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MONOTONE OPERATORS IN MATHEMATICAL FINANCE: NONLINEAR BLACK-SCHOLES EQUATION

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Abstract. We treat nonlinear parabolic Cauchy problems for valuation of options in financial markets, especially problems of Black-Scholes-type with nonlinear diffusion. Typically, methods based on viscosity solutions are used for determining the solvability of such fully nonlinear problems. However, the special form of these problems in Financial Mathematics enables us to transform them into abstract initial value problems with monotone second-order differential operators to which classical results for abstract parabolic Cauchy problems can be applied. The transformation from the unknown option price $P(S,t)$ to its partial derivative $\Delta(S,t) = \frac{\partial P}{\partial S}$, called the *Greek* $\mathbf{\Delta}$, is very simple. The standard theory of monotone operators in Hilbert spaces (of type L^2 with a weight) is applicable to the nonlinear Cauchy problem for the new unknown function $\Delta(S,t)$ of the stock price S at time t.

Keywords: Nonlinear parabolic equation; Weighted Sobolev space; Equation for the Greek Delta; Monotone operators; Abstract nonlinear Cauchy problem; Applications to nonlinear Black-Scholes equations.

AMS classification: 35K55, 35K15, (47H05, 91G80).

§1. Introduction

In this short article we treat a simple application of the well-known classical theory of ***nonlinear monotone operators*** in Hilbert and (reflexive) Banach spaces to nonlinear Black--Scholes-type problems that are abundant in *Mathematical Finance*, such as classical nonlinear Black-Scholes models for option valuation with *transaction costs*. We would like to explain the main idea behind the transformation of (typically) a fully nonlinear parabolic evolutionary problem that is treated mostly by relatively newer methods based on ***viscosity solutions*** in a Banach space of continuous functions into a divergence-type quasi-linear parabolic problem whose ***weak solutions*** are obtained by a standard application of ***nonlinear monotone operators***. To our best knowlwdge, in *Mathematical Finance* this idea was used for the first time in the work of A. Bensoussan, B.-G. Jang, and S. Park [8] and subsequently developed further in V. Barbu [2] and V. Barbu, C. Benazzoli, and L. Di Persio [3] and V. Barbu [4]. We base our method on a single nonlinear Black-Scholes equation that was derived a quarter of a century ago by G. Barles and H. M. Soner [6] for valuation of options with *transaction costs*. The goal of their work was to obtain a precise formula for the ***implied volatility*** $\widehat{\sigma}(S,t) = \widehat{\sigma}\left(S^2 \frac{\partial^2 P}{\partial S^2}, t\right)$ in [6, Eq. (1.2) on p. 372]. This expression for implied volatility replaces the classical constant volatility $\sigma = \text{const} > 0$ that has been used in classical linear Black-Scholes models with the linear Black-Scholes equation for the option price $P = P(S,t)$:

$$\frac{\partial P}{\partial t}(S,t) + \frac{1}{2}\,\sigma^2\, S^2\, \frac{\partial^2 P}{\partial S^2}(S,t) + (r-q)\, S\, \frac{\partial P}{\partial S}(S,t) = r\, P(S,t) \qquad \text{for } 0 < S < \infty \text{ and } -\infty < t < T, \tag{1}$$

with the independent variables S and t, the *stock price* and *time*, respectively, and with the following additional quantities (constants) as given data: the *maturity time* of the option T $(0 < T < \infty)$; the *risk free rate of interest* $r \in \mathbb{R}$; the *instantaneous drift of the stock price returns* $r - q \equiv -q_r \in \mathbb{R}$. Consequently, the classical linear Black-Scholes equation, eq. (1) above, is **transformed** into the following nonlinear parabolic equation (i.e., Eq. (1.2) on p. 372 in [6]):

$$\frac{\partial P}{\partial t}(S,t) + \frac{1}{2}\,\widehat{\sigma}^2\left(S^2\, \frac{\partial^2 P}{\partial S^2}\,, t\right) S^2\, \frac{\partial^2 P}{\partial S^2}(S,t) + (r-q)\, S\, \frac{\partial P}{\partial S}(S,t) = r\, P(S,t) \tag{2}$$

for $S > 0$ and $t < T$, with the following formula for the ***implied volatility*** $\widehat{\sigma}(S,t)$,

$$\widehat{\sigma}(S,t) = \widehat{\sigma}\left(S^2\, \frac{\partial^2 P}{\partial S^2}\,, t\right) = \widehat{\sigma}(0,T)\left[1 + \varsigma\left(\mathrm{e}^{r(T-t)}\, a^2\, S^2\, \frac{\partial^2 P}{\partial S^2}\,(S,t)\right)\right]^{1/2} \tag{3}$$

for $S > 0$ and $t < T$. Here, $\widehat{\sigma}(0,T) > 0$ is a constant, $\varsigma : (-\infty,+\infty) \to \mathbb{R}_+$ is a *nonlinear volatility correction*, a continuous function that is continuously differentiable on $\mathbb{R} \setminus \{0\} = (-\infty,0) \cup (0,+\infty)$ and satisfies the differential equation in G. Barles and H. M. Soner [6, Eq. (3.2), p. 377] subject to the initial condition $\varsigma(0) = 0$. An important property of the function ς is that the function $A \mapsto A\,(1 + \varsigma(A)) : \mathbb{R} \to \mathbb{R}$ is *monotone nondecreasing* which guarantees the *parabolicity hypothesis* $\mathbf{H_{par}}$ formulated below in connection with eq. (2) above. Finally, $a > 0$ is an *"economicaly" relevant parameter* related to the risk aversion factor and the proportional transaction cost (see [6, p. 372]).

Thus, while keeping the nonlinear Barles-Soner equation (2) in mind, with the implied volatility from eq. (3), we will focus on the *nonlinear Black-Scholes equation* of the following more general type:

$$\frac{\partial P}{\partial t}(S,t) + \Sigma\left(S,\; S\, \frac{\partial P}{\partial S}\,,\; S^2\, \frac{\partial^2 P}{\partial S^2}\,,\; t\right) + (r-q)\, S\, \frac{\partial P}{\partial S}(S,t) = r\, P(S,t) \tag{4}$$

for $S > 0$ and $t < T$, with the ***implied volatility*** being included in the function $\Sigma : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times (-\infty,T] \to \mathbb{R}$ which is assumed to satisfy the following basic *parabolicity hypothesis*,

Hypothesis $\mathbf{H_{par}}$. Given any fixed triple $(S, A_1, t) \in \mathbb{R}_+ \times \mathbb{R} \times (-\infty,T]$, the function $A_2 \mapsto \Sigma(S, A_1, A_2, t) : \mathbb{R} \to \mathbb{R}$ is ***monotone increasing***. In other words, the function

$$\begin{gathered} A_2 \longmapsto \Sigma\left(S,\; S\, \frac{\partial P}{\partial S}\,,\; A_2\,,\; t\right) : \mathbb{R} \to \mathbb{R} \quad \text{is monotone increasing in the variable} \\ A_2 = S^2\, \frac{\partial^2 P}{\partial S^2} \in \mathbb{R}\,,\; A_2 = S\, \frac{\partial}{\partial S}\left(S\, \frac{\partial P}{\partial S}\right) - S\, \frac{\partial P}{\partial S} \equiv \left[\left(S\, \frac{\partial}{\partial S}\right)^2 - \left(S\, \frac{\partial}{\partial S}\right)\right] P(S,t)\,. \end{gathered} \tag{5}$$

In our approach to the *nonlinear Black-Scholes equation* (4), we will mostly assume that the restricted function $A_2 \mapsto \Sigma(S, A_1, A_2, t) : \mathbb{R} \to \mathbb{R}$ is continuously differentiable with respect to the variable $A_2 \in \mathbb{R}$. Consequently, our Hypothesis $\mathbf{H_{par}}$ is equivalent with $\frac{\partial \Sigma}{\partial A_2}(S, A_1, A_2, t) \geq 0$ for every quadruple $(S, A_1, A_2, t) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times (-\infty, T]$.

Under the monotonicity hypothesis $\mathbf{H_{par}}$ above, the nonlinear Black-Scholes equation (4) is typically treated by well-known methods using ***viscosity solutions***; see, e.g., the monograph by G. Barles [5] or the article by G. Barles and H. M. Soner [6, Appendix B, pp. 388–398].

In our present work we take advantage of the classical methods using ***nonlinear monotone operators*** in Hilbert and (reflexive) Banach spaces in order to produce ***weak solutions*** to our nonlinear Black-Scholes models of type (4). We will present and explain the main ideas of our approach in the next section.

§2. Preliminary calculations

We denote $x = \log S$ for the stock price $S > 0$ and calculate $S = \mathrm{e}^x$ for the logarithmic stock price $x \in \mathbb{R} = (-\infty, +\infty)$ which yields further for the new function $p(x,t) = P(S,t)$:

$$\frac{\partial p}{\partial x} = S\, \frac{\partial P}{\partial S}\,, \quad \frac{\partial^2 p}{\partial x^2} = S\, \frac{\partial P}{\partial S} + S^2\, \frac{\partial^2 P}{\partial S^2}\,, \quad \text{and} \quad \frac{\partial^2 p}{\partial x^2} - \frac{\partial p}{\partial x} = S^2\, \frac{\partial^2 P}{\partial S^2}\,. \tag{6}$$

Let us consider the following new function which we call the ***flux function***,

$$\mathscr{F} \equiv \mathscr{F}\left(x,\, \frac{\partial p}{\partial x}\,,\, \frac{\partial^2 p}{\partial x^2}\,,\, t\right) = \Sigma\left(S,\, S\, \frac{\partial P}{\partial S}\,,\, S^2\, \frac{\partial^2 P}{\partial S^2}\,,\, t\right) \equiv \Sigma\left(\mathrm{e}^x\,,\, \frac{\partial p}{\partial x}\,,\, \frac{\partial^2 p}{\partial x^2} - \frac{\partial p}{\partial x}\,,\, t\right). \tag{7}$$

It takes into account only the ***sensitivity*** of the option price p depending on the change of the stock price S at time t $(-\infty < t < T < +\infty)$, expressed through the ***Greek "Delta"*** $\boldsymbol{\Delta}$, $\Delta \stackrel{\text{def}}{=} \frac{\partial P}{\partial S}$, at time $t \in (-\infty, T)$. For the meaning of $\boldsymbol{\Delta}$ in a ***hedging strategy*** in Mathematical Finance, the reader is referred to J.-P. Fouque, G. Papanicolaou, and K. R. Sircar [9, §5.3, pp. 99–102] or to J. C. Hull [10, §19.4, pp. 401–407].

In accordance with the sensitivity Δ we introduce the new function $\Delta_x \stackrel{\text{def}}{=} \frac{\partial p}{\partial x} = S\, \frac{\partial P}{\partial S}$ $= S\,\Delta$ at time $t \in (-\infty, T)$; we call it the ***"relative sensitivity"***. We propose to replace the unknown option price P, i.e., the function $P(S,t) = p(x,t)$, governed by the nonlinear parabolic equation (4), by the *relative sensitivity*

$$\Delta_x = \frac{\partial p}{\partial x}(x,t) = S\, \frac{\partial P}{\partial S}(S,t) = S\,\Delta \tag{8}$$

at time $t \in (-\infty, T)$. The corresponding parabolic equation for the unknown function $\Delta_x(x,t)$ is derived by applying the partial derivative $\frac{\partial}{\partial x} = S\, \frac{\partial}{\partial S}$ to equation (4), thus obtaining

$$\frac{\partial}{\partial t}\, \Delta_x(x,t) + \frac{\mathrm{d}}{\mathrm{d}x}\, \mathscr{F}\left(x,\, \Delta_x\,,\, \frac{\partial}{\partial x}\, \Delta_x\,,\, t\right) + (r-q)\, \frac{\partial}{\partial x}\, \Delta_x(x,t) = r\, \Delta_x(x,t) \tag{9}$$

for $x \in \mathbb{R}$ and $t < T$, with the *implied volatility* being included in the functions

$\mathscr{F}\,:\ \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times(-\infty,T]\to\mathbb{R}$ and Σ related by eq. (7) above. We remark that

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\mathscr{F}\left(x,\ \Delta_x\,,\ \frac{\partial}{\partial x}\,\Delta_x\,,\ t\right)=\frac{\partial\mathscr{F}}{\partial x}\left(x,\ \Delta_x\,,\ \frac{\partial}{\partial x}\,\Delta_x\,,\ t\right)$$
$$+\frac{\partial\mathscr{F}}{\partial\Delta_x}\left(x,\ \Delta_x\,,\ \frac{\partial}{\partial x}\,\Delta_x\,,\ t\right)\cdot\frac{\partial}{\partial x}\,\Delta_x+\frac{\partial\mathscr{F}}{\partial\Gamma_x}\left(x,\ \Delta_x\,,\ \frac{\partial}{\partial x}\,\Delta_x\,,\ t\right)\cdot\frac{\partial}{\partial x}\,\Gamma_x$$

with the substitution $\Gamma_x\stackrel{\mathrm{def}}{=}\frac{\partial}{\partial x}\,\Delta_x$ related to the Greek "Gamma" Γ, $\Gamma\stackrel{\mathrm{def}}{=}\frac{\partial^2 P}{\partial S^2}=\frac{\partial\Delta}{\partial S}$ at time $t\in(-\infty,T)$,

$$\Gamma_x=S\,\frac{\partial}{\partial S}\left(S\,\frac{\partial P}{\partial S}\right)=S\,\frac{\partial P}{\partial S}+S^2\,\frac{\partial^2 P}{\partial S^2}=S\,\Delta+S^2\,\Gamma\,,$$

so that

$$\mathscr{F}\,(x,\ \Delta_x\,,\Gamma_x\,,t)\equiv\mathscr{F}\left(x,\ \Delta_x\,,\ \frac{\partial}{\partial x}\,\Delta_x\,,\ t\right)\quad\text{ with }\quad\Gamma_x=\frac{\partial}{\partial x}\,\Delta_x\,.$$

Next, we have to determine a suitable function space H for the function $\Delta_x(\,\cdot\,,t)$ at every time $t\in(0,\infty)$. To this end, we begin with the asymptotic behavior of the function $x\mapsto\Delta_x(x,t)$ as $x\to\pm\infty$. We recall that $\Delta_x(x,t)=S\,\Delta(S,t)=S\,\frac{\partial P}{\partial S}$ with the terminal conditions $P(S,T)=(S-K)^+$ and $P(S,T)=(K-S)^+$ for the European call and put options, respectively, at the expiration time $t=T$. For these two options we have $\Delta(S,T)=0$ for $0\leq S<K$ and $\Delta(S,T)=1$ for $K<S<+\infty$. These conditions are equivalent to $\Delta_x(x,T)=0$ for $-\infty<x<\log K$ and $\Delta_x(x,T)=\mathrm{e}^x$ for $\log K<x<+\infty$, respectively. Consequently, after the substitution

$$u(x,t)\stackrel{\mathrm{def}}{=}\Delta_x(x,t)-\frac{1}{2}\,\mathrm{e}^x\,(1+\tanh x)=\Delta_x(x,t)-\mathrm{e}^x\,\varphi(x)\tag{10}$$
$$\text{for all }\ (x,t)\in\mathbb{R}\times(-\infty,T)\,,\qquad\text{where}\qquad\varphi(x)\stackrel{\mathrm{def}}{=}\frac{\mathrm{e}^x}{\mathrm{e}^x+\mathrm{e}^{-x}}\,,$$

we obtain

$$\frac{u(x,t)}{\mathrm{e}^x}=\frac{\Delta_x(x,t)}{\mathrm{e}^x}-\varphi(x)\ \longrightarrow\ 0\quad\text{ as }\ x\to\pm\infty\,.$$

Thus, we obtain the following asymptotic behavior for the function $u(x,t)$ defined in eq. (10):

$$\frac{u(x,t)}{\mathrm{e}^x}\ \longrightarrow\ 0\quad\text{ as }\ x\to\pm\infty\,.\tag{11}$$

Finally, in order to obtain a parabolic equation for the unknown function $u(x,t)$, we insert the function $\Delta_x(x,t)=\mathrm{e}^x\,\varphi(x)+u(x,t)$ into the semilinear Black-Scholes-type problem in eq. (9), thus arriving at

$$\begin{aligned}&\frac{\partial u}{\partial t}(x,t)+\frac{\mathrm{d}}{\mathrm{d}x}\,\tilde{F}\left(x,\ u(x,t)\,,\ \frac{\partial u}{\partial x}(x,t)\,,\ t\right)+(r-q)\,\frac{\partial u}{\partial x}(x,t)-r\,u(x,t)\\&=r\,\mathrm{e}^x\,\varphi(x)-(r-q)\,\frac{\mathrm{d}}{\mathrm{d}x}\,(\mathrm{e}^x\,\varphi(x))\qquad\text{for }x\in\mathbb{R}\text{ and }t<T,\end{aligned}\tag{12}$$

where we have substituted the following function for the flux,

$$\tilde{F}\left(x,\ u(x,t)\,,\ \frac{\partial u}{\partial x}(x,t)\,,\ t\right)\stackrel{\mathrm{def}}{=}\mathscr{F}\left(x,\ u(x,t)+\mathrm{e}^x\,\varphi(x)\,,\ \frac{\partial u}{\partial x}+\frac{\mathrm{d}}{\mathrm{d}x}\,(\mathrm{e}^x\,\varphi(x))\ ,\ t\right).\tag{13}$$

Let us denote by $\mathfrak{w} : \mathbb{R} \to (0, \infty)$ the *weight function* $\mathfrak{w}(x) \stackrel{\text{def}}{=} \mathrm{e}^{-\mu|x|}$ for $x \in \mathbb{R}$, where $\mu \in (0, \infty)$ is a suitable positive constant that will be specified later. We choose the following space setting for the parabolic equation (12), namely, the weighted L^2-type Lebesgue space $H = L^2(\mathbb{R}; \mathfrak{w})$ which is a Hilbert space endowed with the inner product

$$(f, g)_H \equiv (f, g)_{L^2(\mathbb{R};\mathfrak{w})} \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} f(x)\, g(x) \cdot \mathfrak{w}(x)\, \mathrm{d}x \quad \text{ for } f, g \in H\,.$$

This inner product induces the norm in H, $\|f\|_H \stackrel{\text{def}}{=} (f, f)_H^{1/2} = \left(\int_{-\infty}^{+\infty} |f(x)|^2\, \mathfrak{w}(x)\, \mathrm{d}x\right)^{1/2} < \infty\,.$ In order to guarantee that the terminal value $u_0 : \mathbb{R} \to \mathbb{R}$,

$$u_0(x) = \Delta_x(x, T) - \mathrm{e}^x\, \varphi(x) = \mathrm{e}^x\, \Delta\, (\mathrm{e}^x, T) - \mathrm{e}^x\, \varphi(x) = \mathrm{e}^x\, (\Delta\, (\mathrm{e}^x, T) - \varphi(x))$$

for the European call option belongs to H, from now on we assume that $\mu > 2$.

§3. Main results

In order to rewrite the "backward" Black-Scholes problem (12) as a standard ("forward") evolutionary equation with a prescribed initial value $u_0 \in H$, we relabel the actual time t, $-\infty < t \le T$, by τ and use the letter t to denote the time to maturity, that is, $t = T - \tau \ge 0$. In addition, since we will be concerned only with solutions on the bounded time interval $[0, T]$, from now on we will view the letter T (the *maturity time* of the option) as the terminal time, $0 < T < \infty$, while keeping the initial time at zero. This forces us to replace the unknown function $u(x, \tau)$ by $u(x, t)$ and the operator $\frac{\partial}{\partial \tau}$ by $-\frac{\partial}{\partial t}$. Accordingly, for the flux function $\tilde{F}$ in eq. (13) we substitute

$$\begin{aligned} F\left(x,\, u(x,t)\,,\, \frac{\partial u}{\partial x}(x,t)\,,\, t\right) &\stackrel{\text{def}}{=} \tilde{F}\left(x,\, u(x,t)\,,\, \frac{\partial u}{\partial x}(x,t)\,,\, T-t\right) \\ &= \mathscr{F}\left(x,\, u(x,t) + \mathrm{e}^x\, \varphi(x)\,,\, \frac{\partial u}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}x}\, (\mathrm{e}^x\, \varphi(x))\,\,,\, T-t\right) \end{aligned} \tag{14}$$

whenever $t \in [0, T]$. Consequently, the "backward" Black-Scholes terminal value problem (9) (and (4), as well) becomes the following initial value problem for the unknown function $u : \mathbb{R} \times [0, T] \to \mathbb{R}$, $u = u(x, t)$,

$$\begin{aligned} &\frac{\partial u}{\partial t}(x,t) - \frac{\mathrm{d}}{\mathrm{d}x}\, F\left(x,\, u(x,t)\,,\, \frac{\partial u}{\partial x}(x,t)\,,\, t\right) - (r-q)\, \frac{\partial u}{\partial x}(x,t) + r\, u(x,t) \\ &= \; - r\, \mathrm{e}^x\, \varphi(x) + (r-q)\, \frac{\mathrm{d}}{\mathrm{d}x}\, (\mathrm{e}^x\, \varphi(x)) \qquad \text{for } x \in \mathbb{R} \text{ and } 0 \le t \le T; \end{aligned} \tag{15}$$

$$u(x, 0) = u_0(x) \quad \text{ for } x \in \mathbb{R}\,. \tag{16}$$

This is the kind of problems treated in the monographs by V. Barbu [1] and J.-L. Lions [11].

Our hypotheses on the rather general flux function $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, T] \to \mathbb{R}$ below allow us to take advantage of now classical results in [1, Chapt. III, §§4.2, p. 167] and in [11, Chapt. 2, §1.4, pp. 162–163], Théorème 1.2 and Théorème 1.2 *bis*. It is easy to see from eq. (14) how to reformulate these hypotheses for the function $\mathscr{F} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, T] \to \mathbb{R}$ introduced in eq. (7) in terms of the original flux function Σ.

We impose the following hypotheses on F:

Hypothesis $\mathbf{H}_{\mathbf{cont}}$. For every triple $(A_1, A_2, t) \in \mathbb{R} \times \mathbb{R} \times [0, T]$, the function $F(\,\cdot\,, A_1, A_2, t) : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable. Furthermore, for almost every fixed $x \in \mathbb{R}$, the function $F(x, \cdot, \cdot, \cdot) : \mathbb{R} \times \mathbb{R} \times [0, T] \to \mathbb{R}$ is continuous and it satisfies the *linear growth condition*

$$|F(x, A_1, A_2, t)| \leq C_1 (|A_1| + |A_2|) + C_0 \quad \text{for all } (A_1, A_2, t) \in \mathbb{R} \times \mathbb{R} \times [0, T] \tag{17}$$

with some constants $C_0, C_1 \in (0, \infty)$ which are independent from the variables $(x, A_1, A_2, t) \in \mathbb{R}^3 \times [0, T]$.

Hypothesis $\mathbf{H}_{\mathbf{mono}}$. For almost every fixed $x \in \mathbb{R}$, the function $F(x, \cdot, \cdot, \cdot) : \mathbb{R} \times \mathbb{R} \times [0, T] \to \mathbb{R}$ is continuously (partially) differentiable with respect to the variables A_1 and A_2 with the partial derivatives $\frac{\partial F}{\partial A_1}$ and $\frac{\partial F}{\partial A_2}$, respectively, satisfying

$$\left| \frac{\partial F}{\partial A_1} (x, A_1, A_2, t) \right| \leq c_2 < \infty \quad \text{and} \tag{18}$$

$$0 < c_1 \leq \frac{\partial F}{\partial A_2} (x, A_1, A_2, t) \leq c_2 < \infty \quad \text{for all } (A_1, A_2, t) \in \mathbb{R} \times \mathbb{R} \times [0, T] \tag{19}$$

with some positive constants $c_1, c_2 \in \mathbb{R}$, $0 < c_1 \leq c_2 < \infty$, which are independent from the variables $(x, A_1, A_2, t) \in \mathbb{R}^3 \times [0, T]$; cf. Hypothesis $\mathbf{H}_{\mathbf{par}}$.

Next, we define the nonlinear analogue of the Black-Scholes operator $\mathcal{A}(t) : V \to V'$ (cf. eq. (1)), where V stands for the Sobolev space of all absolutely continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f, f' \in H = L^2(\mathbb{R}; \mathfrak{w})$ endowed with the Sobolev norm $\|f\|_V \stackrel{\text{def}}{=} [(f, f)_H + (f', f')_H]^{1/2} < \infty$.
Naturally, V' denotes the dual space of V with respect to the duality induced by the scalar product $(\,\cdot\,,\,\cdot\,)_H$ on H. Thus, $V \hookrightarrow H = H' \hookrightarrow V'$ is a Gel'fand triple which consists of three Hilbert spaces; see J.-L. Lions [11, Remarque 1.2, Chapt. 2, §1.1, p. 156]. Given any fixed time $t \in [0, T]$, for each $v \in V$ we define $\mathcal{A}(t)v \in V'$ by

$$\begin{aligned} &(\mathcal{A}(t)v, w)_H = \\ &\int_{-\infty}^{+\infty} [F(x,\, v(x),\, v'(x),\, t)\, w'(x) + (r - q)\, v'(x)\, w(x) + r\, v(x)\, w(x)] \cdot \mathfrak{w}(x)\, \mathrm{d}x \\ &- \mu \int_{-\infty}^{+\infty} F(x,\, v(x),\, v'(x),\, t)\, w(x) \operatorname{sgn}(x) \cdot \mathfrak{w}(x)\, \mathrm{d}x \quad \text{for all } w \in V\,. \end{aligned} \tag{20}$$

Of course, we use the symbol $(\,\cdot\,,\,\cdot\,)_H$ also for the unique extension of the inner product on $H \times H$ to the duality on the Cartesian products $V \times V'$ and $V' \times V$. It follows directly from Hypothesis $\mathbf{H}_{\mathbf{cont}}$, ineq. (17), that $\mathcal{A}(t)$ maps V into its dual space V'.

Lemma 1 (The Operator $\mathcal{A}(t)$.). *The mapping $\mathcal{A}(t) : V \to V'$ is* **demicontinuous***, i.e., continuous as a mapping from the strong topology on V to the weak topology on V'. Moreover, there are some constants $\gamma, \gamma_1 \in (0, \infty)$ and $\gamma_0 \in \mathbb{R}$, all independent from time $t \in [0, T]$, such that the mapping $\mathcal{A}(t) + \gamma\mathcal{I} : V \to V'$ is* **monotone** *and* **coercive** *on V, respectively, i.e., we have*

$$(\mathcal{A}(t)v_1 - \mathcal{A}(t)v_2\,,\, v_1 - v_2)_H + \gamma\, \|v_1 - v_2\|_H^2 \geq 0 \quad \textit{for all } v_1, v_2 \in V\,, \tag{21}$$

together with

$$(\mathcal{A}(t)v\,,\,v)_H + \gamma\,\|v\|_H^2 \ge \gamma_1\,\|v\|_V^2 + \gamma_0 \quad \textit{for all } v \in V\,. \tag{22}$$

As usual, $\mathcal{I}$ denotes the identity mapping on V.

Proof of **Lemma 1.** Hypothesis $\mathbf{H_{cont}}$ implies that $\mathcal{A}(t) : V \to V'$ is demicontinuous, by inequality (17) combined with a standard application of Hölder's inequality and Vitali's theorem.

The first inequality, (21), is a direct consequence of the Taylor integral formula

$$\begin{aligned}
&F(x,\, v_1(x),\, v_1'(x),\, t) - F(x,\, v_2(x),\, v_2'(x),\, t) \\
&= \left[\int_0^1 \frac{\partial F}{\partial A_1}\Big(x,\, (1-\theta)\, v_1(x) + \theta\, v_2(x)\,,\, (1-\theta)\, v_1'(x) + \theta\, v_2'(x)\,,\, t\Big)\,\mathrm{d}\theta\right] [v_1(x) - v_2(x)] \\
&+ \left[\int_0^1 \frac{\partial F}{\partial A_2}\Big(x,\, (1-\theta)\, v_1(x) + \theta\, v_2(x)\,,\, (1-\theta)\, v_1'(x) + \theta\, v_2'(x)\,,\, t\Big)\,\mathrm{d}\theta\right] [v_1'(x) - v_2'(x)]\,,
\end{aligned}$$

whenever $v_1, v_2 \in V$, combined with inequalities (18) and (19). An analogous formula with $v_1 = v \in V$ and $v_2 = 0 \in V$, combined with (18) and (19) again and supplemented by ineq. (17) for the function $|F(x, 0, 0, t)| \le C_0$ with $(x, t) \in \mathbb{R} \times [0, T]$, yields the second inequality, (22). We refer the interested reader to the survey article by László Simon [12] for details in calculations leading to the desired inequalities (21) and (22). □

Finally, using these results on the Gel'fand triple $V \hookrightarrow H = H' \hookrightarrow V'$ and the nonlinear mapping $\mathcal{A}(t) : V \to V'$, we rewrite the initial value problem (15), (16) as the corresponding abstract problem

$$\frac{\partial u}{\partial t} - \mathcal{A}(t)u = f(t) \quad \text{for } t \in (0, T)\,; \qquad u(0) = u_0 \in H\,. \tag{23}$$

The function $f : (0, T) \to V'$ is, in fact, equal to the constant (time-independent) function $f(t) \equiv f_0 \in H$, $t \in (0, T)$, given by

$$f_0(x) = \; - r\,\mathrm{e}^x\,\varphi(x) + (r - q)\,\frac{\mathrm{d}}{\mathrm{d}x}\,(\mathrm{e}^x\,\varphi(x)) \quad \text{ for } x \in \mathbb{R}\,. \tag{24}$$

Now we are able to apply the general theorem from J.-L. Lions [11, Chapt. 2, §1.4], Théorème 1.2 on pp. 162–163, to obtain our main result:

Theorem 2 (Existence and uniqueness.)**.** *Let* $T \in (0, \infty)$ *and assume that* Hypotheses $\mathbf{H_{cont}}$ *and* $\mathbf{H_{mono}}$ *are satisfied. Given any initial value* $u_0 \in H$*, there exists a unique* **weak solution** $u : [0, T] \to H$ *to our* **initial value problem** (23) *that has the following properties:*

- **(i)** $u : [0, T] \to H : t \mapsto u(\,\cdot\,, t)$ *is continuous, i.e.,* $u \in C([0, T] \to H)$*, with* $u(0) = u_0$*.*
- **(ii)** $u : (0, T) \to V : t \mapsto u(\,\cdot\,, t)$ *is (strongly) Lebesgue-measurable with the finite integral* $\int_0^T \|u(\,\cdot\,, t)\|_V^2\,\mathrm{d}t < \infty$ *, i.e.,* $u \in L^2((0, T) \to V)$*.*
- **(iii)** *The (weak distributional) derivative* $\frac{\partial u}{\partial t} : (0, T) \to V'$ *is (strongly) Lebesgue--measurable with the finite integral* $\int_0^T \left\|\frac{\partial u}{\partial t}(\,\cdot\,, t)\right\|_{V'}^2\,\mathrm{d}t < \infty$ *, i.e.,* $\frac{\partial u}{\partial t} \in L^2((0, T) \to V')$ *or, equivalently,* $u \in W^{1,2}((0, T) \to V')$*, thanks to* $u \in C([0, T] \to H)$*.*

(iv) *The partial differential equation* (15) *is satisfied in the* **weak sense** *with all terms valued in the dual space V', that is to say, the abstract equation* (23) *holds for* almost every $t \in (0, T)$.

Remark 1 (Proof of Theorem 2.). Let us recall that, by Lemma 1, only the perturbed mapping $\mathcal{A}_\gamma(t) = \mathcal{A}(t) + \gamma\mathcal{I} : V \to V'$ satisfies the conclusions of this lemma, provided $\gamma \in (0, \infty)$ is a sufficiently large constant. To adjust our arguments to this fact, let us consider the function $u_\gamma(x,t) = \mathrm{e}^{\gamma t}\, u(x,t)$ of $(x,t) \in \mathbb{R} \times [0,T]$, with $u(t) \equiv u(\,\cdot\,,t) \in H$ for every $t \in [0,T]$ and $u(t) \in V$ for almost every $t \in (0,T)$. From the equation

$$\mathrm{e}^{\gamma t}\,\frac{\partial}{\partial t}\,u(t) = \frac{\partial}{\partial t}\,u_\gamma(t) - \gamma\, u_\gamma(t) \qquad \text{valued in } V' \text{ for a.e. } t \in (0,T),$$

combined with eq. (23) above, we deduce that the new function $u_\gamma : [0,T] \to H$ satisfies the following analogous abstract problem,

$$\frac{\partial}{\partial t}\,u_\gamma(t) - \mathcal{A}_\gamma(t)\,u_\gamma(t) = \mathrm{e}^{\gamma t}\,f(t) \quad \text{ for } t \in (0,T)\,; \qquad u_\gamma(0) = u_0 \in H\,, \tag{25}$$

where the nonlinear mapping

$$\mathcal{A}_\gamma(t) : V \to V' : v \longmapsto \mathrm{e}^{\gamma t}\,[\mathcal{A}(t) + \gamma\mathcal{I}]\left[\mathrm{e}^{-\gamma t}\,v\right] = \mathrm{e}^{\gamma t}\,\mathcal{A}(t)\left[\mathrm{e}^{-\gamma t}\,v\right] + \gamma\, v$$

possesses all properties of $\mathcal{A}(t) + \gamma\mathcal{I}$ stated in Lemma 1. Consequently, Theorem 2 applies also to problem (25) for u_γ in place of eq. (23) for u. We refer to V. Barbu [1] [1, Chapt. III, §2, §§2.1, pp. 123–138] for perturbations of monotone mappings $V \to V'$ by (real) multiples of the identity $\mathcal{I} : V \to V \hookrightarrow V'$. □

§4. Applications to nonlinear Black-Scholes equations

Example 1. We have started with the nonlinear model (2) due to G. Barles and H. M. Soner [6, Eq. (1.2), p. 372] with (variable) *implied volatility* (3). When rewritten in our notation from Section 3, Theorem 2, this model takes the form of eq. (15) with the flux function F given by

$$\begin{aligned} &F\left(x,\ u(x,t)\,,\ \frac{\partial u}{\partial x}(x,t)\,,\ t\right) \equiv F(x, A_1, A_2, t) \stackrel{\text{def}}{=} \\ &\frac{1}{2}\,\widehat{\sigma}(0,T)^2 \left[1 + \varsigma\left(\mathrm{e}^{rt}\,a^2\,\left\{A_2 + \frac{\mathrm{d}}{\mathrm{d}x}\,(\mathrm{e}^x\,\varphi(x)) - (A_1 + \mathrm{e}^x\,\varphi(x))\right\}\right)\right] \\ &\times \left\{A_2 + \frac{\mathrm{d}}{\mathrm{d}x}\,(\mathrm{e}^x\,\varphi(x)) - (A_1 + \mathrm{e}^x\,\varphi(x))\right\} \\ &= \frac{1}{2}\,\widehat{\sigma}(0,T)^2 \left[1 + \varsigma\left(\mathrm{e}^{rt}\,a^2\,\left\{\frac{\partial u}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}x}\,(\mathrm{e}^x\,\varphi(x)) - (u + \mathrm{e}^x\,\varphi(x))\right\}\right)\right] \\ &\times \left\{\frac{\partial u}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}x}\,(\mathrm{e}^x\,\varphi(x)) - (u + \mathrm{e}^x\,\varphi(x))\right\} \end{aligned} \tag{26}$$

for all $(x, A_1, A_2) \in \mathbb{R}^3$ and for all $t \in [0, T]$. Here, $\widehat{\sigma}(0, T) > 0$ is a constant and $\varsigma : (-\infty, +\infty) \to \mathbb{R}_+$ is a *nonlinear volatility correction* specified in [6, Eq. (3.2), p. 377]. Our Theorem 2 applies to the Barles-Soner model with the *correction* ς modified in the following simple manner: We replace the original correction ς from [6, Theorem 3.1, p. 377] by its simple modification, whenever $A \in \mathbb{R}$,

$$\tilde{\varsigma}(A) \stackrel{\text{def}}{=} \begin{cases} \varsigma(A_*) & \text{if } A \leq A_* \,; \\ \varsigma(A) & \text{if } A_* \leq A \leq A^* \,; \\ \varsigma(A^*) & \text{if } A \geq A^* \,, \end{cases}$$

where $A_*, A^* \in \mathbb{R}$, respectively, are the starting point and the end point for the modification, $-\infty < A_* < 0 < A^* < \infty$. Hence, $\varsigma(A_*) \leq \tilde{\varsigma}(A) \leq \varsigma(A^*)$, by ς being monotone increasing.

We leave the verification of the hypotheses in Theorem 2 to the reader.

Example 2. An interesting highly nonlinear parabolic problem is treated in the work by A. Bensoussan, K. C. Cheung, Y. Li, and S. C. Ph. Yam [7, Eq. (26), p. 836] on mutual--fund management. Taking advantage of an analogous transformation to our substitution $P \mapsto \Delta = \frac{\partial P}{\partial S}$ (the Greek "Delta"), where the unknown function $V(x, t)$ is replaced by its (unknown) partial derivative $\lambda(x, t) = \frac{\partial V}{\partial x}$, the authors obtain more standard semilinear parabolic problems [7, Eq. (31), p. 837] and [7, Eq. (33), p. 838] to which they apply Schauder's fixed point theorem. Similarly as in **Example 1**, [6, Eq. (1.2), p. 372], the semilinear parabolic problem [7, Eq. (31), p. 837] is obtained by an *inter-temporal maximization* of the sum of the *inter-temporal* and the *terminal utilities* of the management fees to be received.

Example 3. A third nonlinear parabolic problem is obtained in the works by V. Barbu [2] and V. Barbu, C. Benazzoli, and L. Di Persio [3] and V. Barbu [4], for a stochastic optimization problem. A convex pay-off functional reflecting a performance criterion in [3, Eq. (1), p. 520] is minimized with respect to an optimal choice of volatility in [3, Eq. (2), p. 520], that is, with respect to a control variable u in the volatility. The resulting nonlinear parabolic equation [3, Eq. (6), p. 521] is a dynamic programming equation to the stochastic optimal control problem [3, Eq. (1), p. 520]. The unknown (smooth) function $\varphi(x, t)$ in [3, Eq. (6), p. 521] is replaced by the new unknown function $\psi = \frac{\partial \varphi}{\partial x}$ which verifies the nonlinear Cauchy problem in [3, Eq. (8), p. 521]. This problem is of similar nature as our problem (15), (16) an thus can be treated by tools suggested in Theorem 2 and Lemma 1; cf. [1, Chapt. III, §§4.2, p. 167], Theorem 4.2, and [11, Chapt. 2, §1.4, pp. 162–163], Théorème 1.2 and Théorème 1.2 *bis*.

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