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NON-ASSOCIATIVE ALGEBRAIC HYPERSTRUCTURES AND THEIR APPLICATIONS TO BIOLOGICAL INHERITANCE

Oyeyemi O. Oyebola and Temitope G. Jaiyeola

Abstract. In this paper, we investigate non-associative properties in algebraic hyperstructures as it plays out in the biological inheritance which is expressed in the genotypic and phenotypic information that are passed to the progenies from the parental traits. This is with the intention to valuate with precision the non-associativity of weak associative properties in algebraic structures derived from some biological inheritance crossing. Examples of biological inheritance crossing which obey the WASS condition $x \cdot (y \cdot z) \cap (x \cdot y) \cdot z \neq \emptyset$ for the $1, 2, 3$-variable forms were found (though the corresponding identities were not obeyed). The structures $(H, \otimes)$ were found to be hypergroupoids or hyperquasigroups which obey 1-variable identity (3-power associativity) or 2-variable identities (LAP, RAP or flexibility) or 3-variable identities (extra-1 or extra-2 or extra-3). Such hyperstructures can be termed to be 3-power associative, flexible, left (right) alternative or extra; in their precise measure of weakness in associativity.

Keywords: hypergroup, hypersemigroup, H_v-group, H_v-semigroup, H_v-structures, Filial generations.

AMS classification: 20N20.

§1. Introduction

The study of algebraic hyperstructures was born in 1934 by F. Marty [6] when he gave the definitions of hypergroups and illustrated with some applications. It had since been a motivating platform for further studies in hyperstructures and its applications to other issues of life. Hyperstructures are algebraic structures equipped with at least one multi-valued operation, called a hyperoperation. The largest classes of hyperstructures are the ones called H_v-structures. Algebraic hyperstructures are suitable generalizations of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Algebraic hyperstructure theory has a multiplicity of applications to other disciplines such as geometry, graphs and hypergraphs, binary relations, lattices, groups, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, C^*-algebras, artificial intelligence and probability theory.

Etherington presented Genetic algebras in 1939, Non-associative algebra and the symbolism of genetics in 1941. Schafer published Structure of genetic algebras in 1949. Mendel authored Experiments in Plant-Hybridization in 1866.

In this work, our main objective is to showcase that non-associativity in hyperstructures is associated with biological inheritance. We explore some properties in algebraic hyperstructures that naturally occur as genetic information gets passed down through generations. Mathematically, the algebraic hyperstructures that arise in genetics are very interesting ones. They are generally commutative but not associative. It is noteworthy that the order in which genes interact in a given filial generation matters, thus, this necessitated the idea of non-associativity. Hence, the need to valuate with precision the relationship that exist between the progenies of each crosses. Thus, the import of the idea of weak associativity property which the study of hyperstructures availed us. Interestingly, many of the algebraic properties of these hyperstructures have genetic import. This work is furtherance to ideas presented by Davvaz et al. [4], contributions made by Al-Tahan and Davvaz [1, 2], Anvariyeh and Momeni [3] and recent compilations of reports in Davvaz and Vougiouklis [5]

§2. Preliminaries and Basic Definitions

In this section, some basic definitions related to hyperstructures and biological inheritance are presented. It is known that an operation ($\circ$) on a set H is any map from $H \times H$ to H. In other words, to any two elements $x, y \in H$ there correspond an element of H which we denote $x \circ y$. This map is written as follows

$$\circ : H \times H \to H : (x, y) \mapsto x \circ y \in H$$

Usual operations are the addition ($+$) and the multiplication ($\cdot$). Hyperoperation or multivalued operation in a set is any operation which maps to any elements x, y of H into a non-empty subset $x * y$ of H. Thus, we write

$$* : H \times H \to P(H) \backslash \emptyset = P^*(H) : (x, y) \mapsto x * y \subset H$$

where $P(H)$ is the power set of H.

A pair $(H, *)$, consisting of a set equipped with a hyperoperation, is called an hypergroupoid. This is the hyperstructure or multivalued structure. Hyperstructure is every algebraic structure in which at least one hyperoperation is defined.

Definition 1. A hypergroup is a pair $(H, \circ)$, where $\circ : H \times H \longrightarrow P^*(H)$, such that the following conditions hold for all x, y, z of H:

1. $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in H$ which means that
$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v$$
2. $H \circ x = x \circ H = H$, where
$$H \circ x = \bigcup_{h \in H} h \circ x \text{ and } x \circ H = \bigcup_{h \in H} x \circ h$$
This condition is called *the reproduction axiom*.

A commutative hypergroup $(H, \circ)$ is a join space if for all x, y, z of H, the following implication holds:

$$x/y \cap z/w \neq \emptyset \Longrightarrow x \circ w \cap y \circ z \neq \emptyset \qquad (transposition\ axiom).$$

where $x/y = \{u \in H | x \in u \circ y\}$.

In 1934, Marty introduced the concept of a hypergroup. The motivation example was the following: Let G be a group and H be any subgroup of G. Then $G/H = \{xH \mid x \in G\}$ becomes a hypergroup where the hyperoperation is defined in a usual manner:

$$xH \circ yH = \{zH \mid z \in xH \cdot yH\},$$

for all $x, y \in G$.

Definition 2. Let $(H, \circ)$ be a hypergroupoid.

(i) An element $e \in H$ is called an identity if, for all $x \in H$, $x \in x \circ e \cap e \circ x$.
An identity e is called scalar identity if, for all $x \in H$, $x \circ e = e \circ x = x$.
An identity e is called partial identity if, for any $x \in H$, $x \in x \circ e$ or $x \in e \circ x$.

(ii) An element $x' \in H$ is called an inverse of $x \in H$ if there is an identity $e \in H$, such that $e \in x \circ x' \cap x' \circ x$.

Definition 3. Let H be a non-empty set and $\cdot : H \times H \longrightarrow P^*(H)$ be a hyperoperation.

(i) Then, the hypergroupoid $(H, \cdot)$ is said to be weakly associative if

$$x \cdot (y \cdot z) \cap (x \cdot y) \cdot z \neq \emptyset$$

WASS: the weak associativity

(ii) Then, the hypergroupoid $(H, \cdot)$ is said to be weakly commutative if

$$x \cdot y \cap y \cdot x \neq \emptyset$$

COW: the weak commutativity

(iii) Then, the hypergroupoid $(H, \cdot)$ is said to be strongly commutative if

$$x \cdot y = y \cdot x$$

Remark 1. If $(H, \cdot)$ is an hypergroupoid with WASS, then, it is called an H_v-semigroup. In addition, if $(H, \cdot)$ has the reproduction axiom, then it is called an H_v-group.

Definition 4. Let $(H, \cdot)$ be an hypergroupoid and let $x, y, z \in H$.

(i) $(H, \cdot)$ is said to have the 3-power associativity property (3-PA) if it obeys the identity $(x \cdot x) \cdot x = x \cdot (x \cdot x)$.

(ii) $(H, \cdot)$ is said to have the left alternative property (LAP) if it obeys the identity $x \cdot (x \cdot y) = (x \cdot x) \cdot y$.

$\otimes$	RY	Ry	rY	ry
RY	$RRYY$	$RRYy$	$RrYY$	$RrYy$
Ry	$RRYy$	$RRyy$	$RrYy$	$Rryy$
rY	$RrYY$	$RrYy$	$rrYY$	$rrYy$
ry	$RrYy$	$Rryy$	$rrYy$	$rryy$

Table 1: Dihybrid crosses with Pea plants

(iii) $(H, \cdot)$ is said to have the right alternative property (RAP) if it obeys the identity $(y \cdot x) \cdot x = y \cdot (x \cdot x)$.

(iv) $(H, \cdot)$ is said to have the flexibility or elasticity if it obeys the identity $(x \cdot y) \cdot x = x \cdot (y \cdot x)$.

(v) $(H, \cdot)$ is said to have the extra-1 law if obeys the identity $((x \cdot y) \cdot z) \cdot x = x \cdot (y \cdot (z \cdot x))$.

(vi) $(H, \cdot)$ is said to have the extra-1 law if it obeys the identity $((x \cdot y) \cdot z) \cdot x = x \cdot (y \cdot (z \cdot x))$.

(vii) $(H, \cdot)$ is said to have the extra-2 law if it obeys the identity $(y \cdot x) \cdot (z \cdot x) = (y \cdot (x \cdot z)) \cdot x$.

(viii) $(H, \cdot)$ is said to have the extra-3 law if it obeys the identity $(x \cdot y) \cdot (x \cdot z) = x \cdot ((y \cdot x) \cdot z)$.

(ix) $(H, \cdot)$ is called an hyperquasigroup if it has the reproduction axiom.

Remark 2. For any other weak law (aside WASS and COW), an hypergroupoid $(H, \cdot)$ with such weak law will be called an H_v-structure.

§3. Examples of Different Genetic Inheritance

In his dihybrid crosses with pea plants, Gregor Mendel simultaneously examined two different genes that controlled two different traits. For instance, in one series of experiments, Mendel began by crossing a plant that was homozygous for both round seed shape and yellow seed color (RRYY) with another plant that was homozygous for both wrinkled seed shape and green seed color (rryy). Then, when Mendel crossed two of the F_1 (First Filial generation) progeny plants with each other ($RrYy \times RrYy$), he obtained an F_2(Second Filial generation).

$$P : (Round\ and\ yellow)\ RRYY\ \otimes\ (wrinkled\ and\ green)\ rryy$$

$$F_1 : RrYy$$

$$F_2 : F_1\ \otimes\ F_1$$

$$F_2 : RrYy\ \otimes\ RrYy$$

Theorem 1. *Let $H = \{RY, Ry, rY, ry\}$ with $\otimes$ defined on H as given in Table 1. Then,*

(i) *$(H, \otimes)$ is a non-associative hyperquasigroup and H_v-group.*

(ii) *$(H, \otimes)$ is an H_v-structure which does not satisfy the left alternative property.*

(iii) *$(H, \otimes)$ is an H_v-structure which does not satisfy the right alternative property.*

(iv) *$(H, \otimes)$ is an H_v-structure which does not satisfy the flexibility property.*

(v) $(H, \otimes)$ *is an H_v-structure which is a* 3*-power associative hyperquasigroup*

Proof. $(H, \otimes)$ is an hyperquasigroup based on the multiplication Table 1.

(i) Let us check if $(H, \otimes)$ is associative or not:

$$(Ry \otimes rY) \otimes ry \neq Ry \otimes (rY \otimes ry)$$

$$RrYy \otimes ry \neq Ry \otimes rrYy$$

$$\{RrYy, Rryy, rrYy, rryy\} \neq \{RrYy, Rryy\} \text{ but, } (Ry \otimes rY) \otimes ry \cap Ry \otimes (rY \otimes ry) \neq \emptyset.$$

Hence, $(H, \otimes)$ is a non-associative hyperquasigroup and H_v-group.

(ii) Let us check if the left alternative property is satisfied:

$$x \cdot xy = xx \cdot y$$

Then,

$$Ry \otimes (Ry \otimes ry) \neq (Ry \otimes Ry) \otimes ry$$

$$Ry \otimes Rryy \neq RRyy \otimes ry$$

$$\{RRyy, Rryy\} \neq \{Rryy, Rryy\} \text{ but, } Ry \otimes (Ry \otimes ry) \cap (Ry \otimes Ry) \otimes ry \neq \emptyset.$$

Hence, $(H, \otimes)$ is an H_v-structure which does not satisfy the left alternative property.

(iii) Let us check if the right alternative property is satisfied:

$$x \cdot yy = xy \cdot y$$

Then,

$$rY \otimes (ry \otimes ry) \neq (rY \otimes ry) \otimes ry$$

$$rY \otimes rryy \neq rrYy \otimes ry$$

$$rrYy \neq \{rrYy, rryy\} \text{ but, } rY \otimes (ry \otimes ry) \cap (rY \otimes ry) \otimes ry \neq \emptyset.$$

Hence, $(H, \otimes)$ is an H_v-structure which does not satisfy the right alternative property.

(iv) Let us check if the flexibility property is satisfied:

$$x \cdot yx = xy \cdot x$$

$$Ry \otimes (rY \otimes Ry) \neq (Ry \otimes rY) \otimes Ry$$

$$Ry \otimes RrYy \neq RrYy \otimes Ry$$

$$\{RRYy, RRyy, Rryy\} \neq \{RRYy, RRyy, RrYy, Rryy\} \text{ but, } Ry \otimes (rY \otimes Ry) \cap (Ry \otimes rY) \otimes Ry \neq \emptyset.$$

Hence, $(H, \otimes)$ is an H_v-structure which does not satisfy the flexibility property.

(v) It can be deduced from the multiplication Table 1 that the 3-power associativity property holds:

$$x \cdot xx = xx \cdot x \ \forall \ x \in H.$$

$$\text{For instance, } Ry \otimes (Ry \otimes Ry) = (Ry \otimes Ry) \otimes Ry$$

$$Ry \otimes RRyy = RRyy \otimes Ry$$

$$RRyy = RRyy. \quad \square$$

Remark 3. Therefore, $(H, \otimes)$ is an H_v-structure, which is a 3-power associative hyperquasigroup whose weakness in associativity is 1-variable measurable and not 2-variable measurable because it failed LAP, RAP and flexibility property.

$\otimes$	AA	Aa	aa
AA	AA	$\{AA, Aa\}$	Aa
Aa	$\{AA, Aa\}$	$\{AA, Aa, aa\}$	$\{Aa, aa\}$
aa	Aa	$\{Aa, aa\}$	aa

Table 2: Hereditary information inherited from crosses

3.1. Simple Mendelian Inheritance

The zygotes AA and aa are called homozygous, since they carry two copies of the same allele. In this case, simple Mendelian inheritance means that there is no chance involved as to what genetic information will be inherited in the next generation; i.e., AA will pass on the allele A and aa will pass on a. However, the zygotes Aa and aA (which are equivalent) each carry two different alleles. These zygotes are called heterozygous. The rules of simple Mendelian inheritance indicate that the next filial generation will inherit either A or a with equal measure. So, when two gametes reproduce, a multiplication is induced which indicates how the hereditary information will be passed down to the next filial generation. This multiplication is given by the following rules:

1. $A \times A = A$
2. $A \times a = \{A, a\}$
3. $a \times A = \{a, A\}$
4. $a \times a = a$

In 1. and 4. above, both gametes carry the same allele, while there is equal presence of the two alleles in 2. and 3.

Theorem 2. *Let $H = \{AA, Aa, aa\}$ with $\otimes$ defined on H as given in Table 2. Then,*

(i) *$(H, \otimes)$ is a non-associative hypergroupoid, not a hyperquasigroup and a H_v-semigroup.*

(ii) *$(H, \otimes)$ is an H_v-structure which does not satisfy the left alternative property.*

(iii) *$(H, \otimes)$ is an H_v-structure which does not satisfy the right alternative property.*

(iv) *$(H, \otimes)$ is an H_v-structure which satisfies the flexibility property.*

(v) *$(H, \otimes)$ is an H_v-structure which is a 3-power associative hypergroupoid.*

(vi) *$(H, \otimes)$ is an H_v-structure which satisfies the extra-1 identity.*

(vii) *$(H, \otimes)$ is an H_v-structure which does not satisfy the extra-2 identity.*

(viii) *$(H, \otimes)$ is an H_v-structure which does not satisfy the extra-3 identity.*

Proof. $(H, \otimes)$ is an hypergroupoid and not a hyperquasigroup based on the multiplication Table 2.

(i) We shall show that the hypergroupoid $(H, \otimes)$ is non-associative:

$$(AA \otimes Aa) \otimes aa \neq AA \otimes (Aa \otimes aa)$$

$$\{AA, Aa\} \otimes aa \neq AA \otimes \{Aa, aa\}$$

$$\{Aa, aa\} \neq \{AA, Aa\} \text{ but, } (AA \otimes Aa) \otimes aa \cap AA \otimes (Aa \otimes aa) \neq \emptyset$$

Hence, $(H, \otimes)$ is non-associative and an H_v-semigroup.

(ii) Let us check if the left alternative property (LAP) is satisfied:

$$xx \cdot y = x \cdot xy$$

$$(Aa \otimes Aa) \otimes aa \neq Aa \otimes (Aa \otimes aa)$$

$$\{AA, Aa, aa\} \otimes aa \neq Aa \otimes \{Aa, aa\}$$

$$\{Aa, aa\} \neq \{AA, Aa, aa\} \text{ but, } (Aa \otimes Aa) \otimes aa \cap Aa \otimes (Aa \otimes aa) \neq \emptyset$$

Hence, LAP is not satisfied by $(H, \otimes)$. So, $(H, \otimes)$ is an H_v-structure.

(iii) Let us check if the right alternative property (RAP) is also satisfied:

$$xy \cdot y = x \cdot yy$$

$$(AA \otimes Aa) \otimes Aa \neq AA \otimes (Aa \otimes Aa)$$

$$\{AA, Aaa\} \otimes Aa \neq AA \otimes \{AA, Aa, aa\}$$

$$\{AA, Aa, aa\} \neq \{AA, Aa\} \text{ but, } (AA \otimes Aa) \otimes Aa \cap AA \otimes (Aa \otimes Aa) \neq \emptyset$$

Hence, RAP is not satisfied by $(H, \otimes)$. So, $(H, \otimes)$ is an H_v-structure.

(iv) We shall show that flexibility property holds in $(H, \otimes)$ by considering the following and others:

$$x \cdot yx = xy \cdot x \ \forall x, y \in H.$$

$$\text{(a)} \qquad AA \otimes (aa \otimes AA) = (AA \otimes aa) \otimes AA$$

$$AA \otimes Aa = Aa \otimes AA$$

$$\{AA, Aa\} = \{AA, Aa\}.$$

$$\text{(b)} \qquad AA \otimes (Aa \otimes AA) = (AA \otimes Aa) \otimes AA$$

$$AA \otimes \{AA, Aa\} = \{AA, Aa\} \otimes AA$$

$$\{AA, Aa\} = \{AA, Aa\}.$$

$$\text{(c)} \qquad Aa \otimes (aa \otimes Aa) = (Aa \otimes aa) \otimes Aa$$

$$Aa \otimes \{Aa, aa\} = \{Aa, aa\} \otimes Aa$$

$$\{AA, Aa, aa\} = \{Aa, Aa, aa\}.$$

Hence, we see that $(H, \otimes)$ satisfies the flexibility property.

(v) We shall now show that the 3-power associative property is true:

$$x \cdot xx = xx \cdot x$$

Then,

$$\begin{aligned} \text{(d)} \qquad AA \otimes (AA \otimes AA) &= (AA \otimes AA) \otimes AA \\ AA &= AA \\ \text{(e)} \qquad Aa \otimes (Aa \otimes Aa) &= (Aa \otimes Aa) \otimes Aa \\ Aa \otimes \{Aa, Aa, aa\} &= \{AA, Aa, aa\} \otimes Aa \\ \{AA, Aa, aa\} &= \{AA, Aa, aa\} \\ \text{(e)} \qquad aa \otimes (aa \otimes aa) &= (aa \otimes aa) \otimes aa \\ aa &= aa \end{aligned}$$

Hence, by (d), (e) and (f), we see that $(H, \otimes)$ satisfies the 3-power associative property.

(vi) We shall show that extra-1 identity holds in $(H, \otimes)$ by considering the following and others:

$$(xy \cdot z)x = x(y \cdot zx) \ \forall \ x, y, z \in H$$

Then,

$$\begin{aligned} ((AA \otimes Aa) \otimes aa) \otimes AA &= AA \otimes (Aa \otimes (aa \otimes AA)) \\ (\{AA, Aa\} \otimes aa) \otimes AA &= AA \otimes (Aa \otimes Aa) \\ \{Aa, aa\} \otimes AA &= AA \otimes \{AA, Aa, aa\} \\ \{AA, Aa\} &= \{AA, Aa\} \end{aligned}$$

Hence, $(H, \otimes)$ satisfies extra-1 identity.

(vii) Let us check if $(H, \otimes)$ satisfies extra-2 identity:

$$yx \cdot zx = (y \cdot xz)x$$

Then,

$$\begin{aligned} (Aa \otimes AA) \otimes (aa \otimes AA) &\neq (Aa \otimes (AA \otimes aa)) \otimes AA \\ \{AA, Aa\} \otimes Aa &\neq (Aa \otimes Aa) \otimes AA \\ \{AA, Aa, aa\} \neq \{AA, Aa\} \text{ but, } (Aa \otimes AA) \otimes (aa \otimes AA) &\cap (Aa \otimes (AA \otimes aa)) \otimes AA \neq \emptyset \end{aligned}$$

Hence, $(H, \otimes)$ does not satisfy extra-2 identity.

(viii) Let us check if $(H, \otimes)$ satisfies extra-3 identity:

$$xy \cdot xz = x(yx \cdot z)$$

Then,

$$\begin{aligned} (AA \otimes Aa) \otimes (AA \otimes aa) &\neq AA \otimes ((Aa \otimes AA) \otimes aa) \\ \{AA, Aa\} \otimes Aa &\otimes AA \otimes ((\{AA, Aa\}) \otimes aa) \\ \{AA, Aa, aa\} &\neq AA \otimes \{AA, Aa, aa\} \\ \{AA, Aa, aa\} \neq \{AA, Aa\} \text{ but, } (AA \otimes Aa) \otimes (AA \otimes aa) &\cap AA \otimes ((Aa \otimes AA) \otimes aa) \neq \emptyset \end{aligned}$$

Hence, $(H, \otimes)$ does not satisfy extra-3 identity. □

Remark 4. Therefore, $(H,\otimes)$ is an H_v-structure, which is a 3-power associative, flexible and extra-1 hyperqroupoid. Its weakness in associativity is $1,2,3$-variable measurable even though it failed LAP and RAP. Depending on the algebraic properties that are satisfied, these can be used to categorise each cross mating that takes place. It can also be used as counsel to guide in experimentation procedures in cross breeding, in order to cut cost, manage time, energy and materials. It gives an added advantage over being probabilistic in experimentation.

3.2. Combs in Chicken

The research conducted by the British geneticists, William Bateson and R. C. Punnett showed that the shape of the comb in chickens was caused by the interaction between two different genes. Bateson and Punnett were aware of the fact that different varieties of chickens possess distinctive combs. For instance, Wyandottes have a “rose" comb, Brahmas have a “pea" comb, and Leghorns have a “single" comb. When Bateson and Punnett crossed a Wynadotte chicken with a Brahma chicken, all of the F_1 progeny had a new type of comb, which the duo termed a “walnut" comb. In this case, neither the rose comb of the Wyandotte nor the pea comb of the Brahma appeared to be dominant, because the F_1 offspring had their own unique phenotype.

$$P : RRpp \otimes rrPP$$

$$F_1 : RrPp$$

$$F_2 : RrPp \otimes RrPp$$

Moreover, when two of these F_1 progeny were crossed with each other, some of the members of the resulting F_2 generation had walnut combs, some had rose combs, some had pea combs, and some had a single comb. Because the four comb shapes appeared in a 9:3:3:1 ratio (i.e., nine walnut chickens per every three rose chickens per every three pea chickens per every one single-comb chicken), it seemed that two different genes must play a role in comb shape. Through continued research, Bateson and Punnett deduced that Wyandotte (rose-combed) chickens must have the genotype *RRpp*, while Brahma chickens must have the genotype *rrPP*. A cross between a Wyandotte and a Brahma would yield offspring that all had the *RrPp* genotype, which manifested as the walnut-comb phenotype. Indeed, any chicken with at least one rose-comb allele (*R*) and one pea-comb allele (*P*) would have a walnut comb. Thus, when two F_1 walnut chickens were crossed, the resulting F_2 generation would yield rose-comb chickens (*RRpp*), pea-comb chickens (*rrPP*), and walnut-comb chickens (*RrPp*), as well as chickens with a new, fourth phenotype—the single-comb phenotype. Based on the process of elimination, it could be assumed that these single-comb chickens had the *rrpp* genotype (Bateson & Punnett, 1905; 1906; 1908).

Lemma 3. *Let $H = \{RP, Rp, rP, rp\}$ with $\otimes$ defined on H as given in Table 3. Then, $(H,\otimes)$ is a non-associative hyperquasigroup and an H_v-group.*

Proof. $(H,\otimes)$ is an hyperquasigroup based on the multiplication Table 3.

$$(RP \otimes rP) \otimes rp \neq RP \otimes (rP \otimes rp)$$

$$RrPP \otimes rp \neq RP \otimes rrPp$$

$\otimes$	RP	Rp	rP	rp
RP	$RRPP$	$RRPp$	$RrPP$	$RrPp$
Rp	$RRPp$	$RRpp$	$RrPp$	$Rrpp$
rP	$RrPP$	$RrPp$	$rrPP$	$rrPp$
rp	$RrPp$	$Rrpp$	$rrPp$	$rrpp$

Table 3: Crosses of Combs in Chicken

$$\{RrPp, rrPp\} \neq \{RrPP, RrPp\} \text{ but, } (RP \otimes rP) \otimes rp \cap RP \otimes (rP \otimes rp) \neq \emptyset$$

Hence, $(H, \otimes)$ is a non-associative hyperquasigroup and an H_v-group. □

§4. Non-associativity of Genetic Inheritance

Algebraic hyperstructure with genetic realization are not necessarily associative but may be weakly associative. It seems logical that the order in which populations mate is significant. i.e., if parents A and B mate and then the resulting progenies mates with C, the resulting progeny is not the same as the offsprings resulting from A mating with the progenies obtained from mating parents B and C originally. Symbolically, $(A \times B) \times C$ is not equal to $A \times (B \times C)$. Epistasis: One set of alleles (a gene) may mask or inhibit the expression of another gene's alleles.

4.1. Epistasis of Dominant Traits in Eye Color

The two allelomorphs responsible for eye color, christened $OCA2$ and $HERC2$ may be represented by Oo and Hh. O and H are dominant over o and h. The alleles interact as shown below:
$Omhh$ and $oomn$ have phenotype blue and $OmHn$ has phenotype brown.
In this case, $m = O$ or o and $n = H$ or h. Hence, we have the result as stated below:

$$P : OOHH \otimes oohh$$

$$F_1 : OoHh$$

and

$$F_1 \otimes F_1 : OoHh \otimes OoHh$$

$$F_2 : Brown, Blue, Blue$$

Brown is represented by D_1, Blue by D_2 and Blue by D_3.

Remark 5. Note that, phenotypically there is no distinction between D_2 and D_3 but there is a clear distinction between their genotypic composition. Hence, the genotypic representation of the resulting offsprings in F_2 is given as:

$$F_2 : \hat{D_1}(of\,genotype\,OOHH), \hat{D_2}(of\,genotype\,OOhh), \hat{D_3}(of\,genotype\,oohh)$$

OOHH (Brown)
OOHh (Brown)
OOhh (Blue)
OoHH (Brown)
OoHh (Brown)
Oohh (Blue)
ooHH (Blue)
ooHh (Blue)
oohh (Blue)

Table 4: Different genetic combinations of eye colors

$\otimes$	OH	Oh	oH	oh
OH	$OOHH$ (Brown)	$OOHh$ (Brown)	$OoHH$ (Brown)	$OoHh$ (Brown)
Oh	$OOHh$ (Brown)	$OOhh$ (Blue)	$OoHh$ (Brown)	$Oohh$ (Blue)
oH	$OoHH$ (Brown)	$OoHh$ (Brown)	$ooHH$ (Blue)	$ooHh$ (Blue)
oh	$OoHh$ (Brown)	$Oohh$ (Blue)	$ooHh$ (Blue)	$oohh$ (Blue)

Table 5: Genes that are far apart or on different chromosomes

Genes come in different versions (or alleles). OCA2 comes in brown (O) and blue (o) versions. HERC2 also comes in two different versions, brown (H) and blue (h). Since people have two copies of each gene, there are nine different possible genetic combinations. This is expressed in Table 4.

Thus, from the result of above experiment, we have that:

$$(\hat{D}_1 \otimes \hat{D}_2) \otimes \hat{D}_3 \neq \hat{D}_1 \otimes (\hat{D}_2 \otimes \hat{D}_3)$$

and

$$(D_1 \otimes D_2) \otimes D_3 \neq D_1 \otimes (D_2 \otimes D_3)$$

Since genes come in different versions, resulting in epistatic representation of the phenotypes, we have that:

$$(\hat{D}_1 \otimes \hat{D}_2) \otimes \hat{D}_3 \;\cap\; \hat{D}_1 \otimes (\hat{D}_2 \otimes \hat{D}_3) \neq \emptyset$$

and

$$(D_1 \otimes D_2) \otimes D_3 \;\cap\; D_1 \otimes (D_2 \otimes D_3) \neq \emptyset$$

Based on Table 4, we have the multiplication table given in Table 5.

Lemma 4. *Let $H = \{OH, Oh, oH, oh\}$ with $\otimes$ defined on H as given in Table 5. Then, $(H, \otimes)$ is a non-associative hyperquasigroup and an H_v-group.*

Proof. $(H, \otimes)$ is an hyperquasigroup based on the multiplication Table 3. Now,

$$(OH \otimes oH) \otimes oh \neq OH \otimes (oH \otimes oh)$$

$$OoHH \otimes oh \neq OH \otimes ooHh$$

$$\{OoHh, ooHh\} \neq \{OoHH, OoHh\}$$

Hence, $(H, \otimes)$ is a non-associative hyperquasigroup. In fact,

$$(OH \otimes oH) \otimes oh \bigcap OH \otimes (oH \otimes oh) \neq \emptyset$$

$$\text{because } \{OoHh, ooHh\} \bigcap \{OoHH, OoHh\} = \{OoHh\}.$$

Thus, considering other triplets as well, $(H, \otimes)$ is a H_v-group. □

§5. Summary and Conclusion

After the introduction of the notion of hyperstructures about 80 years ago, a number of researches, including its applications have been carried out. Vougiouklis (1990) introduced and studied weak hyper-algebraic structures ($H_v -$group) for a pair $(H, \cdot)$ where H is a set and "$\cdot$ " is an hyperoperation, with the axiom

$$x \cdot (y \cdot z) \cap (x \cdot y) \cdot z \neq \emptyset \text{ for all } x, y, z \in H \tag{5.1}$$

some other authors have found the genotypes of F_2-offspring to be a cyclic H_v-semigroup and relationship between algebraic hyperstructures and biological inheritance have been established (Al-Tahan et al. 2017).

The main objective of this paper was to valuate with precision the non-associativity of weak associative properties in algebraic structures derived from some biological inheritance crossing. In this work, examples of biological inheritance crossing which obey axiom (5.1) in the $2, 3$-variable forms were found. Though the corresponding identities were not obeyed. The structure $(H, \otimes)$ were found to be hypergroupoids or hyperquasigroups which obey 1-variable identity (3-power associativity) or 2-variable identities (LAP, RAP or flexibility) or 3-variable identities (extra-1 or extra-2 or extra-3). Such hyperstructures can be termed to be 3-power associative, flexible, left (right) alternative or extra; in their precise measure of weakness in associativity.

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O. O. Oyebola
Department of Mathematics
Federal University of Agriculture Abeokuta
Ogun State, Nigeria.
`oooyeyemi@gmail.com, oyebolaoo@funaab.edu.ng`

T. G. Jaiyeola
Department of Mathematics
Obafemi Awolowo University
Osun State, Nigeria
`tjayeola@oauife.edu.ng, jaiyeolatemitope@yahoo.com`