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SOME TECHNIQUES FOR THE STABILISATION OF THE PRESSURE DISCRETISATION IN REDUCED ORDER MODELS OF INCOMPRESSIBLE FLUIDS

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Abstract. We address the stability of the pressure discretisation for Reduced Order Models (ROMs) of incompressible flows. For Galerkin discretisations of incompressible fluids, the stability of the pressure is guaranteed through the discrete inf-sup condition for the duality velocity - pressure gradient. This property can be extended to ROMs by adding velocity "supremisers" (the Riesz representation of the pressure gradient on the velocity space, cf. [6]). However it is rather costly and several alternative strategies can be carried on. Among them, stabilisation techniques (cf. [1]) or post-processing of the pressure (cf. [4]), that we present here.

Keywords: Keywords separated by commas.

AMS classification: AMS classification codes.

§1. Motivation: pressure stabilisation in ROM

We intend to solve parametric incompressible flow problems, either in laminar or turbulent regimes, with very fast procedures. We actually consider the Smagorinsky turbulence model for incompressible flows. This is the basic Large Eddy Simulation (LES) turbulence model, that solves the large scales of the flow and a part of the inertial spectrum. It is intrinsically linked to a discretisation grid, the sub-grid scale effects are modelled by means of an eddy diffusion term. For simplicity, we introduce it here as a continuous model, where the grid appears in parametric form. Let us consider a bounded domain $\Omega \subset \mathbf{R}^d$ ($d = 2$ or 3), and consider a triangulation $\mathcal{T}_h$ of Ω. We consider the problem: *Find* $\boldsymbol{u} : \Omega \times (0,T) \longrightarrow \mathbb{R}^d$ *and* $p : \Omega \times (0,T) \longrightarrow \mathbb{R}$ *such that:*

$$\left\{ \begin{array}{rcll} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nabla \cdot ((\nu + \nu_t)\, \nabla \boldsymbol{u} + \nabla p & = & \boldsymbol{f} & \text{in } \Omega \times (0,T), \\ \nabla \cdot \boldsymbol{u} & = & 0 & \text{in } \Omega \times (0,T), \\ \boldsymbol{u}(\boldsymbol{x},0) & = & \boldsymbol{u}_0(\boldsymbol{x}) & \text{in } \Omega. \end{array} \right. \tag{1}$$

(plus boundary conditions), where the eddy viscosity is given by

$$\nu_t = \sum_{K \in \mathcal{T}_h} C_S^2\, h_K^2\, |\nabla \boldsymbol{u}|_{K_T},$$

where $C_S = 0.2$ is a universal constant, h_K denotes the diameter of element $K \in \mathcal{T}_h$ and κ_K is the characteristic function of K. The Navier-Stokes equations, that govern incompressible flow in laminar regime, correspond to $\nu_t = 0$.

We actually consider Reduced Order Modelling (ROM), that provides speeds up of computing time typically ranging from tens to thousands. The discretisation of the pressure should be specifically treated, as the standard techniques to build ROMs do not ensure that the pressure discretisation in the ROM is stable. Further, this stabilisation treatment could produce a loss of accuracy.

We start from a standard discretisation, for instance a mixed discretisation by finite elements ("Full Order Model", FOM). We consider homogeneous Dirichlet boundary conditions to avoid non-essential difficulties: *Find* $(\boldsymbol{u}_h, p_h) : (0,T) \longrightarrow \mathbf{X}_h \times Q_h$ *such that for any* $(\boldsymbol{v}_h, q_h) \in \mathbf{X}_h \times Q_h$,

$$\left\{ \begin{array}{rcl} \dfrac{d}{dt}(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) + ((\nu + \nu_t)\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) - (p_h, \nabla \cdot \boldsymbol{v}_h) &=& (\boldsymbol{f}, \boldsymbol{v}_h) \quad \text{in } \mathcal{D}'(0,T) \\ (\nabla \cdot \boldsymbol{u}_h, q_h) &=& 0, \\ \boldsymbol{u}_h(0) &=& \boldsymbol{u}_{0h}, \end{array} \right. \tag{2}$$

The pair of discrete spaces $(\mathbf{X}_h, Q_h) \subset H_0^1(\Omega)^d \times L^2(\Omega)/\mathbf{R}$ is assumed to satisfy the discrete inf-sup condition: There exists $\beta > 0$ such that

$$\beta \, \|q_h\|_{L^2(\Omega)} \le \sup_{\boldsymbol{v}_h \in \mathbf{X}_h} \frac{(q_h, \nabla \cdot \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{H^1(\Omega)^d}}, \;\; \forall q_h \in Q_h.$$

The standard numerical analysis of mixed problems ensures that the discrete problem is well posed, in particular the pressure is bounded in $L^2(\Omega)$.

1.1. The Reduced Order Model

Assume we have constructed a pair of velocity-pressure spaces $\boldsymbol{X}_r$, Q_s of very low dimension (the "Reduced Spaces") that approximates the varieties $\{\boldsymbol{u}_h(t),\ t \in [0,T]\}$ and $\{p_h(t),\ t \in [0,T]\}$, respectively. We then consider the ROM discretisation:
Find $(\boldsymbol{u}_r, p_s) : (0,T) \longrightarrow \boldsymbol{X}_r \times Q_s$ *s. t.* for any $(\boldsymbol{\varphi}, q_s) \in \boldsymbol{X}_r \times Q_s$,

$$\left\{ \begin{array}{rcl} \dfrac{d}{dt}(\boldsymbol{u}_r, \boldsymbol{\varphi}) + b(\boldsymbol{u}_r, \boldsymbol{u}_r, \boldsymbol{\varphi}) + ((\nu + \nu_t)\nabla \boldsymbol{u}_r, \nabla \boldsymbol{\varphi}) - (p_s, \nabla \cdot \boldsymbol{\varphi}) &=& (\boldsymbol{f}, \boldsymbol{\varphi}) \quad \text{in } \mathcal{D}'(0,T), \\ (\nabla \cdot \boldsymbol{u}_r, q_s) &=& 0, \\ \boldsymbol{u}_r(0) &=& \boldsymbol{u}_{0r}, \end{array} \right. \tag{3}$$

The reduced pressure p_s may eventually be eliminated if the reduced velocities of $\boldsymbol{X}_r$ are weakly divergence-free.

A very popular technique to construct the reduced spaces is the Proper Orthogonal Decomposition (POD). It starts from velocity "snapshots" $\chi^v = \text{span}\left\{\boldsymbol{u}_h^1, \ldots, \boldsymbol{u}_h^N\right\} \subset \mathbf{X}_h$, from the solution of the FOM at discrete times t_n, $n = 1, \ldots, N$ (and similarly for pressure). The POD builds an optimal subspace $\boldsymbol{X}_r = \text{span}\{\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_r\}$ that provides the best approximation of the velocity snapshots with respect to the discrete $l^2(\mathcal{H})$ norm (where $\mathcal{H}$ is some subspace of $H^1(\Omega)^d$ or $\mathcal{H} = L^2(\Omega)^d$), among all sub-spaces of χ^v of dimension r. The POD projection error is given by $\left\| \left\{ \boldsymbol{u}_h^n - \Pi_{\mathcal{H}} \boldsymbol{u}_h^n \right\}_{n=0}^N \right\|_{l^2(\mathcal{H}_v)}^2 = \sum_{i=r+1}^{N} \lambda_i$, where the λ_i are the eigenvalues of the correlation matrix $\left[(\boldsymbol{u}_h^n, \boldsymbol{u}_h^m)_{\mathcal{H}} \right]_{n,m=1}^N$ and $\Pi_{\mathcal{H}}$ is the orthogonal projection on $\mathcal{H}$. Typically, the λ_i decrease with exponential rate so if $r << N$, X_r provides a very good approximation of $\mathbf{X}_h$.

1.2. The reduced pressure treatment

Similary, the POD provides a reduced space for pressure $Q_s \subset Q_h$ with $dim(Q_s) << dim(Q_h)$. However, there are no reasons for the pair of spaces $(\boldsymbol{X}_r, Q_s)$ to satisfy the discrete inf-sup condition, and consequently the pressure discretisation on Q_s besides the velocity discretisation on $\mathbf{X}_h$ is unstable. Att present, the standard way to overcome this difficulty (cf. [6]) is to enrich the velocity space $\boldsymbol{X}_r$ with the Riesz representatives in $\boldsymbol{X}_h$ of the reduced pressure gradients ∇q_s in the $H_0^1(\Omega)^d$-$H^{-1}(\Omega)^d$ duality,

$$\psi_s(q_s) \in \boldsymbol{X}_h, \;\; (\nabla \psi_s, \nabla \boldsymbol{v}_h)_{L^2(\Omega)} = \langle \nabla q_s, \boldsymbol{v}_h \rangle_{\Omega)} = -(q_s, \nabla \cdot \boldsymbol{v}_h)_\Omega, \;\; \forall \boldsymbol{v}_h \in \boldsymbol{X}_h. \tag{4}$$

$\psi_s(q_s)$ also is a supremizer in $\boldsymbol{X}_h$ of the normalised duality $\dfrac{\langle \nabla q_s, \boldsymbol{v}_h \rangle_\Omega}{\|\nabla \boldsymbol{v}_h\|_{L^2(\Omega)}}$. The pair of spaces $(\boldsymbol{X}_r^s, Q_s)$ satisfies the discrete inf-sup condition, where $\boldsymbol{X}_r^s$ is the enriched space

$$\boldsymbol{X}_r^s = \boldsymbol{X}_r \cup \{\psi_s(q_s), \;\; \forall q_s \in Q_s\}. \tag{5}$$

However, this requires reduced velocity spaces with larger dimension.

1.3. Reduced pressure recovery

As the reduced velocities are weakly divergence free (second equation in (3)), the reduced velocity can be decoupled from that of the reduced pressure. The pressure can be recovered in a two-step solution of the ROM:

Step 1: Computation of the reduced velocity. *Find* $\boldsymbol{u}_r : (0,T) \longrightarrow \boldsymbol{X}_r$ *s. t.* for any $\boldsymbol{\varphi} \in \boldsymbol{X}_r$,

$$\left\{ \begin{array}{rcll} \dfrac{d}{dt}(\boldsymbol{u}_r, \boldsymbol{\varphi}) + b(\boldsymbol{u}_r, \boldsymbol{u}_r, \boldsymbol{\varphi}) + ((\nu + \nu_t)\nabla \boldsymbol{u}_r, \nabla \boldsymbol{\varphi}) & = & (\boldsymbol{f}, \boldsymbol{\varphi}) & \text{in } \mathcal{D}'(0,T), \\ \boldsymbol{u}_r(0) & = & \boldsymbol{u}_{0r}. & \end{array} \right.$$

Note that this problem can be formulated as three independent ordinary differential systems, one for each velocity component.

Step 2: Computation of the reduced pressure. Several methods are available, depending on the pressure equation used, either Pressure gradient equation or Poisson pressure equation. The pressure gradient equation is directly obtained from the Navier-Stokes equations,

$$\nabla p = -\partial_t \boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla \cdot ((\nu + \nu_t)\nabla \boldsymbol{u}) - \boldsymbol{f} \;\text{ in }\; \Omega \times (0,T).$$

The reduced pressure can be recovered by testing with the Riesz transformed-enriched reduced velocity space: $p_s(t) \in Q_s$ satisfies,$\forall \boldsymbol{\varphi}_r \in \boldsymbol{X}_h^s$,

$$-(p_s(t), \nabla \cdot \boldsymbol{\varphi}_r)_{L^2(\Omega)} = \frac{d}{dt}(\boldsymbol{u}_r(t), \boldsymbol{\varphi}_r) - b(\boldsymbol{u}_r(t); \boldsymbol{u}_r, \boldsymbol{\varphi}_r) - ((\nu + \nu_t)\nabla \boldsymbol{u}(t), \nabla \boldsymbol{\varphi}_r) - (\boldsymbol{f}(t), \boldsymbol{\varphi}_r),$$

This provides a stable recovery of the pressure. Also, $l^2(L^2)$ error estimates for the pressure hold in terms of the eigenvalues corresponding to the neglected modes (cf. [5])

An alternative procedure is to build a pressure Poisson equation, by testing the Navier-Stokes equations with pressure gradients. This yields *Find* $p_s \in Q_s$ *such that, for all* $q_s \in Q_s$,

$$\sum_{K\in\mathcal{T}_h} \tau_K(\nabla p_r, \nabla q_s)_K = -\sum_{K\in\mathcal{T}_h} \tau_K(\partial_t \boldsymbol{u}_r + \boldsymbol{u}_r\cdot\nabla\boldsymbol{u}_r - \nabla\cdot((\nu+\nu_t)\nabla\boldsymbol{u}_r) - \boldsymbol{f}, \nabla q_s)_K. \qquad (6)$$

Here, the τ_K are stabilisation coefficients of order h_K^2. This equation implies a discretisation of the natural boundary conditions for pressure,

$$\partial_{\mathbf{n}} p_r = (\partial_t \boldsymbol{u}_r + \boldsymbol{u}_r\cdot\nabla\boldsymbol{u}_r - \nabla\cdot((\nu+\nu_t)\nabla\boldsymbol{u}_r) - \boldsymbol{f})\cdot\mathbf{n} \text{ on } \partial\Omega.$$

This provides a stable recovery of the pressure. For Navier-Stokes equations ($\nu_t = 0$), error estimates for the pressure in terms of the sums of the eigenvalues corresponding to the neglected modes in velocity in $H^1(\Omega)$ norm, as well as in pressure in the norm $\|p\|_h = \left(\sum_{i=1}^{N-1} h_K^2 \|\nabla p\|_{0,K}^2\right)^{1/2}$, are proved in [4]. When $d = 3$, with truncated convection velocity in (6) to h_K^{-1} in 3D; that is, the term $\boldsymbol{u}_r\cdot\nabla\boldsymbol{u}_r$ is changed into $\tilde{\boldsymbol{u}}_r\cdot\nabla\boldsymbol{u}_r$, where $\tilde{\boldsymbol{u}}_r$ is a truncated approximation to $\boldsymbol{u}_r$ with $L^\infty(\Omega)$ norm smaller than $Constant \times h_K^{-1}$.

§2. Local Projection Stabilisation (LPS) Reduced Basis model.

The Local Projection Stabilisation provides an alternative way of stabilising the pressure discretisation, both in the FOM and the ROM models, with further reduction of computational cost. We state it in terms of the Reynolds number $\Re = UL/\nu$ with U and L a characteristic velocity and length of the flow. We discretise these equations by: *Find* $U_h = (\mathbf{u}_h, p_h) \in \mathbf{X}_h \times Q_h$ *such that for all* $V_h = (\mathbf{v}_h, q_h) \in \mathbf{X}_h \times Q_h$ *it holds*

$$A(U_h; U_h, V_h; \Re) \quad = \langle f, \mathbf{v}_h\rangle \qquad (7)$$

where $A(Z; U, V; \Re) = A_{Conv}(Z, U, V) + A_{Dif}(U, V; \Re) + A_{Div}(U, V) + A_{Pres}(U, V)$, with

$$A_{Conv}(Z, U, V) = \int_\Omega (\mathbf{z}\cdot\nabla\mathbf{u})\mathbf{v}\, d\Omega,\ A_{Dif}(U, V; \Re) = \int_\Omega \left(\frac{1}{\Re} + \nu_t(\Re, \mathbf{u})\right)\nabla\mathbf{u} : \nabla\mathbf{v}\, d\Omega,$$
$$A_{Div}(U, V) = \int_\Omega [(\nabla\cdot\mathbf{u})q - (\nabla\cdot\mathbf{v})p]\, d\Omega,\ A_{Pres}(U, V) = \int_\Omega \tau_p\, \sigma_h^*(\nabla p)\, \sigma_h^*(\nabla q)\, d\Omega.$$

Here, $\sigma_h^* = Id - \sigma_h$ is a fluctuation operator, σ_h is a restriction operator from $L^2(\Omega)$ to a coarse-grid FE space. Also, $\tau_p(\boldsymbol{x}) = \tau_K,\ \forall \boldsymbol{x} \in,\ \forall K \in \mathcal{T}_h$, where $\tau_K = O(h_K^2)$ is the stabilisation coefficient. The pressure stabilisation term $A_{Pres}(U, V)$ controls the small scale components of ∇p that are not represented in X_h by $\int_\Omega (\nabla\cdot\mathbf{v})p\, d\Omega$. Actually,

$$\|p_h\| = \sup_{\mathbf{v}\in Q_h} \frac{(\nabla\cdot\mathbf{v}, p_h)}{\|\mathbf{v}\|_{H_0^1(\Omega)^d}} + \left(\int_\Omega \tau_p\, \left|\sigma_h^*(\nabla p)\right|^2 d\Omega\right)^{1/2}$$

is a norm on Q_h equivalent to the $L^2(\Omega)$ norm.

2.1. Construction of Reduced Basis by Greedy Algorithm

We intend to approximate the parametric variety $\{U(\mathfrak{R}),\ \mathfrak{R} \in \mathcal{D} = [\mathfrak{R}_1, \mathfrak{R}_2]\}$. We build a "Reduced Basis"(as an alternative to the reduced space provided by the POD), by means of a greedy algoritm, as follows. For simplicity of notation, we set $Y = H_0^1(\Omega)^d \times L^2(\Omega)$:

1. **Initialization**.
   - Choose a discrete set of parameters $\mathcal{D}_{train}$ that approximates $\mathcal{D}$.
   - Randomly choose $\mathfrak{R}_1 \in \mathcal{D}_{train}$ and set $B_1 = U_h(\mathfrak{R}_1) = (\boldsymbol{u}_h(\mathfrak{R}_1), p_h(\mathfrak{R}_1))$, $Y_1 = \mathtt{Span}\{B_1\}$.
2. **Enrichment**. Assuming known B_{N-1}, compute

$$\mathfrak{R}_N = \text{argmax}_{\mathfrak{R}\in\mathcal{D}_{train}} \|U_{N-1}(\mathfrak{R}) - U_h(\mathfrak{R})\|_Y$$

   and set

$$B_N = \{B_{N-1}, U_h(\mathfrak{R}_N)\}, \quad Y_N = \mathtt{Span}\{B_N\}.$$

For evolution problems a further reduction of the discrete space by POD is needed. That is, once $\mathfrak{R}_N$ is chosen, we define

$$\tilde{B}_N = \{B_{N-1}, U_h(\mathfrak{R}_N, t_1), \cdots, , U_h(\mathfrak{R}_N, t_M)\},$$

where $t_1, \cdots, t_M$ are the time steps. Then B_N is obtained from the POD analysis of the elements of $\tilde{B}_N$, by retaining the more energetic modes up to a preset level.

The Greedy Algorithm is oriented to minimize the distance in $L^\infty(\mathcal{D}, X)$ between the reduced and the trust solutions. In practice, the exact error $\|U_N(\mathfrak{R}) - U_{FOM}(\mathfrak{R})\|_Y$ is approximated by an a-posteriori estimator Δ_N. The greedy algorithm to construct the reduced basis with the error estimator instead of the exact error is called "weak" greedy algorithm.

2.2. Reduced basis problem with LPS treatment of Pressure

The Reduced Basis spaces $Y_N = X_N \times Q_N$ are constructed by a weak greedy algorithm using the error estimator $\Delta_N(\mu)$. The corresponding reduced problem, that now we state for the Smagorinsky turbulence model, is then set as follows: *Compute* $U_N(\mathfrak{R}) = (\boldsymbol{u}, p) \in Y_N$ *by*

$$A^*(U_N(\mathfrak{R}), V_N; \mathfrak{R}) = F(V_N) \quad \forall V_N \in Y_N, \tag{8}$$

where the form A^* is an approximation of the form A constructed by replacing $\nu_T(\mathbf{z}; \mathfrak{R})$ and $\tau_p(\boldsymbol{x})$ by reduced approximations $\nu_T^*(\mathbf{z}; \mathfrak{R})$ and $\tau_p^*(\boldsymbol{x})$, obtained by means of the empirical interpolation method (EIM). The stability of the reduced problem is ensured as

$$\|p_h\| = \sup_{\mathbf{v}\in X_N} \frac{(\nabla\cdot\mathbf{v}, p_h)}{\|\mathbf{v}\|_{H_0^1(\Omega)^d}} + \left(\int_\Omega \tau_p(\boldsymbol{x})\ \left|\sigma_h^*(\nabla p)\right|^2 d\Omega\right)^{1/2}$$

again is a norm on the reduced pressure space M_N for suitable choices of the projection operator σ_h (cf. [3]).

§3. A posteriori error estimation

In this section we discuss the construction of the a posteriori error indicator $\Delta_N(\mathfrak{R})$ that we use to build the reduced basis for problem (8). We use the Brezzi-Rappaz-Raviart theory for approximation of branches of non-singular solutions of non-linear problems [2]. The building of the error indicator is based upon the Lipschitz continuity of the tangent operator (cf. [3]):

Theorem 1. *There exists a positive constant ρ_T such that*

$$\left|\partial_1 A(U_h^1, V_h; \mathfrak{R})(Z_h) - \partial_1 A(U_h^2, V_h; \mathfrak{R})(Z_h)\right| \le \rho_T \|U_h^1 - U_h^2\|_X \|Z_h\|_Y \, \|V_h\|_Y,$$

for all $U_h^1, U_h^2, Z_h, V_h \in Y_h$.

Let $\langle \mathcal{R}(U_N(\mathfrak{R}), \mathfrak{R}), V_h\rangle = F(V_h; \mathfrak{R}) - A(U_N(\mathfrak{R}), V_h; \mathfrak{R})$ be the residual of the trial solution $U_N(\mathfrak{R}) \in Y_N$, β_N be the uniform (in parameter) coercivity constant of $\partial_1 A(U_N(\mathfrak{R}))$, given by

$$\beta_N = \min_{\mathfrak{R}\in\mathcal{D}} \sup_{V_N\in Y_N} \frac{\partial_1 A(U_N(\mathfrak{R}), V_h; \mathfrak{R})}{\|V_N\|_Y},$$

and $\tau_N(\mathfrak{R}) = \dfrac{4\rho_T}{\beta_N^2} \|\mathcal{R}(\cdot; \mathfrak{R})\|_{Y_h'}$. Then it holds

Theorem 2. *If $\beta_N > 0$ and $\tau_N(\mathfrak{R}) \le 1$, then there exists a unique solution $U_h(\mathfrak{R})$ to (FE) such that*

$$\left(2\frac{\rho_T}{\beta_N} + \tau_N\right)^{-1} \Delta_N(\mathfrak{R}) \le \|U_h(\mathfrak{R}) - U_N(\mathfrak{R})\|_Y \le \Delta_N(\mathfrak{R})$$

$$\text{with} \quad \Delta_N(\mathfrak{R}) = \frac{\beta_N}{2\rho_T}\left[1 - \sqrt{1 - \tau_N(\mathfrak{R})}\right].$$

This estimate means that the norm of the error in the natural space $Y = H_0^1(\Omega)^d \times L^2(\Omega)$ is driven by the dual error of the residual $\left\|\mathcal{R}(U_N(\mathfrak{R}), \mathfrak{R})\right\|_{Y_h'}$, which is amplified in terms of the Lipschitz continuity and the coercivity of the tangent Smagorinsky operator $\partial_1 A(U_N(\mathfrak{R}),\cdot;\mathfrak{R})$. The condition $\tau_N(\mathfrak{R}) \le 1$ holds if the residual of the trial solution $U_N(\mathfrak{R})$ is small enough. If not, we still may use a linearised form of $\Delta_N(\mathfrak{R})$ as a function of $\tau_N(\mathfrak{R})$.

§4. Numerical results

This section deals with the application of the LPS stabilised reduced basis approximation of the steady Smagorinsky model.

We actually compare its performances with the reduced basis method constructed with the spaces (Y_N', Q_N) obtained by adding to the reduced velocity space Y_N the supremisers of the gradients of the pressure basis functions of Q_N, in the same way as the space $\boldsymbol{X}_r^s$ is obtained from $\boldsymbol{X}_r$ by (5).

We consider the 2D lid-driven cavity flow, with the following computational setting

- Domain: $\Omega = (0, 1) \times (0, 1)$.
- Boundary conditions: $\boldsymbol{u} = 0$ at the bottom and lateral boundaries, $\boldsymbol{u} = 1$ on the cavity lid.

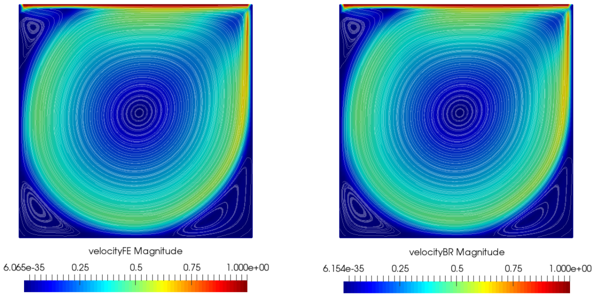
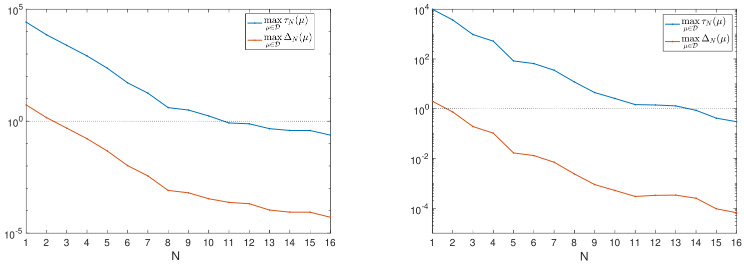


Figure 1: FE (left) and RB (right) velocity solution for $\Re$ = 4521



(a) LPS stabilisation (b) Supremisers stabilisation

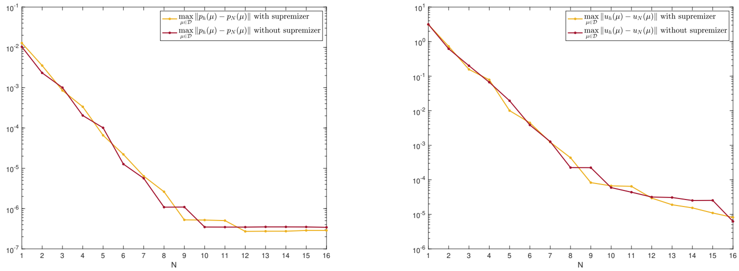
Figure 2: Convergence history of Greedy algorithm for LPS and Supremisers stabilisation of pressure

- Reynolds range: $\Re \in [1000, 5100]$
- Velocity-pressure finite element pair: $(\mathbb{P}2 - \mathbb{P}2)$ (non inf-sup stable).
- Regular mesh (2601 nodes and 5000 triangles)

This corresponds to 30.603 degrees of freedom (dof) for the FOM (finite element discretisation). The EIM approximations of the eddy viscosity ν_T^* and of the stabilised coefficients τ^* respectively require 52 and 48 dofs. The reduced velocity-pressure spaces are of dimension 32 for the LPS stabilised method, and of 48 for the supremisers stabilised method. This last dimension is necessarily larger, as we include the pressure gradient supremisers in the reduced velocity basis.

In Figure 1 we compare the velocity FOM solution with the ROM solution obtained with the LPS stabilised reduced method. We observe a very small error among both. Figure 2 shows the convergence history of the Greedy algorithm for both ROMs (with LPS (left) and supremisers (right) pressure stabilisation). A uniform decay of both the estimators and the dual residual norm is observed, with a decrease velocity similar for both methods.

Figure 3 shows the comparison of the decay of errors in natural norms between the full order and the reduced order solutions, for pressure (left) and velocity (right), as the dimension



(a) Pressure error (b) Velocity error

Figure 3: Comparison of errors between pressure and velocity for LPS and Supremisers stabilisation of pressure

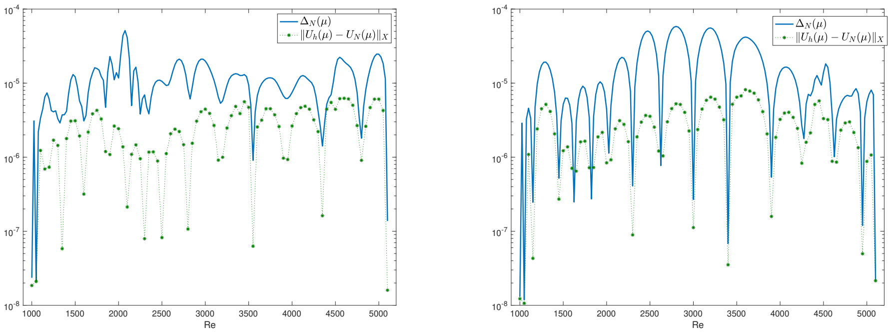


Figure 4: *A posteriori* error bounds at $N = N_{\max}$. The parameter $\mathfrak{R}$ is denoted by μ.

of the reduced spaces increases. We observe quite close error levels for both ROMs.

In Figure 4 we compare the actual true error with the estimated error for the full range of $\mathfrak{R} \in \mathcal{D}$, for both reduced methods. We observe that the ROMs with supremiser pressure stabilisation behave more smoothly than the ROM with LPS pressure stabilisation, while the efficiencies (error/estimator rate) are typically of two orders of magnitude.

In Table 1 we present the error levels and the computing times when solving the Smagorinsky model with both ROMs for some test values of the parameter $\mathfrak{R}$ that are not used to build the reduced basis. We observe quite similar error levels, with computing times for the LPS stabilised method nearly 30% below the computing times required by the supremisers stabilised method. We obtain speeds-up of several thousand for both methods, with an increased speed-up for the LPS stabilised method.

§5. Conclusions

In this work we have described several techniques to treat the pressure discretisation in reduced order modelling of incompressible flows. Besides a post-processing to recover the

LPS stabilisation

Data	$\mathfrak{R} = 1610$	$\mathfrak{R} = 2751$	$\mathfrak{R} = 3886$	$\mathfrak{R} = 4521$
T_{FE}	2259.04s	3008.81s	4756.93s	6171.41s
T_{online}	0.7s	0.75s	0.78s	0.81s
speedup	3227	4011	7146	7619
$\|\|\mathbf{u}_h - \mathbf{u}_N\|\|_{H^1}$	$5.4 \cdot 10^{-7}$	$1.44 \cdot 10^{-6}$	$1.49 \cdot 10^{-6}$	$5.44 \cdot 10^{-6}$
$\|\|p_h - p_N\|\|_{L^2}$	$1.34 \cdot 10^{-8}$	$3.35 \cdot 10^{-8}$	$1.55 \cdot 10^{-7}$	$2.25 \cdot 10^{-8}$

Supremiser stabilisation

Data	$\mathfrak{R} = 1610$	$\mathfrak{R} = 2751$	$\mathfrak{R} = 3886$	$\mathfrak{R} = 4521$
T_{FE}	2259.04s	3008.81s	5574.5s	6171.41s
T_{online}	0.99s	1.03s	1.26s s	1.23s
speedup	2267	2895	4391	5016
$\|\|\mathbf{u}_h - \mathbf{u}_N\|\|_{H^1}$	$4.94 \cdot 10^{-7}$	$4.51 \cdot 10^{-6}$	$6.52 \cdot 10^{-7}$	$3.52 \cdot 10^{-6}$
$\|\|p_h - p_N\|\|_{L^2}$	$5.23 \cdot 10^{-8}$	$3.22 \cdot 10^{-8}$	$6.38 \cdot 10^{-8}$	$8.45 \cdot 10^{-8}$

Table 1: Comparison of errors and speeds-up of computing time for ROMs with LPS and supremiser pressure stabilisation.

pressure from ROMs that initially only compute the velocity, we have reviewed the construction of a reduced basis approximation of the Smagorinsky turbulence model, in which the pressure discretisation is stabilised by the Local Projection Stabilisation technique. The reduced basis is built by a greedy algorithm based upon an error indicator specifically built for this discretisation.

Some numerical tests for 2d lid-driven cavity flow show that the LPS stabilised ROM error levels are quite close to those obtained with today standard reduced methods to solve incompressible flow problems, with enhanced reduction of the computing times.

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