Monografías Matemáticas *García de Galdeano* **43**, 21–29 (2024)

ESTIMATING THE DISTANCE BETWEEN THE INVARIANT MANIFOLDS OF L_3 IN THE RCP3BP USIGN HIGH PRECISION METHODS

Inmaculada Baldomá, Ernesto Lanchares and Mercè Ollé

Abstract. Recently an asymptotic formula for the distance of the invariant manifolds of L_3 in the RCP3BP when the mass parameter tends to zero was proven. In this study we numerically check the formula using high precision routines and give approximate values for the constants involved.

Keywords: invariant manifold, Parameterization method, inner equation.

AMS classification: AMS classification codes.

§1. Introduction

Let us consider a dynamical system consisting of three bodies interacting gravitationally in free space. Using Newton's equations, you obtain the system

$$
\begin{aligned}
m_1\ddot{q}_1 &= G\frac{m_1 m_2}{\|q_1-q_2\|^3}(q_2-q_1)+G\frac{m_1 m_S}{\|q_S-q_1\|^3}(q_S-q_1)\\
m_2\ddot{q}_2 &= G\frac{m_2 m_1}{\|q_1-q_2\|^3}(q_1-q_2)+G\frac{m_2 m_S}{\|q_S-q_2\|^3}(q_S-q_2)\\
m_S\ddot{q}_S &= G\frac{m_S m_1}{\|q_1-q_S\|^3}(q_1-q_S)+G\frac{m_S m_2}{\|q_2-q_S\|^3}(q_2-q_S),
\end{aligned}
$$

where m_1, m_2, m_S, q_1, q_2, q_S are the mass and positions with respect to some fixed origin of the considered bodies. Now, we make several assumptions in order to reduce the complexity of the system.

First, let the bodies lay in the same plane of motion. Also assume $m_S \ll m_1, m_2$ therefore we can uncouple the movement of the first two objects from the third. As the two body problem is integrable, we can consider a particular solution and use it in order to determine the motion of the third body. If we consider a circular solution, we will have the Restricted Circular Planar 3 Body Problem (RCP3BP for short). Which can be summarized in the following ODE

$$
\ddot{q}_S = G\frac{m_1}{\|q_1(t)-q_S\|^3}(q_1(t)-q_S)+G\frac{m_2}{\|q_2(t)-q_S\|^3}(q_2(t)-q_S).
$$

We can take rotating coordinates, also called synodic coordinates, such that the positions of the primaries are fixed at points $(\mu, 0)$ and $(\mu-1, 0)$. In this coordinate system, the dynamics of the secondary body are described by the Hamiltonian system given by Hamiltonian

$$H(q,p) = \frac{||p||}{2} - q^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} p - \frac{1-\mu}{||q-(\mu,0)||} - \frac{\mu}{||q-(\mu-1,0)||} \tag{1}$$

with $q = (q_x, q_y)$, $p = (p_x, p_y) = (\dot{q}_x - q_y, \dot{q}_y + q_x)$ and we have dropped the subindices avoiding a cumbersome notation. The equations of motion associated with (1) are

$$\begin{cases} \dot{q}_x = \partial_{p_x} H(q,p) = p_x + q_y \\ \dot{q}_y = \partial_{p_y} H(q,p) = p_y - q_x \\ \dot{p}_x = -\partial_{q_x} H(q,p) = p_y + \frac{(1-\mu)(\mu-q_x)}{\left((\mu-q_x)^2+q_y^2\right)^{\frac{3}{2}}} + \frac{\mu(\mu-q_x-1)}{\left((\mu-q_x-1)^2+q_y^2\right)^{\frac{3}{2}}} \\ \dot{p}_y = -\partial_{q_y} H(q,p) = -p_x - q_y \left[\frac{1-\mu}{\left((\mu-q_x)^2+q_y^2\right)^{\frac{3}{2}}} + \frac{\mu}{\left((\mu-q_x-1)^2+q_y^2\right)^{\frac{3}{2}}} \right]. \end{cases} \tag{2}$$

This is a widely studied system and some of its properties are well known. Here we will describe some of them. We leave most of the proofs out as they are more involved but we provide references so any interested reader can go and check the proofs by themselves.

One can note that (see [12]) the system has a symmetry in the equations

$$(q_x, q_y, p_x, p_y; t) \leftrightarrow (q_x, -q_y, -p_x, p_y; -t). \tag{3}$$

Also, the system has 5 equilibrium points L_1, L_2, L_3, L_4 and L_5, the celebrated Lagrange points. The points L_4 and L_5 are called the triangular Lagrange points because they form an equilateral triangle with the two primaries. And L_1, L_2 and L_3 are called the collinear Lagrange points because they are located in the same line as the primaries $\Sigma = \{q_y = p_x = q_x - p_y = 0\}$.

The collinear equilibrium points (L_1, L_2 and L_3) are of type center-saddle and for small values of μ (more precisely $\mu \leq \mu_R = \frac{1}{2}\left(1 - \frac{\sqrt{69}}{9}\right)$) the two triangular ones (L_4 and L_5) are of type center-center (see [13]). Since L_1, L_2 and L_3 are of type center-saddle, they each have associated a 1D stable and unstable manifolds.

The points L_1, L_2, L_4 and L_5 are widely studied mostly due to the astronomical interest. L_4 and L_5 are stable in the Lyapunov sense. For this reason, it is common to find objects near them. For instance Trojan and Greek asteroids in the Jupiter-Sun system. The points L_1 and L_2 (and their associated invariant manifolds) have also been studied (for instance the recently launched James-Webb telescope was launched to the point L_1 in the Earth-Sun system or more recently [1]). However, the point L_3 being "at the other side" of the big primary, has received somewhat less attention. This work will be concerned with the stable and unstable manifolds of the equilibrium point L_3.

The point L_3 is the solution of

$$q_x + \frac{(1-\mu)(\mu-q_x)}{|q_x-\mu|^2} + \frac{\mu(\mu-q_x-1)}{|q_x-\mu+1|^2} = 0 \tag{4}$$

with $q_x > 1$ and $q_y = 0$. As stated, this equilibrium point is of type center-saddle and it can be proven (see for instance [13]) that the eigenvalues associated with the linearized system

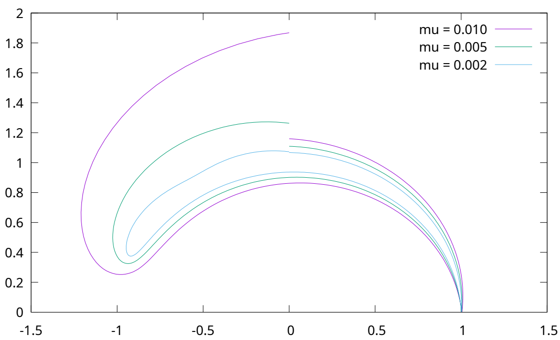

Figure 1: Branches $W^{s,+}$ and $W^{s,+}$ of the stable and unstable manifolds of L_3 for different values of μ continued to the Poincaré section Σ.

around L_3 are

$$\lambda_{1,2,3,4} = \{\pm\sqrt{\mu}\rho(\mu), \pm i\omega(\mu)\}, \quad \text{with} \quad \begin{cases} \rho(\mu) = \sqrt{\frac{21}{8}} + O(\mu) \\ \omega(\mu) = 1 + \frac{7}{8}\mu + O(\mu^2). \end{cases} \tag{5}$$

This implies the existence of a 1d stable manifold and a 1d unstable manifold. Due to the symmetry (3), these manifolds have two branches each: one that circumvents L_4 which we will denote as $W^{s,+}$, $W^{u,+}$ for the stable and unstable manifolds respectively and another one that circumvents L_5 which we will denote as $W^{s,-}$ and $W^{u,-}$. The positive stable branch $W^{s,+}$ is symmetric to $W^{u,-}$ and $W^{s,-}$ is symmetric to $W^{u,+}$. Therefore, one can restrict the study to just the positive branches of the manifolds and the results will follow for the symmetric counterparts.

Let us consider the transversal Poincaré section $\Sigma = \{q_x = 0, q_y > 1\}$ and denote P^u, P^s the first intersections of $W^{u,+}$, $W^{s,+}$ with Σ respectively. (Note that we can consider $\Sigma' = \{q_x = 0\}$ and P^u will be the second intersection between Σ' and $W^{u,+}$). One can visually check (by doing numerical integration) that P^u and P^s are closer and closer as we decrease the values of μ, see Figure 1. In fact, in [3] and [4] it is proven that

$$\text{dist}_\Sigma(P^u, P^s) = \sqrt[3]{4}\mu^{\frac{1}{3}} e^{\frac{-A}{\sqrt{\mu}}} \left[|\Theta| + O\left(\frac{1}{\log\mu}\right) \right], \tag{6}$$

where $\Theta \in \mathbb{C}$ is an unknown constant constant, usually called the Stokes constant, and A is given by

$$A = \int_0^{\frac{\sqrt{2}-1}{2}} \frac{2}{1-x}\sqrt{\frac{x}{3(x+1)(1-4x-4x^2)}}\,\mathrm{d}x.$$

In this work we will use two methodologies in order to obtain an approximation of Θ.

§2. First approach

The first step is to accurately compute the equilibrium point L_3. As we already stated, the y coordinate of L_3 is 0 and the x coordinate is the solution of (4), from which we can derive

$$\mathcal{P}(\xi) = \xi^5 + (\mu+2)\xi^4 + (2\mu+1)\xi^3 - (1-\mu)\xi^2 - 2(1-\mu)\xi - (1-\mu) = 0. \tag{7}$$

Which can be easily solved with high precision using any numerical software. In our case, we used Maple [11] with a tolerance of 10^{-250}.

To compute approximations to W^{u} and W^{s} we will use the celebrated parameterization method [6, 7, 8] that allows us to obtain expansions of the local invariant manifolds of the equilibrium point L_3. This method is much more generic and can be used to obtain approximations to any kind of invariant manifolds, not just the ones associated with equilibrium points (such as invariant tori, periodic orbits, etc.) (see [9]).

By applying this method, we can obtain $W^{\mathrm{u}}(s)$ and $W^{\mathrm{s}}(s)$ which (for small values of s) approximate the stable and unstable manifolds up to $O(|s|^p)$, where p can be increased by simply iterating the method.

Now that we have obtained good approximations of the stable and unstable manifolds near the equilibrium point L_3, we extend them until the Poincaré section $\Sigma = \{q_x = 0, q_y > 1\}$ in order to compute the distance between them.

To that end, we numerically integrate system (2) with initial conditions $W^{\mathrm{u,s}}(s)$, for a small value of the parameter s, until reaching the Poincaré section.

One important aspect of the numerical integration is not to lose significant digits of precision while doing the integration. If we recall, from formula (6), we want to compute the distance between the stable and unstable manifolds. Therefore, we need to subtract two quantities that are exponentially close together. In that process we will lose a significant amount of digits of precision and only a few will remain after the subtraction. To this end, all of the computations done with the Taylor integrator (see [10]) had a working tolerance of 10^{-250}.

The last step is computing the approximations of $|\Theta|$. First, we compute the distance between P^{u} and P^{s} from the values obtained by numerical integration. Recall that the asymptotic behaviour for this distance is given by

$$\mathrm{dist}_\Sigma(P^{\mathrm{u}}, P^{\mathrm{s}}) = \sqrt[3]{4}\mu^{\frac{1}{3}} e^{\frac{-A}{\sqrt{\mu}}} \left[|\Theta| + O\left(\frac{1}{\log\mu}\right)\right], \tag{8}$$

and therefore, we solve for Θ to compute the aforementioned approximations

$$\frac{\mathrm{dist}_\Sigma(P^{\mathrm{u}}), P^{\mathrm{s}})e^{\frac{A}{\sqrt{\mu}}}}{\sqrt[3]{4}\mu^{\frac{1}{3}}} \approx |\Theta|\,. \tag{9}$$

In the Figure 2, we can observe the plot of the computed value of $|\Theta|$ given by formula (9) for different values of μ.

We can observe the values keep oscillating back and forth around in a seemingly random way (note the scale of the plot). This is likely due to the two times scales present in the problem. The fact that we are not exactly in the invariant manifold but very close to it, makes us stay trapped in an invariant tube around the actual invariant manifold (this will be the

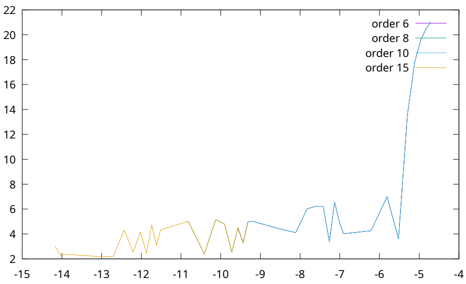

Figure 2: $\log \mu$ against the computed value of $|\Theta|$ for different orders of the parameterization method.

invariant manifold associated to a Lyapunov periodic orbit very close to L_3). This means that upon arriving at the Poincaré section the orbit we are computing has been doing circles around the invariant manifold at a very high speed. This creates some noise (that is practically random due to the difference in time scales) in the calculation of the distance. However, being the distance so small this noise gets greatly amplified giving rise to the errors seen in Figure 2.

Precisely this Figure motivates us to apply an alternative method for computing approximations on the value of the constant Θ.

§3. Inner equation

In our second method, we follow the approach described in [3, 4] to attack this problem and derive the inner equation.

The idea is, by means of some changes of variables and singular scalings, to decouple (at first order) the saddle and the center behaviour and write the Hamiltonian of the system as one close to integrable.

This close to integrable system is analyzed using singular perturbation theory to obtain the distance distance between manifolds. One of the key tools in this analysis is the so called inner equation (see [2, 5]) from which the constant Θ is defined.

In [3] the inner equation for this problem is derived:

$$H^{\text{in}} = \mathcal{H}(U, W, X, Y) + H_1^{\text{in}}(U, W, X, Y; \delta) \tag{10}$$

with $H_1(U, W, X, Y; 0) = 0$ and

$$\begin{aligned}
\mathcal{H}(U, W, X, Y) =& W + XY + \mathcal{K}(U, W, X, Y),\\
\mathcal{K}(U, W, X, Y) =& \frac{-3}{4} U^{\frac{2}{3}} W^2 - \frac{1}{3U^{\frac{2}{3}}}\left(\frac{1}{\sqrt{1 + \mathcal{J}(U, W, X, Y)}} - 1\right),\\
\mathcal{J}(U, W, X, Y) =& \frac{4W^2}{9U^{\frac{2}{3}}} - \frac{16W}{27U^{\frac{4}{3}}} + \frac{16}{81U^2} + \frac{4(X+Y)}{9U}\left(W - \frac{2}{3U^{\frac{2}{3}}}\right)\\
& - \frac{4i(X-Y)}{3U^{\frac{2}{3}}} - \frac{X^2 + Y^2}{3U^{\frac{4}{3}}} + \frac{10XY}{9U^{\frac{4}{2}}}.
\end{aligned}$$

The equations of motion of $\mathcal{H}$ are given by

$$\begin{cases} \dot{U} = 1 + \partial_W \mathcal{K} \\ \dot{W} = -\partial_U \mathcal{K} \\ \dot{X} = iX + i\partial_Y \mathcal{K} \\ \dot{Y} = -iY - i\partial_X \mathcal{K} \end{cases} \tag{11}$$

And we look for solutions as graphs of "time" u, i.e. solutions of the form

$$Z^{\mathrm{u,s}}(U) = (W^{\mathrm{u,s}}(U), X^{\mathrm{u,s}}(U), Y^{\mathrm{u,s}}(U)),$$

such that

$$\lim_{\Re U \to +\infty} Z^{\mathrm{s}}(U) = 0, \qquad \lim_{\Re U \to -\infty} Z^{\mathrm{u}}(U) = 0. \tag{12}$$

That is, we are interested in solutions that satisfy the aforementioned asymptotic conditions.

Combining the previous equation with the equations of motion, one can deduce the invariance equation

$$\partial_U Z(U) = \frac{\mathcal{A} Z(U) + f(U, Z(U))}{1 + \partial_W \mathcal{K}(U, Z(U))} \tag{13}$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}, \qquad f(U,Z) = \begin{pmatrix} -\partial_U \mathcal{K}(U,Z) \\ i\partial_Y \mathcal{K}(U,Z) \\ -i\partial_X \mathcal{K}(U,Z) \end{pmatrix}.$$

It can be shown that (13) has analytical solutions for $Z^{\mathrm{u,s}}(U)$ in appropriate complex domains. Moreover there exists a function $\chi = (\chi_1, \chi_2, \chi_3)$ satisfying (12) such that

$$\Delta Z(U) = Z^{\mathrm{u}}(U) - Z^{\mathrm{s}}(U) = \Theta e^{-iU} \left((0,0,1) + \chi(U) \right).$$

and

$$\left| U^{\frac{7}{3}} \chi_1(U) \right| \le b_2, \quad \left| U^2 \chi_2(U) \right| \le b_2, \quad |U \chi_3(U)| \le b_2,$$

for some constant $b_2 > 0$. This implies that

$$\Theta = \lim_{\Im U \to -\infty} \Delta Y(U) e^{iU}. \tag{14}$$

Our approach to approximate Θ then will be to compute

$$\Theta_\rho = |\Delta Y(-i\rho)| \, e^{\rho} \tag{15}$$

for ρ big enough then $\Theta_\rho \approx |\Theta|$.

The first thing we focus in is to obtain a good approximation of $Z^{\mathrm{u,s}}(U)$. To this end, we will look for expansions of $Z^{\mathrm{u,s}}(U)$ as power series in $U^{-\frac{1}{3}}$.

The first terms of the series are

$$\begin{aligned} W^{\mathrm{u,s}}(U) &= \frac{4}{243 U^{\frac{8}{3}}} - \frac{172}{2187 U^{\frac{14}{3}}} + \mathcal{O}\left(U^{-\frac{20}{3}}\right), \\ X^{\mathrm{u,s}}(U) &= -\frac{2i}{9U^{\frac{4}{3}}} + \frac{28}{81 U^{\frac{7}{3}}} - \frac{20i}{27 U^{\frac{10}{3}}} - \frac{16424}{6561 U^{\frac{13}{3}}} + \mathcal{O}\left(U^{-\frac{16}{3}}\right), \\ Y^{\mathrm{u,s}}(U) &= \frac{2i}{9U^{\frac{4}{3}}} + \frac{28}{81 U^{\frac{7}{3}}} + \frac{20i}{27 U^{\frac{10}{3}}} - \frac{16424}{6561 U^{\frac{13}{3}}} + \mathcal{O}\left(U^{-\frac{16}{3}}\right). \end{aligned} \tag{16}$$

κ	ρ	$\Delta Y(-i\rho)$	$\lvert\pi_U(\xi^{\mathrm{u}}_\Sigma) - \pi_U(\xi^{\mathrm{s}}_\Sigma)\rvert$	e^ρ	Θ_ρ
1000	15	$5.043092 \cdot 10^{-7}$	$1.7 \cdot 10^{-249}$	$3.2422 \cdot 10^{6}$	1.635062
1000	20	$3.383404 \cdot 10^{-9}$	$2.1 \cdot 10^{-250}$	$4.8218 \cdot 10^{8}$	1.631402
1000	23	$1.681640 \cdot 10^{-10}$	$5.2 \cdot 10^{-250}$	$9.6926 \cdot 10^{9}$	1.629951
1000	24	$6.183395 \cdot 10^{-11}$	$1.4 \cdot 10^{-249}$	$2.6353 \cdot 10^{10}$	1.629522
1000	25	$2.273650 \cdot 10^{-11}$	$1.9 \cdot 10^{-249}$	$7.1650 \cdot 10^{10}$	1.629074
1000	30	$1.511938 \cdot 10^{-13}$	$2.6 \cdot 10^{-249}$	$1.0642 \cdot 10^{13}$	1.609096
1000	35	$6.856063 \cdot 10^{-16}$	$2.3 \cdot 10^{-249}$	$1.5804 \cdot 10^{15}$	1.083555
10000	15	$5.043101 \cdot 10^{-7}$	$2.0 \cdot 10^{-249}$	$3.2422 \cdot 10^{6}$	1.635062
10000	20	$3.383414 \cdot 10^{-9}$	$2.0 \cdot 10^{-249}$	$4.8218 \cdot 10^{8}$	1.631403
10000	23	$1.681663 \cdot 10^{-10}$	$1.2 \cdot 10^{-249}$	$9.6926 \cdot 10^{9}$	1.629968
10000	24	$6.183593 \cdot 10^{-11}$	$3.6 \cdot 10^{-249}$	$2.6353 \cdot 10^{10}$	1.629569
10000	25	$2.273835 \cdot 10^{-11}$	$1.1 \cdot 10^{-249}$	$7.1650 \cdot 10^{10}$	1.629203

Table 1: Results of the experimentation for various values of κ and ρ and truncated series at order $U^{-\frac{16}{3}}$.

κ	ρ	$\Delta Y(-i\rho)$	$\lvert\pi_U(\xi^{\mathrm{u}}_\Sigma) - \pi_U(\xi^{\mathrm{s}}_\Sigma)\rvert$	e^ρ	Θ_ρ
1000	30	$1.511937 \cdot 10^{-13}$	$4.6 \cdot 10^{-250}$	$1.0643 \cdot 10^{13}$	1.609096
1000	35	$6.856094 \cdot 10^{-16}$	$2.1 \cdot 10^{-250}$	$1.5804 \cdot 10^{15}$	1.083560
10000	30	$1.529456 \cdot 10^{-13}$	$3.2 \cdot 10^{-249}$	$1.0643 \cdot 10^{13}$	1.627734
10000	35	$1.029279 \cdot 10^{-15}$	$2.1 \cdot 10^{-249}$	$1.5804 \cdot 10^{15}$	1.626700
10000	40	$6.938445 \cdot 10^{-18}$	$4.4 \cdot 10^{-249}$	$2.3466 \cdot 10^{17}$	1.628174
10000	45	$5.634902 \cdot 10^{-20}$	$2.4 \cdot 10^{-249}$	$3.4838 \cdot 10^{19}$	1.963118
50000	45	$4.665288 \cdot 10^{-20}$	$4.2 \cdot 10^{-249}$	$3.4838 \cdot 10^{19}$	1.625318

Table 2: Results of the experimentation for various values of κ and ρ and truncated series at order $U^{-\frac{25}{3}}$.

In order to compute $\Delta Y(-i\rho)$, we take as initial condition

$$\begin{aligned}
\xi^{\mathrm{s}}_0 &= \left(\kappa - i\rho, [W^{\mathrm{u}}(\kappa - i\rho)]_p, [X^{\mathrm{u}}(\kappa - i\rho)]_p, [Y^{\mathrm{u}}(\kappa - i\rho)]_p\right), \\
\xi^{\mathrm{u}}_0 &= \left(-\kappa - i\rho, [W^{\mathrm{s}}(-\kappa - i\rho)]_p, [X^{\mathrm{s}}(-\kappa - i\rho)]_p, [Y^{\mathrm{s}}(-\kappa - i\rho)]_p\right),
\end{aligned}$$

i.e. the power series truncated at order p at points $U = \kappa - i\rho$ for $Z^{\mathrm{s}}(U)$ and $U = -\kappa - i\rho$ for $Z^{\mathrm{u}}(U)$ and some big value of κ. We will take these points as initial conditions for the system of ODEs (11) and integrate until the section

$$\Sigma_{\mathrm{in}} = \{(U, W, X, Y) \in \mathbb{C}^4 \mid \Re U = 0\}.$$

The results obtained are summarised in tables 1 and 2.

The first thing that one notices is that all the values suggest that the first two digits of the Stokes constant are 1.6. So we can be happy that indeed Θ seems to be different from zero.

Looking more closely at the data, one can clearly see that for a fixed value of κ, we can only increment ρ up to a certain point, otherwise we start to get incorrect results that do not match the expectations. For instance, see the results obtained for $\kappa = 1000$ and $\rho = 35$. Even increasing the order in the series in the initial condition, the error is dominated by the small value of κ.

This behaviour occurs due to the nature of the series (16). That series is divergent, as we have already discussed, and the coefficients grow very fast. Indeed, the coefficient of $U^{\frac{-38}{3}}$ is already bigger than $9 \cdot 10^4$. This implies that getting more values of the series does not guarantee an improvement in the initial approximation. It is clear that to overcome this problem one has to take bigger values of κ as the error from the series will be dominant otherwise.

Although the results obtained seem to suggest that $|\Theta|$ is close (or relatively close) to 1.6, we think that there is not enough evidence to corroborate that. In our opinion, a more involved study has to be made in order to determine the value of the Stokes constant Θ. However, we think that with just a little bit[1] more computing power one can ensure that $\Theta \neq 0$ which is the important thing. The actual value of Θ is not that important apart from being different from zero (otherwise, (6) will prove useless). Nonetheless, a more rigorous study doing interval analysis should be done.

References

[1] Abad, A., Arribas, M., Palacios, M., and Elipe, A. Evolution of the characteristic curves in the restricted three-body problem in terms of the mass parameter. *Celestial Mechanics and Dynamical Astronomy 135*, 2 (Feb 2023), 7. Available from: `https://doi.org/10.1007/s10569-022-10118-z`.

[2] Baldomá, I. The inner equation for one and a half degrees of freedom rapidly forced hamiltonian systems. *Nonlinearity 19* (06 2006). `doi:10.1088/0951-7715/19/6/011`.

[3] Baldomá, I., Giralt, M., and Guardia, M. Breakdown of homoclinic orbits to l3 in the rpc3bp (i). complex singularities and the inner equation. `doi:10.48550/ARXIV.2107.09942`.

[4] Baldomá, I., Giralt, M., and Guardia, M. Breakdown of homoclinic orbits to l3 in the rpc3bp (ii). an asymptotic formula. `doi:10.48550/ARXIV.2107.09941`.

[5] Baldomá, I., and Seara, T. M. The inner equation for generic analytic unfoldings of the hopf-zero singularity. *Discrete and Continuous Dynamical Systems - B 10*, 2&3, September (2008), 323–347.

[6] Cabré, X., Fontich, E., and de la Llave, R. The parameterization method for invariant manifolds. I. Manifolds associated to non-resonant subspaces. *Indiana Univ. Math. J. 52*, 2 (2003), 283–328. Available from: `https://doi.org/10.1512/iumj.2003.52.2245`.

[7] Cabré, X., Fontich, E., and de la Llave, R. The parameterization method for invariant manifolds. II. Regularity with respect to parameters. *Indiana Univ. Math. J. 52*,

[1] This is a joke.

2 (2003), 329–360. Available from: https://doi.org/10.1512/iumj.2003.52.2407.

[8] Cabré, X., Fontich, E., and de la Llave, R. The parameterization method for invariant manifolds. III. Overview and applications. *J. Differential Equations 218*, 2 (2005), 444–515. Available from: https://doi.org/10.1016/j.jde.2004.12.003.

[9] Haro, Àlex, and Canadell, Marta and Figueras, Jordi-Lluis and Luque, Alejandro and Mondelo, Josep Maria (auth.). *The Parameterization Method for Invariant Manifolds: From Rigorous Results to Effective Computations*, 1 ed. Applied Mathematical Sciences 195. Springer International Publishing, 2016. Available from: http://gen.lib.rus.ec/book/index.php?md5=13d3835c1d7e35831356fdd2cf146270.

[10] Jorba, A., and Zou, M. A software package for the numerical integration of odes by means of high-order taylor methods. *Experimental Mathematics 14* (01 2005), 99–117. doi:10.1080/10586458.2005.10128904.

[11] Maplesoft, a division of Waterloo Maple Inc.. Maple [online]. Available from: https://www.maplesoft.com/.

[12] Meyer, K., and Offin, D. *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem*. 01 2017. doi:10.1007/978-3-319-53691-0.

[13] Szebehely, V. Theory of orbits: The restricted problem of three bodies. *American Journal of Physics 36*, 4 (1968), 375–375. doi:10.1119/1.1974535.

Inmaculada Baldomá and Mercè Ollé
Departament de Matemàtiques
Universitat Politècnica de Catalunya
immaculada.baldoma@upc.edu

Ernesto Lanchares
Departamento de Matemáticas y Computación
Universidad de La Rioja
ernesto.lanchares@unirioja.es