$$

for $i = 1, \dots, 3$ and the $W_{i,t}$ are independent standard real valued Wiener processes. We denote by σ the diagonal matrix $\boldsymbol{\sigma} = \sigma(X)\,\mathrm{Id}_d$, where Id_d is the identity matrix of $\mathbb{R}^3$.

A stochastic process of the form (2) is said to be **isotropic in space**. We denote by $\mathscr{D}(x)$ the quantity

$$\mathscr{D}(x) = \frac{\sigma^2(x)}{2},$$

associated to the classical diffusion parameter when X reduces to the Brownian motion. **Nelson's stochastic derivatives** (see [9]) are defined by $D_+[X] = \mathbf{b}(t, X)$ and $D_-[X] = \mathbf{b}_\star(t, X) = \mathbf{b}(t, X) - 2\mathscr{D}(x) - 2\frac{1}{p_t(X)}\mathscr{D}(x)\nabla[p]$ where p_t is the density of X_t. The **stochastic derivative** is defined by (see [6])

$$\mathbf{D}[X] = \left(\frac{D_+[X] + D_-[X]}{2}\right) - i\left(\frac{D_+[X] - D_-[X]}{2}\right).$$

As a consequence, we obtain $\mathbf{D}[X] = \mathbf{v} - i\mathbf{u}$ with $\mathbf{v} = \mathbf{v}_{0,\sigma} - \nabla(\mathscr{D}(x))$ and $\mathbf{u} = \mathbf{u}_{0,\sigma} + \nabla(\mathscr{D}(x))$ where $\mathbf{v}_{0,\sigma} = \mathbf{b}(t, X) - \mathscr{D}(x)\frac{\nabla(p_t)}{p_t}$ and $\mathbf{u}_{0,\sigma} = \mathscr{D}(x)\frac{\nabla(p_t)}{p_t}$.

The stochastic derivative has the following properties:

- *Chain rule*: Let $f(t, X)$ be a real valued function then

  $$\mathbf{D}[f(t, X)] = \partial_t f(t, X) + \langle \mathbf{D}[X], \nabla[f] \rangle - i\,\mathscr{D}(X)\,\Delta[f] = \mathbb{L}[X],$$

  the gradient ∇ and the Laplacian Δ of a function $f(t, X)$ being always taken with respect to the spatial variables X and the differential operator $\mathbb{L}$ is defined by

  $$\mathbb{L}[\bullet] = \partial_t[\bullet] + \langle \mathbf{D}[X], \nabla[\bullet] \rangle - i\mathscr{D}(X)\Delta[\bullet].$$

- *Leibniz's rule*: Let X, Y be two real valued stochastic processes then

  $$\frac{d}{dt}\left[\mathbb{E}\,(X \cdot Y)\right] = \mathbb{E}\left(\mathbf{D}[X] \cdot Y + X \cdot \overline{\mathbf{D}}[Y]\right),$$

  where the operator $\overline{\mathbf{D}}$ is defined by $\overline{\mathbf{D}}[X] = \mathbf{v} + i\mathbf{u} = \mathbf{D}[X] + 2i\mathbf{u}$.

§3. Stochastic virial theorem and induced potential

In this section, we denote by $\mathbf{K}$ the kinetic energy defined for all $v \in \mathbb{R}^d$ by $\mathbf{K}(v) = \frac{1}{2}m\langle v, v \rangle$ and U an homogeneous potential of order γ. We look for stochastic processes solution of the **stochastic Newton equation** (see [7])

$$m\mathbf{D}\left[\mathbf{D}[X]\right] = -\nabla U(X).$$

This equation is used as a model for motion of particles of mass m under a potential force $-\nabla U(X)$ covering in particular the gravitational case accepting that trajectories can be non regular and in particular stochastic. As a consequence, we do not had a noise on the classical Newton's equation but instead extend the meaning of this equation by extending the differential calculus under use via the stochastic derivative. This point of view is formalized in the stochastic embedding theory developed in [6].

When σ is constant, it can be proved [6, 7] that solutions of the stochastic Newton equation (3) corresponds to **Nelson's diffusion** as introduced by E. Nelson in [9]. Existence of solutions for Nelson's diffusion were proved by E.A. Carlen in [2] and different notions of uniqueness are studied by L. Wu [11]. Existence and uniqueness in a more general setting is open up to our knowledge. Explicit solutions are, as usual, difficult to obtain. However, qualitative properties of the solutions can be obtain looking for stochastic version of classical results like the virial theorem or Noether's theorem.

3.1. A stochastic virial theorem

Theorem 1 (Stochastic virial theorem)**.** *Let X be a solution of the stochastic Newton equation* (3)*, then we have the following equality*

$$\mathbf{D}^2[m \parallel X \parallel^2] = 4\mathbf{K}(\mathbf{D}[X]) - 2\gamma U(X) - i4m\mathscr{D}(X)\ \mathrm{div}(\mathbf{D}[X]) - i6m\mathbf{D}[\mathscr{D}(X)].$$

When σ is constant, we recover the usual stochastic virial theorem proved in [7].

Proof. Let $f(X) = \parallel X \parallel^2$ with $X \in \mathbb{R}^3$, then as $\Delta[f(X)] = 6$ we have $\mathbf{D}[X^2] = 2\langle \mathbf{D}[X], X\rangle - i3\sigma^2(X)$. As a consequence, we deduce $\mathbf{D}^2\left[mX^2\right] = 2m\mathbf{D}\left[\langle \mathbf{D}[X], X\rangle\right] - im3\mathbf{D}\left[\sigma^2(X)\right]$. Using the chain rule formula, we obtain $\mathbf{D}\left[\sigma^2(X)\right] = \langle \mathbf{D}[X], \nabla(\sigma^2)\rangle - i\frac{1}{2}\sigma^2\Delta(\sigma^2)$. The quantity $\mathbf{D}\left[\langle \mathbf{D}[X], X\rangle\right]$ is decomposed as

$$\mathbf{D}\left[\langle \mathbf{D}[X], X\rangle\right] = \mathbf{D}\left[\langle \mathbf{v} + i\mu\mathbf{u}, X\rangle\right] = \mathbf{D}\left[\langle \mathbf{v}, X\rangle\right] - i\mathbf{D}\left[\langle \mathbf{u}, X\rangle\right].$$

In order to compute $\mathbf{D}\left[\langle \mathbf{v}, X\rangle\right]$, we apply the chain rule formula with $f(t, X) = \langle X, \mathbf{v}(t, X)\rangle$. We have

$$\mathbf{D}\left[\langle \mathbf{v}, X\rangle\right] = \langle X, \partial_t \mathbf{v}\rangle + \langle \mathbf{D}[X], \mathbf{v}\rangle + \langle X, \langle \mathbf{D}[X], \nabla[\mathbf{v}]\rangle\rangle - i\mathscr{D}(X)\left[2\,\mathrm{div}(\mathbf{v}) + \langle X, \Delta[\mathbf{v}]\rangle\right],$$

where $\nabla[\mathbf{v}] = (\nabla(v_1), \dots, \nabla(v_3)) \in \mathbb{R}^{3\times 3}$, $\langle \mathbf{D}[X], \nabla[\mathbf{v}]\rangle = (\langle \mathbf{D}[X], \nabla(v_1)\rangle, \dots, \langle \mathbf{D}[X], \nabla(v_3)\rangle) \in \mathbb{R}^d$ and $\Delta[\mathbf{v}] = (\Delta[v_1], \dots, \Delta[v_3]) \in \mathbb{R}^3$. A similar formula is obtained for $\mathbf{D}\left[\langle \mathbf{u}, X\rangle\right]$ which gives

$$\mathbf{D}\left[\langle \mathbf{u}, X\rangle\right] = \langle X, \partial_t \mathbf{u}\rangle + \langle \mathbf{D}[X], \mathbf{u}\rangle + \langle X, \langle \mathbf{D}[X], \nabla[\mathbf{u}]\rangle\rangle - i\mathscr{D}(X)\left[2\,\mathrm{div}(\mathbf{u}) + \langle X, \Delta[\mathbf{u}]\rangle\right].$$

We deduce that $\mathbf{D}\left[\langle \mathbf{D}[X], X\rangle\right]$ is given by

$$\mathbf{D}\left[\langle \mathbf{D}[X], X\rangle\right] = \langle X, \mathbb{L}[\mathbf{D}[X]]\rangle + \langle \mathbf{D}[X], \mathbf{D}[X]\rangle - i\sigma^2\,\mathrm{div}(\mathbf{D}[X]),$$

By (2), we have $\mathbb{L}[\mathbf{D}[X]]) = \mathbf{D}[\mathbf{D}[X]]$. As X satisfies the stochastic Newton equation, we deduce that $m\mathbf{D}[\mathbf{D}[X]] = -\nabla U(X)$. As a consequence, we obtain

$$m\mathbf{D}\left[\langle \mathbf{D}[X], X\rangle\right] = -\langle X, \nabla U\rangle + m\langle \mathbf{D}[X], \mathbf{D}[X]\rangle - i\,m\sigma^2\,\mathrm{div}(\mathbf{D}[X]),$$

and

$$\mathbf{D}^2\left[mX^2\right] = -2\langle X, \nabla U\rangle + 2m\langle \mathbf{D}[X], \mathbf{D}[X]\rangle - i2\,m\sigma^2\,\mathrm{div}(\mathbf{D}[X]) - im\,3\mathbf{D}\left[\sigma^2(X)\right].$$

By the Euler theorem, we have $\langle X, \nabla U\rangle = \gamma U$, where γ is the homogeneity order. As a consequence, we deduce that

$$\mathbf{D}^2\left[mX^2\right] = -2\gamma U + 2m\langle \mathbf{D}[X], \mathbf{D}[X]\rangle - i2\,m\sigma^2\,\mathrm{div}(\mathbf{D}[X]) - im3\mathbf{D}\left[\sigma^2(X)\right].$$

As by definition of the kinetic energy $2m\langle \mathbf{D}[X], \mathbf{D}[X]\rangle = 4\mathbf{K}(\mathbf{D}[X])$, this completes the proof. □

3.2. Equilibrium and induced potential

Mimicking the classical case, we say that the system is at **equilibrium** if $\mathbf{D}[mX^2] = 0$. Using the virial theorem, we deduce that equilibrium is equivalent to

$$2K(\mathbf{D}[X]) = \gamma U(X) + i2m\mathscr{D}(X)\ \mathrm{div}(\mathbf{D}[X]) + im3\mathbf{D}[\mathscr{D}(X)].$$

Using that $\mathbf{D}[X] = \mathbf{v} - i\mathbf{u}$, we obtain $2K(\mathbf{D}[X]) = m(\mathbf{v}^2 - \mathbf{u}^2 - 2i\langle \mathbf{v}, \mathbf{u}\rangle)$ and $\mathrm{div}(\mathbf{D}[X]) = \mathrm{div}[\mathbf{v}] - i\,\mathrm{div}[\mathbf{u}]$. By the chain rule formula

$$\begin{aligned}\mathbf{D}[\mathscr{D}(X)] &= \langle \mathbf{D}[X], \nabla(\mathscr{D}(X))\rangle - i\mathscr{D}(X)\Delta[\mathscr{D}(X)],\\ &= \langle \mathbf{v}, \nabla(\mathscr{D}(X))\rangle - i\left[\langle \mathbf{u}, \nabla(\mathscr{D}(X))\rangle + \mathscr{D}(X)\Delta[\mathscr{D}(X)]\right].\end{aligned}$$

We obtain the following set of equations:

Lemma 2. *The real part of equation* $\mathbf{D}[mX^2] = 0$ *corresponds to* $m\mathbf{v}^2 = \gamma U(X) - U_\sigma(t, X)$ *where the potential* $U_\sigma(t, X)$ *is called the* **stochastic induced potential** *and is defined by*

$$U_\sigma(t, X) = -m(\mathbf{u}^2 + 2\mathscr{D}(X)\,\mathrm{div}[\mathbf{u}]) - m3\left(\langle \mathbf{u}, \nabla(\mathscr{D}(X))\rangle + \mathscr{D}(X)\Delta(\mathscr{D}(X))\right).$$

A more explicit form can be obtained expressing each term as functions of p_t and $\mathscr{D}$.

Lemma 3 (Stochastic induced potential)**.** *The stochastic induced potential is given by*

$$U_\sigma = -4m\mathscr{D}^2\frac{\Delta\left(\sqrt{p_t}\right)}{\sqrt{p_t}} - m\left[7\mathscr{D}\langle\nabla(\mathscr{D}), \frac{\nabla(p_t)}{p_t}\rangle + 4\langle\nabla(\mathscr{D}), \nabla(\mathscr{D})\rangle + 5\mathscr{D}\Delta[\mathscr{D}]\right].$$

When the diffusion coefficient is constant, we recover the classical induced potential obtained in [7]:

$$U_{0,\sigma} = -4m\mathscr{D}^2\frac{\Delta\left(\sqrt{p_t}\right)}{\sqrt{p_t}},$$

which coincides with the **Bohm or quantum potential** introduced by D. Bohm in his non-local hidden variable theory for quantum mechanics (see [5], p.168).

Proof. In order to obtain an explicit form of the stochastic induced potential, one can use the formula for $\mathbf{D}[X]$ in order to explicit $\mathbf{u}$ as a function of σ and $p_t(X)$. We denote by $\mathbf{c}_\sigma$ the vector of $\mathbb{R}^d$ defined by $\mathbf{c}_\sigma = \nabla(\mathscr{D}(X))$. We have

$$\begin{aligned} \mathrm{div}[\mathbf{u}] = {} & \mathrm{div}\,[\mathbf{u}_{0,\sigma} + \mathbf{c}_\sigma] = \langle \nabla\,(\mathscr{D}(X))\,, \frac{\nabla(p_t)}{p_t}\rangle + \mathscr{D}(X)\,\mathrm{div}\left[\frac{\nabla(p_t)}{p_t}\right] + \mathrm{div}(\mathbf{c}_\sigma), \\ = {} & \langle \nabla\,(\mathscr{D}(X))\,, \frac{\nabla(p_t)}{p_t}\rangle + \mathscr{D}(X)\left(-\frac{\langle \nabla(p_t), \nabla(p_t)\rangle}{p_t^2} + \frac{1}{p_t}\Delta[p_t]\right) + \mathscr{D}(X), \end{aligned}$$

and $\mathbf{u}^2 = \mathbf{u}_{0,\sigma}^2 + 2\langle \mathbf{u}_{0,\sigma}, \mathbf{c}_\sigma\rangle + \mathbf{c}_\sigma^2$. As a consequence, one obtains

$$\begin{aligned} U_{0,\sigma} = {} & -m\left[\mathbf{u}_{0,\sigma}^2 + \frac{\sigma^4}{2p_t}\Delta[p_t] - \frac{\sigma^4}{2}\frac{\langle \nabla(p_t), \nabla(p_t)\rangle}{p_t^2}\right] \\ & -m\left[2\langle \mathbf{u}_{0,\sigma}, \mathbf{c}_\sigma\rangle + \mathbf{c}_\sigma^2 + \sigma^2\langle \nabla(\mathscr{D}), \frac{\nabla(p_t)}{p_t}\rangle + \mathscr{D}\Delta[\sigma^2]\right], \\ = {} & -m\sigma^4\frac{\Delta\left(\sqrt{p_t}\right)}{\sqrt{p_t}} - m\left[\sigma^2\langle \nabla(\sigma^2), \frac{\nabla(p_t)}{p_t}\rangle + \langle \nabla(\mathscr{D}), \nabla(\mathscr{D})\rangle + \mathscr{D}\Delta[\sigma^2]\right] \end{aligned}$$

Moreover, we have $\langle \mathbf{u}, \nabla(\sigma^2)\rangle = \mathscr{D}\langle \frac{\nabla(p_t)}{p_t}, \nabla(\sigma^2)\rangle + 2\langle \nabla(\mathscr{D}), \nabla(\mathscr{D})\rangle$ which gives

$$\langle \mathbf{u}, \nabla(\sigma^2)\rangle + \mathscr{D}\Delta[\sigma^2] = \mathscr{D}\langle \frac{\nabla(p_t)}{p_t}, \nabla(\sigma^2)\rangle + 2\langle \nabla(\mathscr{D}), \nabla(\mathscr{D})\rangle + \mathscr{D}\Delta[\sigma^2].$$

We finally obtain

$$U_\sigma = -m\sigma^4\frac{\Delta\left(\sqrt{p_t}\right)}{\sqrt{p_t}} - m\left[7\mathscr{D}\langle \nabla(\mathscr{D}), \frac{\nabla(p_t)}{p_t}\rangle + 4\langle \nabla(\mathscr{D}), \nabla(\mathscr{D})\rangle + 5\mathscr{D}\Delta[\mathscr{D}]\right].$$

This concludes the proof. □

§4. Action functional and the stochastic Hamilton-Jacobi equation

In the constant diffusion case, we prove in [7] following a previous result of E. Nelson [9] that if X_t is a solution of the stochastic Newton's equation then the stochastic derivative of X is a gradient, meaning that the stochastic derivative can be written as

$$\mathbf{D}[X] = \frac{\nabla[\mathscr{A}](t, X_t)}{m},$$

where the complex valued functional $\mathscr{A}(t, X) = S(t, X) + iR(t, X)$ is called the **action functional**. Defining the function

$$\psi(t, X) = e^{i\mathscr{A}(t,X)/2m\mathscr{D}},$$

called the wave function, we prove that ψ is a solution of the (linear) Schrödinger equation.

In this Section, we study what is preserved from this construction in the non-constant diffusion case.

4.1. Explicit conditions for a gradient stochastic derivative

In the constant diffusion case, solutions X of the stochastic Newton equation satisfy the **gradient conditions** (4). This is trivial for the $\mathbf{u}$ component but follows from the reality of the right-hand side of the stochastic Newton equation for $\mathbf{v}$ (see [7],§.6, Lemma 5). A necessary condition ensuring that $\mathbf{u}$ and $\mathbf{v}$ are gradient is given by the **curl-conditions** $\operatorname{curl}(\mathbf{v}) = 0$ and $\operatorname{curl}(\mathbf{u}) = 0$. We have the following result:

Lemma 4 (Gradient conditions)**.** *Let X be a solution of the stochastic Newton equation. The* curl-*conditions are satisfied if and only if* $\nabla[\mathscr{D}] \wedge \nabla[p] = 0$ *and* $\operatorname{curl}(\mathbf{b}) = 0$.

Proof. We have $\mathbf{u} = \nabla[\mathscr{D}]+\mathscr{D}\nabla[\ln(p)]$. We then have using the identity $\operatorname{curl}(\phi\mathbf{a}) = \phi \operatorname{curl}(\mathbf{a})+ \nabla[\phi] \wedge \mathbf{a}$ that $\operatorname{curl}(\mathbf{u}) = \operatorname{curl}(\mathscr{D}\nabla[\ln(p)]) = \mathscr{D}\operatorname{curl}(\nabla[\ln(p)]) + \nabla[\mathscr{D}] \wedge \nabla[\ln(p)] = \nabla[\mathscr{D}] \wedge \nabla[\ln(p)]$. For $\mathbf{v}$, the curl-condition reads $\operatorname{curl}(\mathbf{v}) = \operatorname{curl}(\mathbf{b} - \mathscr{D}\nabla[\ln(p)] - \nabla[\mathscr{D}]) = \operatorname{curl}(\mathbf{b})$. The curl conditions then imply that one must have $\nabla[\mathscr{D}] \wedge \nabla[\ln(p)] = 0$ and $\operatorname{curl}(\mathbf{b}) = 0$. This concludes the proof. □

The second condition is directly satisfied if X has a gradient drift. As $\nabla[p] \neq 0$, the first condition is satisfied if $\nabla[\mathscr{D}] = 0$ or $\nabla[\mathscr{D}] = \lambda\nabla[p]$ for $\lambda \in \mathbb{R}$. If σ is constant then the first condition is automatically satisfied.

4.2. Stochastic Hamilton-Jacobi equation

We assume that the gradient condition are satisfied. We then obtain a generalization of the **stochastic Hamilton-Jacobi** proved in [7]:

Theorem 5 (Stochastic Hamilton-Jacobi equation)**.** *The action functional $\mathscr{A}$ satisfies the equation*

$$\nabla\left[\partial_t\mathscr{A} + \frac{1}{2m}\langle\nabla[\mathscr{A}],\nabla[\mathscr{A}]\rangle + i\mathscr{D}\Delta[\mathscr{A}] + U\right] = -i\Delta[\mathscr{A}]\nabla[\mathscr{D}].$$

The proof follows exactly the same lines as in [7]. When σ is constant, then $\mathscr{D}$ is constant and $\mathscr{D}\nabla\,[\Delta[\mathscr{A}]] = \nabla[\mathscr{D}\Delta[\mathscr{A}]]$. Equation (5) reduces to $\partial_t\mathscr{A}+\frac{1}{2m}\langle\nabla[\mathscr{A}],\nabla[\mathscr{A}]\rangle+i\mathscr{D}\Delta[\mathscr{A}] = -U$ obtained in [7].

4.3. A Schrödinger equation

The function ψ satisfies a partial differential equation given by:

Theorem 6. *The function ψ satisfies for all $j = 1,\dots,3$, the non linear partial differential equation*

$$\partial_{x_j}(A(\mathscr{D},\psi) + B(\mathscr{D},\psi)) = \partial_{x_j}(\mathscr{D})\Delta[\mathscr{D}\ln(\psi)],$$

where

$$A(\mathscr{D},\psi) = \frac{1}{\psi}\left[-i\mathscr{D}\partial_t(\psi) + (\mathscr{D})^2\Delta[\psi] + \frac{U}{2m}\psi\right]$$

and

$$B(C,\mathscr{D},\psi) = \ln(\psi)\left(\mathscr{D}\Delta[\mathscr{D}] + 2\mathscr{D}(1+\ln(\psi))\frac{\langle\nabla[\mathscr{D}],\nabla[\psi]\rangle}{\psi\ln(\psi)} + \ln(\psi)\langle\nabla[\mathscr{D}],\nabla[\mathscr{D}]\rangle\right)$$

When $\mathscr{D}$ is constant, we have $B = 0$ and we recover the fact that ψ satisfies the linear Schrödinger equation (see [7]):

$$-im\mathscr{D}\partial_t(\psi) - m\mathscr{D}^2\Delta[\psi] + U\psi = 0.$$

The density $p_t(x)$ of the process X is related to the modulus of ψ as follows:

Lemma 7. *Let* $\rho(x,t) = |\psi(x)|^2$ *then* $\nabla[\mathscr{D}\ln(\rho)] = \dfrac{\nabla[\mathscr{D}p]}{p}$.

When σ is constant then $\rho = p$.

Proof. We have $\psi\bar{\psi} = e^{i\frac{(\mathscr{A}-\bar{\mathscr{A}})}{2m\mathscr{D}}}$ and as a consequence $\psi\bar{\psi} = e^{\frac{R}{m\mathscr{D}}}$ so that $\ln(|\psi|^2) = \dfrac{R}{m\mathscr{D}}$ and $\nabla[m\mathscr{D}\ln(|\psi|^2)] = \nabla[R]$ which gives $\nabla[m\mathscr{D}\ln(|\psi|^2)] = \mathbf{u}$. As $\rho = \psi\bar{\psi}$ and using the expression of $\mathbf{u} = \dfrac{\nabla[\mathscr{D}p]}{p}$, we then obtain $\nabla[\mathscr{D}\ln(\rho)] = \dfrac{\nabla[\mathscr{D}p]}{p}$.This concludes the proof. □

4.4. Proof of theorem 6

We have $m\mathbf{D}X = \nabla[\mathscr{A}]$ by definition and as $\mathscr{A} = -i2m\mathscr{D}\ln(\psi)$, we deduce that $\mathbf{D}X = -2i\nabla[\mathscr{D}\ln(\psi)]$. The stochastic Newton equation $m\mathbf{D}^2X = -\nabla[U]$ then reads as

$$\mathbf{D}\left[-i2m\nabla[\mathscr{D}\ln(\psi)]\right] = -\nabla[U].$$

We compute the left hand side using the chain rule formula for $\mathbf{D}$. We have for all $j \in \{1,\dots,3\}$, that

$$\mathbf{D}\left[-i\partial_{x_j}[\mathscr{D}\ln(\psi)]\right] = -i\left(\partial_t\left(\partial_{x_j}[\ln(\psi)]\right) + \langle \mathbf{D}X, \nabla[\partial_{x_j}[\mathscr{D}\ln(\psi)]]\rangle - i\mathscr{D}\Delta[\partial_{x_j}[\mathscr{D}\ln(\psi)]]\right).$$

We now compute each term of the right hand side. Assuming enough regularity, the Schwartz lemma ensure that $\partial_t\left(\partial_{x_j}[\mathscr{D}\ln(\psi)]\right) = \partial_{x_j}(\partial_t[\mathscr{D}\ln(\psi)])$. Moreover, we have

$$\langle \mathbf{D}X, \nabla[\partial_{x_j}[\mathscr{D}\ln(\psi)]]\rangle = -i\partial_{x_j}\langle \nabla[\mathscr{D}\ln(\psi)], \nabla[\mathscr{D}\ln(\psi)]\rangle.$$

For the third term, we easily have that

$$-i\mathscr{D}\Delta[\partial_{x_j}[\mathscr{D}\ln(\psi)] = -i\mathscr{D}\partial_{x_j}\Delta[\mathscr{D}\ln(\psi)] = \partial_{x_j}(-i\mathscr{D}\Delta[\mathscr{D}\ln(\psi)]) + i\partial_{x_j}(\mathscr{D})\Delta[\mathscr{D}\ln(\psi)].$$

Regrouping these terms, we obtain

$$\begin{aligned}\mathbf{D}\left[-i\partial_{x_j}[\mathscr{D}\ln(\psi)]\right] = {} & \partial_{x_j}(-i\partial_t[\mathscr{D}\ln(\psi)] + \langle \nabla[\mathscr{D}\ln(\psi)], \nabla[\mathscr{D}\ln(\psi)]\rangle + \mathscr{D}\Delta[\mathscr{D}\ln(\psi)]) \\ & - \partial_{x_j}(\mathscr{D})\Delta[\mathscr{D}\ln(\psi)].\end{aligned}$$

The stochastic Newton equation is then equivalent to

$$\partial_{x_j}\left(-i\partial_t[\mathscr{D}\ln(\psi)] + \langle \nabla[\mathscr{D}\ln(\psi)], \nabla[\mathscr{D}\ln(\psi)]\rangle + \mathscr{D}\Delta[\mathscr{D}\ln(\psi)] + \frac{U}{2m}\right) - \partial_{x_j}(\mathscr{D})\Delta[\mathscr{D}\ln(\psi)] = 0$$

We can explicit the term in the ∂_{x_j} part. We have using the fact that $\mathscr{D}$ does not depend on t that $\partial_t[\mathscr{D}\ln(\psi)] = \mathscr{D}\dfrac{\partial_t(\psi)}{\psi}$. Moreover, as $\nabla(\mathscr{D}\ln(\psi)) = \nabla[\mathscr{D}]\ln(\psi) + \mathscr{D}\dfrac{\nabla[\psi]}{\psi}$, we obtain

$$\begin{aligned}&\langle\nabla[\mathscr{D}\ln(\psi)], \nabla[\mathscr{D}\ln(\psi)]\rangle\\ &\qquad = (\ln(\psi))^2\langle\nabla[\mathscr{D}], \nabla[\mathscr{D}]\rangle + \frac{\mathscr{D}^2}{\psi^2}\langle\nabla[\psi], \nabla[\psi]\rangle + 2\frac{\mathscr{D}\ln(\psi)}{\psi}\langle\nabla[\mathscr{D}], \nabla[\psi]\rangle.\end{aligned}$$

Properties of the Laplacian implies that

$$\Delta[\mathscr{D}\ln(\psi)] = \ln(\psi)\Delta[\mathscr{D}] + \mathscr{D}\Delta[\ln(\psi)] + \frac{2}{\psi}\langle\nabla[\mathscr{D}], \nabla[\psi]\rangle.$$

As $\Delta(\ln(\psi)) = \dfrac{\Delta[\psi]}{\psi} - \dfrac{\langle\nabla[\psi], \nabla[\psi]\rangle}{\psi^2}$, we finally obtain

$$\Delta[\mathscr{D}\ln(\psi)] = \ln(\psi)\Delta[\mathscr{D}] + \frac{\mathscr{D}\Delta[\psi]}{\psi} - \frac{\mathscr{D}}{\psi^2}\langle\nabla[\psi], \nabla[\psi]\rangle + \frac{2}{\psi}\langle\nabla[\mathscr{D}], \nabla[\psi]\rangle.$$

As a consequence, we have, regrouping first terms depending on ψ and $\mathscr{D}$, and then terms depending on $\nabla[\mathscr{D}]$ or $\Delta[\mathscr{D}]$ that

$$-i\partial_t[\mathscr{D}\ln(\psi)] + \langle\nabla[\mathscr{D}\ln(\psi)], \nabla[\mathscr{D}\ln(\psi)]\rangle + \mathscr{D}\Delta[\mathscr{D}\ln(\psi)] + \frac{U}{2m} = (I) + (II) + U$$

where the terms (I) and (II) are respectively defined by

$$\begin{aligned}(I) &= -i\mathscr{D}\frac{\partial_t(\psi)}{\psi} + \frac{\mathscr{D}^2}{\psi^2}\langle\nabla[\psi], \nabla[\psi]\rangle + \mathscr{D}\left(\frac{\mathscr{D}\Delta[\psi]}{\psi} - \frac{\mathscr{D}}{\psi^2}\langle\nabla[\psi], \nabla[\psi]\rangle\right),\\ &= \frac{\mathscr{D}}{\psi}\left[-i\partial_t(\psi) + \mathscr{D}\Delta[\psi]\right],\end{aligned}$$

and

$$\begin{aligned}(II) = &\left((\ln(\psi))^2\langle\nabla[\mathscr{D}], \nabla[\mathscr{D}]\rangle + 2\frac{\mathscr{D}\ln(\psi)}{\psi}\langle\nabla[\mathscr{D}], \nabla[\psi]\rangle\right)\\ &+ \mathscr{D}\left(\ln(\psi)\Delta[\mathscr{D}] + \frac{2}{\psi}\langle\nabla[\mathscr{D}], \nabla[\psi]\rangle\right),\\ = &\ln(\psi)\left(\mathscr{D}\Delta[\mathscr{D}] + 2\mathscr{D}(1 + \ln(\psi))\frac{\langle\nabla[\mathscr{D}], \nabla[\psi]\rangle}{\psi\ln(\psi)} + \ln(\psi)\langle\nabla[\mathscr{D}], \nabla[\mathscr{D}]\rangle\right).\end{aligned}$$

This concludes the proof.

§5. Conclusion and perspectives

The generalization of the theory developed in [7] to study stochastic Newtonian dynamics does not lead to a tractable evaluation of the induced potential due to the complexity of the partial differential equation satisfied by the wave function whose modulus is the density of a solution. An alternative would be to follow the approach suggested by A. Albeverio and al. in [1] where in a framework similar to [7], they are able to reproduce the Hubble morphology classification of galaxies by considering suitable combination of solutions of the linear Schrodinger equation.

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