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Renormalized solutions for a stochastic p-Laplace equation with L^1 initial data

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Abstract. For $1 < p < \infty$, we consider a stochastic p-Laplace equation on a bounded domain with homogeneous Dirichlet boundary conditions. The technical difficulties arise from the L^1 random initial data under consideration. We introduce the notion of renormalized solutions.

Keywords: Renormalized solutions, stochastic forcing, L^1 random initial data.

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§1. Introduction

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in[0,T]}, (\beta_t)_{t\in[0,T]})$ be a stochastic basis with a complete, countably generated probability space $(\Omega, \mathcal{F}, P)$, a filtration $(\mathcal{F}_t)_{t\in[0,T]} \subset \mathcal{F}$ satisfying the usual assumptions and a real valued, $\mathcal{F}_t$-Brownian motion $(\beta_t)_{t\in[0,T]}$. Let $D \subset \mathbb{R}^d$ a bounded Lipschitz domain, $T > 0$, $Q_T = (0,T) \times D$ and $1 < p < \infty$. Furthermore, let $u_0 : \Omega \to L^1(D)$ be $\mathcal{F}_0$-measurable and $\Phi \in L^2(\Omega \times Q_T)$ be progressively measurable. In this contribution, we study the nonlinear evolution problem:

$$\begin{aligned} du - \operatorname{div}(|\nabla u|^{p-2}\nabla u)\,dt &= \Phi\,d\beta && \text{in } \Omega \times Q_T, \\ u &= 0 && \text{on } \Omega \times (0,T) \times \partial D, \\ u(0,\cdot) &= u_0 && \in\ L^1(\Omega \times D). \end{aligned} \tag{1.1}$$

The diffusion operator in our equation is the p-Laplace operator for $1 < p < \infty$, i.e.,

$$\Delta_p(u) := \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

Obviously, $\Delta_2 = \Delta$, while Δ_p is a nonlinear monotone operator for $p \neq 2$. In the last decades, there has been an extensive study on (1) (see, e.g., [16], [15], [17], [14] and [4]). In our case, the main technical difficulty arises from the random initial data in $L^1(\Omega \times D)$. In this setting, variational solutions are out of range and therefore we consider the more general notion of renormalized solutions which has been introduced by [11] for the study of global existence and weak stability of the Boltzmann equation. Renormalized solutions of (1) with a deterministic right hand side have been studied by many authors, (see, e.g., [7], [5], [8]). Later, this solution concept has been extended to more general problems of parabolic, elliptic-parabolic and hyperbolic type (see, e.g., [9],[10], [6], [1]). For stochastic conservation laws the notion of entropy solutions has been considered in [3]. For a quasilinear, degenerate hyperbolic-parabolic SPDE with L^1 random initial data, the well-posedness and regularity of kinetic solutions has been studied in [13], but, to the best of our knowledge, these results do not apply in the situation of (1). Our aim is to extend the notion of renormalized solutions for the stochastic setting. The well-posedness of (1.1) in the framework of renormalized solutions is the subject of a forthcoming research article.

The well-posedness for $\mathcal{F}_0$-measurable initial data $u_0 \in L^2(\Omega \times D)$ is an easy consequence of classical well-posedness results:

Theorem 1. *Let the conditions in the introduction be satisfied. Furthermore, let* $u_0 \in L^2(\Omega \times D)$*. Then there exists a unique strong solution to (1.1), i.e., there is an* $\mathcal{F}_t$*-adapted stochastic process* $u : \Omega \times [0,T] \to W_0^{1,p}(D)$ *such that* $u \in L^p(\Omega; L^p(0,T; W_0^{1,p}(D))) \cap L^2(\Omega; C([0,T]; L^2(D)))$, $u(0,\cdot) = u_0$ *in* $L^2(\Omega \times D)$ *and*

$$u(t) - u_0 - \int_0^t \operatorname{div}(|\nabla u|^{p-2}\nabla u)\, ds = \int_0^t \Phi\, d\beta$$

in $W^{-1,p'}(D) + L^2(D)$ *for all* $t \in [0,T]$ *and a.s. in* Ω.

Remark 1. Since we know from all terms except the term $\int_0^t \operatorname{div}(|\nabla u|^{p-2}\nabla u)\, ds$ that these terms are elements of $L^2(D)$ for all $t \in [0,T]$ and a.s. in Ω it follows that $\int_0^t \operatorname{div}(|\nabla u|^{p-2}\nabla u)\, ds \in L^2(D)$ for all $t \in [0,T]$ and a.s. in Ω. Therefore this equation is an equation in $L^2(D)$.

Proof. This result is a consequence of [14], Chapter II, Theorem 2.1 and Corollary 2.1. We only have to check the assumptions of this theorem. Following the notations therein, we set $V = W_0^{1,p}(D) \cap L^2(D)$ in the case $1 < p < 2$ and $V = W_0^{1,p}(D)$ in the case $p \geq 2$, $H = L^2(D)$, $E = \mathbb{R}$, $A : V \to V^*$, $A(u) = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $B = \Phi$, $f(t,\omega) = 2 + \|B(t,\omega)\|_2^2$ for almost each $(t,\omega) \in (0,T) \times \Omega$ and $z = 0$. Then we have $\mathcal{L}_Q(E; H) = \mathcal{L}_2(\mathbb{R}, L^2(D)) = L^2(D)$.
We remark that A does not depend on $(t,\omega) \in [0,T]\times\Omega$ and that B does not depend on $u \in V$. Obviously, conditions (A1), (A2) and (A5) in [14] are satisfied. Moreover, in the case $p \geq 2$ the validity of conditions (A3) and (A4) is well known in the theory of monotone operators. Therefore we only consider the case $1 < p < 2$.
In this case we check condition (A3). Using the norms

$$\|v\|_V := \left(\|v\|^p_{W_0^{1,p}(D)} + \|v\|_2^p\right)^{\frac{1}{p}}, \ \|v\|_{W_0^{1,p}(D)} := \|\nabla v\|_{L^p(D)^d}$$

we have

$$\begin{aligned} |B|_Q^2 + 2\|v\|_V^p &= \|B\|_2^2 + 2\|v\|_V^p = f - 2 + 2\|v\|_V^p \\ &= f - 2 + 2\|v\|^p_{W_0^{1,p}(D)} + 2\|v\|_2^p = f - 2 + 2\|v\|_2^p + 2\langle Av, v\rangle_{V^*,V} \\ &\leq f + \|v\|_2^2 + 2\langle Av, v\rangle_{V^*,V} \end{aligned}$$

for all $v \in V$ since $x^p \leq 1 + x^2$ for all $x \geq 0$. This proves condition (A3) for $\alpha = K = 2$.
Now we check condition (A4). We estimate

$$\|A(u)\|_{V^*} \leq \|A(u)\|_{W^{-1,p'}(D)} \leq \|\nabla u\|^{p-1}_{L^p(D)^d} \leq \|u\|_V^{p-1}.$$

Therefore [14], Chapter II, Theorem 2.1, Corollary 2.1 and Theorem 2.2 provide the existence of a strong solution to (1.1). □

§2. Itô formula and renormalization

For two Banach spaces X, Y, let $L(X;Y)$ denote the Banach space of bounded, linear operators from X to Y.
In order to find an appropriate notion of renormalized solutions to (1.1), we use the methods of [12] to prove a particular version of the Itô formula. For the sake of completeness, we recall the following regularization procedure:

Lemma 2. *Let $D \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, $1 \leq p < \infty$ and $r = \min\{p, 2\}$. There exists a sequence of operators*

$$\Pi_n : W^{-1,p'}(D) + L^r(D) \to W_0^{1,p}(D) \cap L^2(D),\ n \in \mathbb{N}$$

such that

i.) $\Pi_n(v) \in W_0^{1,p}(D) \cap L^2(D) \cap C^\infty(\overline{D})$ *for all $v \in W^{-1,p'}(D) + L^r(D)$ and all $n \in \mathbb{N}$*

ii.) For any $n \in \mathbb{N}$ and any Banach space

$$F \in \{W_0^{1,p}(D),\ L^2(D),\ W^{-1,p'}(D),\ W_0^{1,p}(D) \cap L^2(D),\ W^{-1,p'}(D) + L^2(D)\}$$

$\Pi_n : F \to F$ *is a bounded linear operator such that* $\lim_{n\to\infty} \Pi_{n|F} = I_F$ *pointwise in F, where I_F is the identity on F.*

Proof. We follow the ideas of [12], p. 200, Exemple 2.1 and let $\Pi_n(v) := (\phi_n \cdot v) * \rho_n$ be the convolution of the multiplication of $v \in W^{-1,p'}(D) + L^r(D)$ with an appropriate cutoff function ϕ_n and a standard mollifier ρ_n with support in $B_{1/n}(0)$ for $n \in \mathbb{N}$. Then, the assertion follows using Hardy and Young inequality. □

Proposition 3. *Let $G \in L^{p'}(\Omega \times Q_T)^d$, $\Phi \in L^2(\Omega \times Q_T)$ be progressively measurable, $u_0 \in L^2(\Omega \times D)$ be $\mathcal{F}_0$-measurable and $u \in L^2(\Omega; C([0,T]; L^2(D))) \cap L^p(\Omega; L^p(0,T; W_0^{1,p}(D)))$ satisfying the equality*

$$u(t) - u_0 - \int_0^t \operatorname{div} G\, ds = \int_0^t \Phi\, d\beta \tag{2.1}$$

in $L^2(D)$ for all $t \in [0,T]$ and a.s. in Ω.
Then, for all $\psi \in C^\infty([0,T]\times\overline{D})$ and all $S \in C^2(\mathbb{R})$ with $\operatorname{supp}(S'')$ *compact such that $S'(0) = 0$ or $\psi(t,x) = 0$ for all $(t,x) \in [0,T] \times \partial D$ we have*

$$\begin{aligned}&\int_D S(u(t))\psi(t) - S(u_0)\psi(0)\, dx + \int_0^t \int_D S''(u)\nabla u G\psi\, dx\, ds + \int_0^t \int_D S'(u) G\nabla\psi\, dx\, ds\\ &= \int_0^t \int_D S'(u)\psi\Phi\, dx\, d\beta + \int_0^t \int_D S(u)\psi_t\, dx\, ds + \frac{1}{2}\int_0^t \int_D S''(u)\psi\Phi^2\, dx\, ds \end{aligned}\tag{2.2}$$

for all $t \in [0,T]$ and a.s. in Ω.
Especially for $\psi \in C^\infty(\overline{D})$ not depending on t we get

$$\begin{aligned}&\int_D (S(u(t)) - S(u_0))\psi\, dx + \int_0^t \int_D S''(u)\nabla u G\psi\, dx\, ds + \int_0^t \int_D S'(u) G\nabla\psi\, dx\, ds\\ &= \int_0^t \int_D S'(u)\psi\Phi\, dx\, d\beta + \frac{1}{2}\int_0^t \int_D S''(u)\psi\Phi^2\, dx\, ds\end{aligned}$$

for all $t \in [0,T]$ and a.s. in Ω.

Proof. We choose the regularizing sequence (Π_n) according to Lemma 2 and set $u_n := \Pi_n(u)$, $u_0^n := \Pi_n(u_0)$, $(\operatorname{div} G)_n := \Pi_n(\operatorname{div} G)$ and $\Phi_n := \Pi_n(\Phi)$. We apply the operator Π_n to both sides of this equality. Since $\Pi_n \in L(W^{-1,p'}(D) + L^2(D); W_0^{1,p}(D) \cap L^2(D))$, we may conclude

$$u_n(t) - u_0^n - \int_0^t (\operatorname{div} G)_n\,ds = \int_0^t \Phi_n\,d\beta$$

in D, for all $t \in [0,T]$ and a.s. in Ω. Now we apply pointwise in $x \in D$ the classic Itô formula for $h(t,u) := S(u)\psi(t,x)$ with respect to the time variable t. Integration over D afterwards yields

$$\begin{aligned}&\int_D S(u_n(t))\psi(t) - S(u_0^n)\psi(0)\,dx - \int_0^t \langle(\operatorname{div} G)_n, S'(u_n)\psi\rangle_{W^{-1,p'}(D),W_0^{1,p}(D)}\,ds\\ &= \int_0^t\int_D S'(u_n)\psi\Phi_n\,dx\,d\beta + \int_0^t\int_D S(u_n)\psi_t\,dx\,ds + \frac{1}{2}\int_0^t\int_D S''(u_n)\psi\Phi_n^2\,dx\,ds\end{aligned}$$

for all $t \in [0,T]$ and a.s. in Ω. Again by [12] we may pass to the limit with $n \to \infty$. Thus, we get

$$\begin{aligned}&\int_D S(u(t))\psi(t) - S(u_0)\psi(0)\,dx - \int_0^t \langle\operatorname{div} G, S'(u)\psi\rangle_{W^{-1,p'}(D),W_0^{1,p}(D)}\,ds\\ &= \int_0^t\int_D S'(u)\psi\Phi\,dx\,d\beta + \int_0^t\int_D S(u)\psi_t\,dx\,ds + \frac{1}{2}\int_0^t\int_D S''(u)\psi\Phi^2\,dx\,ds\end{aligned}$$

for all $t \in [0,T]$ and a.s. in Ω. This concludes the equality

$$\begin{aligned}&\int_D S(u(t))\psi(t) - S(u_0)\psi(0)\,dx + \int_0^t\int_D S''(u)\nabla u G\psi\,dx\,ds + \int_0^t\int_D S'(u)G\nabla\psi\,dx\,ds\\ &= \int_0^t\int_D S'(u)\psi\Phi\,dx\,d\beta + \int_0^t\int_D S(u)\psi_t\,dx\,ds + \frac{1}{2}\int_0^t\int_D S''(u)\psi\Phi^2\,dx\,ds\end{aligned}$$

for all $t \in [0,T]$ and a.s. in Ω. □

§3. Renormalized solution

Let us assume that there exists a strong solution u to (1.1) in the sense of Theorem 1. We observe that for initial data u_0 merely in L^1, the Itô formula for the square of the norm (see, e.g., [15]) can not be applied and consequently the natural a priori estimate for ∇u in $L^p(\Omega \times Q_T)^d$ is not available. Choosing $\psi \equiv 1$ and

$$S(u) = \int_0^u T_k(r)\,dr$$

in (2.2), where $T_k : \mathbb{R} \to \mathbb{R}$ is the truncation function at level $k > 0$ defined by

$$T_k(r) = \begin{cases} r & , \ |r| \le k,\\ k\operatorname{sign}(r) & , \ |r| > k,\end{cases}$$

we find that there exists a constant $C(k) \geq 0$ depending on the truncation level $k > 0$, such that

$$\mathbb{E}\int_0^T \int_D |\nabla T_k(u)|^p \, dx\, ds \leq C(k).$$

As in the deterministic case, the notion of renormalized solutions takes this information into account.

Definition 1. Let the assumptions in the introduction be fulfilled with $u_0 \in L^1(\Omega \times D)$. Then $u \in L^1(\Omega; C([0,T]; L^1(D)))$ is called a renormalized solution to (1.1) with initial value u_0, if and only if

(i) $T_k(u) \in L^p(\Omega; L^p(0,T; W_0^{1,p}(D)))$ for all $k > 0$.

(ii) For all $\psi \in C^\infty([0,T] \times \bar{D})$ and all $S \in C^2(\mathbb{R})$ such that S' has compact support with $S'(0) = 0$ or $\psi(t,x) = 0$ for all $(t,x) \in [0,T] \times \partial D$ the equality

$$\begin{aligned}
&\int_D S(u(t))\psi(t) - S(u_0)\psi(0)\, dx + \int_0^t \int_D S''(u)|\nabla u|^p \psi \, dx\, ds \\
&+ \int_0^t \int_D S'(u)|\nabla u|^{p-2}\nabla u \cdot \nabla\psi \, dx\, ds \\
&= \int_0^t \int_D S'(u)\psi\Phi \, dx\, d\beta + \int_0^t \int_D S(u)\psi_t \, dx\, ds + \frac{1}{2}\int_0^t \int_D S''(u)\psi\Phi^2 \, dx\, ds
\end{aligned} \tag{3.1}$$

holds true for all $t \in [0,T]$ and a.s. in Ω.

(iii) The following energy dissipation condition holds true:

$$\lim_{k\to\infty} \mathbb{E}\int_{\{k<|u|<k+1\}} |\nabla u|^p \, dx\, dt = 0.$$

Several remarks about Definition 1 are in order: Let u be a renormalized solution in the sense of Definition 1. Since $\operatorname{supp}(S') \subset [-M, M]$, it follows that S is constant outside $[-M, M]$ and for all $k \geq M$, $S(u(t)) = S(T_k(u(t)))$ a.s. in $\Omega \times D$ for all $t \in [0,T]$. In particular, we have

$$S(u) \in L^p(\Omega; L^p(0,T; W^{1,p}(D))) \cap L^\infty(\Omega \times Q_T).$$

From the chain rule for Sobolev functions it follows that

$$S'(u)(|\nabla u|^{p-2}\nabla u) = S'(u)\chi_{\{|u|<M\}}(|\nabla u|^{p-2}\nabla u) = S'(T_M(u))(|\nabla T_M(u)|^{p-2}\nabla T_M(u)) \tag{3.2}$$

a.s. in $\Omega \times Q_T$ and therefore from (i) it follows that all the terms in (3.1) are well-defined. In general, for the renormalized solution u, ∇u may not be in $L^p(\Omega \times Q_T)^d$ and therefore (iii) is an additional condition which can not be derived from (ii). However, for $u \in L^1(\Omega \times Q_T)$ satisfying (i), we can define a generalized gradient (still denoted by ∇u) by setting

$$\nabla u(\omega, t, x) := \nabla T_k(u(\omega, t, x))$$

a.s. in $\{|u| < k\}$ for all $k > 0$. From (ii) it follows that u satisfies the equation

$$\begin{aligned}
&S(u(t)) - S(u(0)) - \int_0^t \operatorname{div}(S'(u)|\nabla u|^{p-2}\nabla u)\, ds \\
&= -\int_0^t S''(u)|\nabla u|^p \, ds + \int_0^t \Phi S'(u)\, d\beta + \frac{1}{2}\int_0^t S''(u)\Phi^2 \, ds,
\end{aligned} \tag{3.3}$$

or equivalently the SPDE

$$dS(u) - \operatorname{div}(S'(u)|\nabla u|^{p-2}\nabla u)\,dt + S''(u)|\nabla u|^p\,dt = \Phi S'(u)\,d\beta + \frac{1}{2}S''(u)\Phi^2\,dt \tag{3.4}$$

in $L^1(D)$ for all $t \in [0,T]$, a.s. in Ω and for any $S \in C^2(\mathbb{R})$ such that $S'(0) = 0$ with $\operatorname{supp}(S')$ compact.

Remark 2. Let u be a renormalized solution to (1.1) with $\nabla u \in L^p(\Omega \times Q_T)^d$. For fixed $l > 0$, let $h_l : \mathbb{R} \to \mathbb{R}$ be defined by

$$h_l(r) = \begin{cases} 0 & , \ |r| \geq l+1 \\ l+1-|r| & , \ l < |r| < l+1 \\ 1 & , \ |r| \leq l. \end{cases}$$

Taking $S(u) = \int_0^u h_l(r)\,dr$ as a test function in (3.5), we may pass to the limit with $l \to \infty$ and we find that u is a strong solution to (1.1).

3.1. The Itô product rule

In the well-posedness theory of renormalized solutions in the deterministic setting (see, e.g., [7]), the product rule is a crucial part. In the following Lemma, we propose an Itô product rule for strong solutions to (1.1). In the following, we will call a function $f : \mathbb{R} \to \mathbb{R}$ *piecewise continuous*, iff it is continuous except for finitely many points.

Proposition 4. *For* $1 < p < \infty$, $u_0, v_0 \in L^2(\Omega\times D)$ $\mathcal{F}_0$*-measurable let u be a strong solution to* (1.1) *with initial datum* u_0 *and v be a strong solution to* (1.1) *with initial datum* v_0 *respectively. Then, for any* $H \in C_b^2(\mathbb{R})$ *and any* $Z \in W^{2,\infty}(\mathbb{R})$ *with* Z'' *piecewise continuous such that* $Z(0) = Z'(0) = 0$

$$\begin{aligned}(Z((u-v)(t)), H(u(t)))_2 &= (Z(u_0-v_0), H(u_0))_2 \\ &+ \int_0^t \langle \Delta_p(u) - \Delta_p(v), H(u)Z'(u-v)\rangle_{W^{-1,p'}(D),W_0^{1,p}(D)}\,ds \\ &+ \int_0^t \langle \Delta_p(u), H'(u)Z(u-v)\rangle_{W^{-1,p'}(D),W_0^{1,p}(D)}\,ds + \int_0^t (\Phi H'(u), Z(u-v))_2\,d\beta \\ &+ \frac{1}{2}\int_0^t\int_D \Phi^2 H''(u)Z(u-v)\,dx\,ds \end{aligned} \tag{3.5}$$

for all $t \in [0,T]$ *a.s. in* Ω.

Proof. We fix $t \in [0,T]$. Since u, v are strong solutions to (1.1), it follows that

$$u(t) = u_0 + \int_0^t \Delta_p(u)\,ds + \int_0^t \Phi\,d\beta, \tag{3.6}$$

$$v(t) = v_0 + \int_0^t \Delta_p(v)\, ds + \int_0^t \Phi\, d\beta$$

and consequently

$$(u - v)(t) = u_0 - v_0 + \int_0^t \Delta_p(u) - \Delta_p(v)\, ds \tag{3.7}$$

holds in $L^2(D)$, a.s. in Ω. For $n \in \mathbb{N}$ we define Π_n according to Lemma 2 and set $\Phi_n := \Pi_n(\Phi)$, $u_0^n := \Pi_n(u_0)$, $v_0^n := \Pi_n(v_0)$, $u_n := \Pi_n(u)$, $v_n := \Pi_n(v)$, $g_n := \Pi_n(\Delta_p(u))$, $h_n := \Pi_n(\Delta_p(v))$. Applying Π_n on both sides of (3.7) yields

$$(u_n - v_n)(t) = u_0^n - v_0^n + \int_0^t g_n - h_n\, ds \tag{3.8}$$

and applying Π_n on both sides of (3.6) yields

$$u_n(t) = u_0^n + \int_0^t g_n\, ds + \int_0^t \Phi_n\, d\beta \tag{3.9}$$

in $W_0^{1,p}(D) \cap L^2(D) \cap C^\infty(\overline{D})$ a.s. in Ω. The pointwise Itô formula in (3.8) and (3.9) leads to

$$Z(u_n - v_n)(t) = Z(u_0^n - v_0^n) + \int_0^t (g_n - h_n) Z'(u_n - v_n)\, ds \tag{3.10}$$

and

$$H(u_n)(t) = H(u_0^n) + \int_0^t g_n H'(u_n)\, ds + \int_0^t \Phi_n H'(u_n)\, d\beta + \frac{1}{2}\int_0^t \Phi_n^2 H''(u_n)\, ds \tag{3.11}$$

in D, a.s. in Ω. From (3.10), (3.11) and the product rule for Itô processes, which is just and easy application of the two-dimensional classical Itô formula (see, e.g., [2], Proposition 8.1, p. 218), applied pointwise in t for fixed $x \in D$ it follows that

$$\begin{aligned} Z(u_n - v_n)(t)H(u_n)(t) &= Z(u_0^n - v_0^n)H(u_0^n) \\ &+ \int_0^t (g_n - h_n)Z'(u_n - v_n)H(u_n)\, ds + \int_0^t g_n H'(u_n) Z(u_n - v_n)\, ds \\ &+ \int_0^t \Phi_n H'(u_n) Z(u_n - v_n)\, d\beta + \frac{1}{2}\int_0^t \Phi_n^2 H''(u_n) Z(u_n - v_n)\, ds \end{aligned} \tag{3.12}$$

in D, a.s. in Ω. Integration over D in (3.12) yields

$$I_1 = I_2 + I_3 + I_4 + I_5 + I_6 \tag{3.13}$$

where

$$
\begin{aligned}
I_1 &= (Z((u_n - v_n)(t)), H((u_n)(t))_2 \\
I_2 &= (Z(u_0^n - v_0^n), H(u_0^n))_2 \\
I_3 &= \int_0^t \int_D (g_n - h_n) Z'(u_n - v_n) H(u_n)\, dx\, ds \\
I_4 &= \int_0^t \int_D g_n H'(u_n) Z(u_n - v_n)\, dx\, ds \\
I_5 &= \int_0^t (\Phi_n H'(u_n), Z(u_n - v_n))_2\, d\beta \\
I_6 &= \frac{1}{2} \int_0^t \int_D \Phi_n^2 H''(u_n) Z(u_n - v_n)\, dx\, ds
\end{aligned}
$$

a.s. in Ω. For any fixed $s \in [0, t]$ and almost every $\omega \in \Omega$, $u_n(\omega, s) \to u(\omega, s)$ and $v_n(\omega, s) \to v(\omega, s)$ for $n \to \infty$ in $L^2(D)$. Since Z, H, H' are continuous and bounded functions, it follows that

$$\lim_{n\to\infty} I_1 = (Z((u - v)(t)), H'(u(t))_2, \tag{3.14}$$

$$\lim_{n\to\infty} I_2 = (Z(u_0 - v_0), H'(u_0))_2 \tag{3.15}$$

in $L^2(\Omega)$ and a.s. in Ω. Note that

$$I_3 = \int_0^t \langle (g_n - h_n), Z'(u_n - v_n) H(u_n) \rangle_{W^{-1,p'}(D), W_0^{1,p}(D)}\, ds$$

a.s. in Ω and from the properties of Π_n it follows that

$$\lim_{n\to\infty} g_n(\omega, s) - h_n(\omega, s) = \Delta_p(u(\omega, s)) - \Delta_p(v(\omega, s))$$

in $W^{-1,p'}(D)$ for all $s \in [0, t]$ and a.e. $\omega \in \Omega$. Recalling the convergence result for (Π_n) from Lemma 2, there exists a constant $C_1 \geq 0$ not depending on s, ω and $n \in \mathbb{N}$ such that

$$
\begin{aligned}
\|g_n(\omega, s) - h_n(\omega, s)\|_{W^{-1,p'}(D)} &= \|\Pi_n(\Delta_p(u(\omega, s)) - \Delta_p(v(\omega, s)))\|_{W^{-1,p'}(D)} \\
&\leq C_1 \|\Delta_p(u(\omega, s)) - \Delta_p(v(\omega, s))\|_{W^{-1,p'}(D)}.
\end{aligned}
$$

Since the right-hand side of the above equation is in $L^{p'}(\Omega \times (0, t))$, from Lebesgue's dominated convergence theorem it follows that

$$\lim_{n\to\infty} g_n - h_n = \Delta_p(u) - \Delta_p(v)$$

in $L^{p'}(\Omega \times (0, t); W^{-1,p'}(D))$ and, with a similar reasoning, also in $L^{p'}(0, t; W^{-1,p'}(D))$ a.s. in Ω. From the chain rule for Sobolev functions it follows that

$$\nabla(Z'(u_n - v_n) H(u_n)) = Z''(u_n - v_n) \nabla(u_n - v_n) H(u_n) + Z'(u_n - v_n) H'(u_n) \nabla u_n \tag{3.16}$$

a.s. in $(0,t)\times\Omega$. Moreover, there exists a constant $C_2 = C_2(\|Z'\|_\infty, \|Z''\|_\infty, \|H\|_\infty, \|H'\|_\infty) \geq 0$ such that

$$\int_0^t \|\nabla(Z'(u_n - v_n)H(u_n))\|_p^p \, ds \leq C_2 \int_0^t (\|\nabla u\|_p^p + \|\nabla v\|_p^p) \, ds \tag{3.17}$$

a.s. in Ω. Consequently, for almost every $\omega \in \Omega$ there exists $\chi(\omega) \in L^p(0,t; W_0^{1,p}(D))$ such that, passing to a not relabeled subsequence that may depend on $\omega \in \Omega$,

$$Z'(u_n - v_n)H(u_n) \rightharpoonup \chi(\omega) \tag{3.18}$$

weakly in $L^p(0,t; W_0^{1,p}(D))$. Since in addition,

$$\lim_{n\to\infty} Z'(u_n - v_n)H(u_n) \to Z'(u - v)H(u)$$

in $L^p((0,t)\times D)$ a.s. in Ω, we get

$$\chi(\omega) = Z'(u - v)H(u) \tag{3.19}$$

in $L^p(0,t; W_0^{1,p}(D))$ a.s. in Ω and the weak convergence in (3.18) holds for the whole sequence. Therefore,

$$Z'(u_n - v_n)H(u_n) \rightharpoonup Z'(u - v)H(u)$$

for $n \to \infty$ weakly in $L^p(0,t; W_0^{1,p}(D))$ for almost every $\omega \in \Omega$. Resuming the above results it follows that

$$\lim_{n\to\infty} I_3 = \int_0^t \langle \Delta_p(u) - \Delta_p(v), Z'(u - v)H(u)\rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} \, ds \tag{3.20}$$

a.s. in Ω. With analogous arguments we get

$$\lim_{n\to\infty} I_4 = \int_0^t \langle \Delta_p(u), H'(u)Z(u - v)\rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} \, ds \tag{3.21}$$

a.s. in Ω. By Itô isometry,

$$\begin{aligned}
&\mathbb{E}\left| \int_0^t \int_D \Phi_n H'(u_n)Z(u_n - v_n) - \Phi H'(u)Z(u - v) \, dx \, d\beta \right|^2 \\
&= \mathbb{E} \int_0^t \int_D |\Phi_n H'(u_n)Z(u_n - v_n) - \Phi H'(u)Z(u - v)|^2 \, dx \, ds.
\end{aligned}$$

From the convergence

$$\Phi_n H'(u_n)Z(u_n - v_n) \to \Phi H'(u)Z(u - v)$$

in $L^2(D)$ for $n \to \infty$ a.s. in $\Omega \times (0,t)$ and since, for almost any (ω, s), there exists a constant $C_3 \geq 0$ not depending on the parameters n, s, ω such that

$$\|\Phi_n(\omega, s)H'(u_n(\omega, s))Z(u_n(\omega, s) - v_n(\omega, s))\|_2 \leq C_3\|\Phi(\omega, s)\|_2$$

for all $n \in \mathbb{N}$, a.s. in $\Omega \times (0,t)$, it follows that

$$\lim_{n\to\infty} \Phi_n H'(u_n) Z(u_n - v_n) = \Phi H'(u) Z(u - v)$$

in $L^2(\Omega \times (0,t) \times D)$ and consequently

$$\lim_{n\to\infty} I_5 = \int_0^t \int_D \Phi H'(u) Z(u-v)\, dx\, d\beta \tag{3.22}$$

in $L^2(\Omega)$ and, passing to a subsequence if necessary, also a.s. in Ω. According to the properties of (Π_n), $\Phi_n^2 \to \Phi^2$ in $L^1((0,t)\times D)$ for $n \to \infty$ a.s. in Ω. From the boundedness and the continuity of H'' and Z we get

$$\lim_{n\to\infty} H''(u_n) Z(u_n - v_n) = H''(u) Z(u-v)$$

in $L^q((0,t)\times D)$ for all $1 \le q < \infty$ and weak-$*$ in $L^\infty((0,t)\times D)$ a.s. in Ω, thus it follows that

$$\lim_{n\to\infty} I_6 = \frac{1}{2}\int_0^t \int_D \Phi^2 H''(u) Z(u-v)\, dx\, ds \tag{3.23}$$

a.s. in Ω. Passing to a subsequence if necessary, taking the limit in (3.12) for $n \to \infty$ a.s. in Ω the assertion follows from (3.14)-(3.23). $\square$

Corollary 5. *Proposition 4 still holds true for $H \in W^{2,\infty}(\mathbb{R})$ such that H'' is piecewise continuous.*

Proof. There exists an approximating sequence $(H_\delta)_{\delta>0} \subset C_b^2(\mathbb{R})$ such that $\|H_\delta\|_\infty \le \|H\|_\infty$, $\|H_\delta'\|_\infty \le \|H'\|_\infty$, $\|H_\delta''\|_\infty \le \|H''\|_\infty$ for all $\delta > 0$ and $H_\delta \to H$, $H_\delta' \to H'$ uniformly on compact subsets, $H_\delta'' \to H''$ pointwise in $\mathbb{R}$ for $\delta \to 0$. With this convergence we are able to pass to the limit with $\delta \to 0$ in (3.5). $\square$

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