Monografías Matemáticas *García de Galdeano* **42**, 173–181 (2019)

LIPSCHITZ SPACES ASSOCIATED TO THE HARMONIC OSCILLATOR

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Abstract.

We define Lipschitz classes adapted to the Harmonic Oscillator

$$\mathcal{H} = -\Delta + |x|^2, \quad x \in \mathbb{R}^n.$$

These classes will be defined either through a pointwise condition or through some integral conditions, in this case by using a semigroup approach. We will prove that the different definitions are equivalent. The semigroup approach will allow us to prove regularity properties of some Bessel operators associated to $\mathcal{H}$.

Keywords: Harmonic Oscillator, Lipschitz Hölder spaces, Semigroups.

AMS classification: 35R11, 35R09, 34A08, 26A33.

§1. Introduction

Along this note, we shall denote by $\mathcal{H}$ the Harmonic Oscillator

$$\mathcal{H} = -\Delta + |x|^2, \quad x \in \mathbb{R}^n.$$

Our purpose is to define Lipschitz classes adapted to $\mathcal{H}$. These classes will be defined either through a pointwise condition or through some integral conditions, in this case by using a semigroup approach. We will prove that the different definitions are equivalent. The semigroup approach will allow us to prove regularity properties of some Bessel operators associated to $\mathcal{H}$. Several of the results contained in this note can be found in [1] and [2].

Lipschitz (also called Hölder) spaces are classes of smooth functions which are basic in functional analysis, Fourier analysis and partial differential equations. Roughly speaking, for certain $k \in \mathbb{N} \cup \{0\}$ and $k < \alpha < k+1$, the space Lipschitz-α is the class of functions that are more regular than C^k (the space of functions whose k-order derivatives are continuous) and less regular than C^{k+1}. Lipschitz spaces are usually defined through pointwise estimates but this approach is not convenient when we want to prove regularity results of some differential operators, because in most of cases it leads to quite involved computations. However, the semigroup description of Lipschitz spaces is really useful for this purpose. This approach was introduced by Taibleson and Stein in the 60's, see [8, 13, 14, 15]. They characterized classical Lipschitz spaces through the heat semigroup, $e^{y\Delta}$ and the Poisson semigroup $e^{-y\sqrt{-\Delta}}$. These characterizations raise the question of analyzing some Hölder spaces associated to different Laplacians and to find the pointwise and semigroup estimate characterizations. In the case of the Ornstein-Ulhenbeck operator $\mathcal{O} = -\frac{1}{2}\Delta + x \cdot \nabla$, in [3] some Lipschitz classes were defined by means of its Poisson semigroup, $e^{-y\sqrt{\mathcal{O}}}$, and in [4] a pointwise characterization was obtained for $0 < \alpha < 1$. In the case of the classical parabolic operator $\partial_t - \Delta$, in [12] Lipschitz classes adapted to this operator were characterized through the Poisson semigroup. In the case of the Hermite operator on $\mathbb{R}^n$, $n \geq 1$, $\mathcal{H} = -\Delta + |x|^2$, adapted Hölder classes were defined pointwise in [11]. These last spaces were characterized in [1] by means of the Poisson semigroup, $e^{-y\sqrt{\mathcal{H}}}$, also in the parabolic case. Laplacians More recently, see [2], some spaces have been defined in the case of Schrödinger operators $\mathcal{L} = -\Delta + V$ in $\mathbb{R}^n$, $n \geq 3$, where V is a nonnegative potential satisfing

$$\left(\frac{1}{|B|}\int_B V(y)^q dy\right)^{1/q} \leq \frac{C}{|B|}\int_B V(y)dy, \ q > n/2, \text{ for every ball } B. \tag{1}$$

It could be said that the breakdown of the analysis of Schrödinger operators was the paper by Shen, [7]. It relays on estimates of the heat kernel of $e^{-t\mathcal{L}}$. However this method only covers the range $n \geq 3$.

The Harmonic Oscillator is probably the most important example among the family of Schrödinger operators. It has the advantage that the kernel of the heat semigroup, $e^{-t\mathcal{H}}f(x)$ is known explicitly. Along this note we shall show how this fact allows us to built a satisfactory theory of Lipschitz spaces for all $n \geq 1$. In this way we shall complement some results of [1] and [2]. We do not want to be exhaustive in this presentation. However we shall remark those results that are new. Sometimes the proofs will be only suggested.

Definition 1 (Hermite Hölder spaces)**.** Let $0 < \alpha < 2$. We consider the space of functions

$$C^\alpha_{\mathcal{H}}(\mathbb{R}^n) = \left\{f : (1 + |\cdot|)^\alpha f(\cdot) \in L^\infty(\mathbb{R}^n), \text{ and } \sup_{|z|>0} \frac{\|f(\cdot + z) + f(\cdot - z) - 2f(\cdot)\|_\infty}{|z|^\alpha} < \infty.\right\}$$

with associated norm

$$\|f\|_{C^\alpha_{\mathcal{H}}} = [f]_{M^\alpha} + [f]_{C^\alpha_{\mathcal{H}}}.$$

Where $[f]_{M^\alpha} = \|(1 + |\cdot|)^\alpha f(\cdot)\|_\infty$ and $[f]_{C^\alpha_{\mathcal{H}}} = \sup_{|z|>0} \frac{\|f(\cdot + z) + f(\cdot - z) - 2f(\cdot)\|_\infty}{|z|^\alpha}$.

Remark 1. This definition was already considered for Schrödinger operators $\mathcal{L}$, where the function $1 + |x|$ was substituted by the inverse of the so called critical radius $\rho(x)$, see (4). It can be seen that, if $0 < \alpha < 1$, the last space coincides with the space such that $[f]_{M^\alpha} < \infty$ and $\sup_{|z|>0} \frac{\|f(\cdot+z)-f(\cdot)\|_\infty}{|z|^\alpha}$. This space was defined in [10], [11].

Definition 2. Let $e^{-y\mathcal{H}} = W_y$ and $e^{-y\sqrt{\mathcal{H}}} = P_y$ be the heat and Poisson semigroups associated to $\mathcal{H}$. For $\alpha > 0$ we define the spaces $\Lambda^W_{\alpha/2}$ and Λ^P_α as

(A) $\Lambda^W_{\alpha/2} = \left\{f : [f]_{M^\alpha} < \infty \text{ and } \left\|\partial^k_y W_y f\right\|_{L^\infty(\mathbb{R}^n)} \leq C_k y^{-k+\alpha/2},\ k = [\alpha/2] + 1\right\}$. We shall denote by $S^W_\alpha[f]$ the infimum of the constants C_k above.

(B) $\Lambda^P_\alpha = \left\{f : M^P[f] = \int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+1}}dx < \infty, \text{ and } \left\|\partial^k_y P_y f\right\|_{L^\infty(\mathbb{R}^n)} \leq B_k y^{-k+\alpha},\ k = [\alpha] + 1\right\}$. We shall denote by $S^P_\alpha[f]$ the infimum of the constants B_k above.

We observe that condition $[f]_{M^\alpha} < \infty$ implies that in particular the function f must be bounded. Moreover, if $f \in \Lambda^P_\alpha$ then $\rho(\cdot)^{-\alpha} f \in L^\infty(\mathbb{R}^n)$, see [2] Theorem 1.9. For $\mathcal{H}$ it is known that $\rho(x) = \frac{1}{1+|x|}$. Therefore we get that $f \in L^\infty(\mathbb{R}^n)$, so Λ^P_α coincides with the space defined in [1]. We will also see that this condition is natural as soon as either $S^P_\alpha[f] < \infty$ or $S^W_\alpha[f] < \infty$.

The main Theorem of this note is the following

Theorem 1. *Let* $0 < \alpha < 2$, $n \geq 1$. *The following statements are equivalent:*

$$(1)\, f \in C^\alpha_{\mathcal{H}}(\mathbb{R}^n), \qquad (2)\, f \in \Lambda^W_{\alpha/2}, \qquad (3)\, f \in \Lambda^P_\alpha.$$

Moreover, the norms of the function f in these spaces are equivalent.

The Theorem was proved to be true in the case $\mathcal{L}$ for $n \geq 3$ and $0 < \alpha \leq 2 - \frac{n}{q}$, where q is the exponent in (1). In [1] it was proved that (1) is equivalent to (3). On the other hand, the proof of (2) implies (3) was given in [2] and it remains valid in this case for any $\alpha > 0$. We include it here as Theorem 6, we believe that is of independent interest. Finally, we sketch the proof of (1) implies (2) at the end of Section 2.

Our second aim is to study the regularity of operators in the Lipschitz spaces. As example of the technique we shall present here the *Bessel potential of order* $\beta > 0$,

$$(Id + \mathcal{H})^{-\beta/2} f(x) = \frac{1}{\Gamma(\beta/2)} \int_0^\infty e^{-t} e^{-t\mathcal{H}} f(x) t^{\beta/2} \frac{dt}{t}.$$

Theorem 2. *Let* $\alpha, \beta > 0$. *Then, the Bessel potential satisfies*

(i) $\|(Id + \mathcal{H})^{-\beta/2} f\|_{\Lambda^W_{\frac{\alpha+\beta}{2}}} \leq C\|f\|_{\Lambda^W_{\alpha/2}}$.

(ii) $\|(Id + \mathcal{H})^{-\beta/2} f\|_{\Lambda^W_{\beta/2}} \leq C\|f\|_\infty$.

All the results in this note have been proved for the elliptic operator $\mathcal{H}$ but they can be done in the parabolic case parallely, as we did in [1] for the Poisson case.

§2. Proof of Theorem 1.

It is well-known that, for $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, see [9], the heat semigroup associated to $\mathcal{H}$ is given by the Mehler's formula

$$e^{-y\mathcal{H}} f(x) = W_y f(x) = \int_{\mathbb{R}^n} W_y(x,z) f(x-z)dz = \int_{\mathbb{R}^n} \frac{e^{-\frac{|z|^2 \coth y}{4}} e^{-\frac{|2x-z|^2 \tanh y}{4}}}{(2\pi \sinh(2y))^{n/2}} f(x-z)dz, \qquad (2)$$

In addition, by Bochner subordination, for $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, the Poisson semigroup is given by

$$e^{-y\sqrt{\mathcal{H}}} f(x) = P_y f(x) = \frac{y}{2\sqrt{\pi}} \int_0^\infty \int_{\mathbb{R}^n} e^{-y^2/4\tau} \frac{e^{-\frac{|x-z|^2}{4}\coth\tau} e^{-\frac{|x+z|^2}{4}\tanh\tau}}{(2\pi \sinh 2\tau)^{n/2}} f(z)\, dz \frac{d\tau}{\tau^{3/2}}. \qquad (3)$$

See [1] for more details.

Remark 2. The following results will be use along the paper. Let $\tau > 0$.

(1) If $\tau < 1$, then $\sinh\tau \sim \tau$, $\cosh\tau \sim C$, $\coth\tau \sim \frac{1}{\tau}$ and $\tanh\tau \sim \tau$.

(2) If $\tau > 1$, then $\sinh\tau \sim e^\tau$, $\cosh\tau \sim e^\tau$, $\coth\tau \sim C$ and $\tanh\tau \sim C$.

(3) Let $z \geq 0$ and $\alpha \geq 0$ there exist a constant $C_\alpha > 0$ such that $z^\alpha e^{-z} \leq C_\alpha e^{-z/2}$.

As usual, by $A \sim B$ we mean there exist constants C_1, C_2 such that $C_1 A \leq B \leq C_2 A$.

We shall also need some expressions of hat kernel acting over constants functions. The reader can be found the proofs of the following formulas in [1].

Lemma 3. *For each $x \in \mathbb{R}^n$ and $\tau > 0$, we have:*

(1) $e^{-y\mathcal{H}} 1(x) = \dfrac{e^{-\frac{\tanh(2y)}{2}|x|^2}}{(\cosh(2y))^{n/2}}$.

(2) $|\partial_y e^{-y\mathcal{H}} 1(x)| \leq C(\min\{y, 1\} + |x|^2) \dfrac{e^{-\frac{\tanh(2y)}{2}|x|^2}}{(\cosh(2y))^{n/2}}$.

Lemma 4. *Let $k \in \mathbb{N}$. Then for every $x \in \mathbb{R}^n$ and $y > 0$,*

$$\Big| \int_{\mathbb{R}^n} \partial_y^k W_y(x, z) dz \Big| \leq \frac{C_k}{y^k}.$$

Proof. For $k = 1$, the proof follows easily by using Remark 2 and the estimate

$$|\partial_y W_y(x, z)| \leq C \frac{e^{-\frac{|2x-z|^2 \tanh y}{c}} e^{-\frac{|z|^2 \coth y}{c}}}{(\sinh(2y))^{n/2} y}.$$

For $k \geq 1$ the proof is parallel. □

Remark 3. Observe that for bounded functions f, Lemma 4 assures that $\left\| \partial_y W_y f \right\|_{L^\infty(\mathbb{R}^n)} \leq C \|f\|_\infty y^{-1}$. Therefore we can assume in the definition of $\Lambda^W_{\alpha/2}$ that $y < 1$. By subordination the same fact occurs for Λ^P_α.

Proposition 5. *Let $\alpha > 0$,*

- *If $k = [\alpha/2] + 1$ and f is a function satisfying $M^\alpha[f] < \infty$, then $\|\partial_y^k W_y f\|_{L^\infty(\mathbb{R}^n)} \leq C_\alpha y^{-k+\alpha/2}$ if, and only if, for $m \geq k$, $\|\partial_y^m W_y f\|_{L^\infty(\mathbb{R}^n)} \leq C_m y^{-m+\alpha/2}$. Moreover, for each m, C_m and C_α are comparable.*
- *If $k = [\alpha] + 1$ and f is a function satisfying $M^P[f] = \int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+1}} dx < \infty$, then, $\|\partial_y^k P_y f\|_{L^\infty(\mathbb{R}^n)} \leq C_k y^{-k+\alpha}$ if, and only if, for $m \geq k$, $\|\partial_y^m P_y f\|_{L^\infty(\mathbb{R}^n)} \leq C_m y^{-m+\alpha}$.*

Proof. Let $m \geq [\alpha/2] + 1 = k$. By the semigroup property and Lemma 4 we have

$$\left| \partial_y^m W_y f(x) \right| = C \left| \partial_y^{m-k} W_{y/2} (\partial_u^k W_u f(x) \big|_{u=y/2}) \right| \leq C'_\alpha \frac{1}{y^{m-k}} y^{-k+\alpha/2} = C_m y^{-m+\alpha/2}.$$

For the converse, the fact $|\partial_y^\ell W_y f(x)| \to 0$ as $y \to \infty$, allows us to integrate on y as many times as we need to get $\|\partial_y^k W_y f\|_{L^\infty(\mathbb{R}^n)} \leq C_\alpha \, y^{-k+\alpha/2}$.

For the Poisson semigroup the proof is parallel. □

The following result appears for the first time in [2] in the case of Schrödinger operators.

Theorem 6. *Let $\alpha > 0$. If $f \in \Lambda^W_{\alpha/2}$, then $f \in \Lambda^P_\alpha$. Moreover, $S^P_\alpha[f] \leq C S^W_\alpha[f]$.*

Proof. Let $k = [\alpha/2]+1$ and $f \in \Lambda^W_{\alpha/2}$, then $[\alpha]+1 = [\alpha/2+\alpha/2]+1 \leq [\alpha/2]+[\alpha/2]+2 = 2k$. By Proposition 5 it is enough to prove that $\|\partial_y^{2k} P_y f\|_\infty \leq C y^{-(2k)+\alpha}$.

Since $\partial_y^2\Big(\frac{ye^{-\frac{y^2}{4\tau}}}{\tau^{3/2}}\Big) = \partial_\tau\Big(\frac{ye^{-\frac{y^2}{4\tau}}}{\tau^{3/2}}\Big)$, k-times integration by parts give

$$
\begin{aligned}
|\partial_y^{2k} P_y f(x)| &= \left|\frac{1}{2\sqrt{\pi}}\int_0^\infty \partial_y^{2k}\left(\frac{ye^{-\frac{y^2}{4\tau}}}{\tau^{3/2}}\right) e^{-\tau\mathcal{H}} f(x)d\tau\right| = \left|\frac{1}{2\sqrt{\pi}}\int_0^\infty \partial_\tau^{k}\left(\frac{ye^{-\frac{y^2}{4\tau}}}{\tau^{3/2}}\right) e^{-\tau\mathcal{H}} f(x)d\tau\right| \\
&= \frac{1}{2\sqrt{\pi}}\left|\int_0^\infty (-1)^k \left(\frac{ye^{-\frac{y^2}{4\tau}}}{\tau^{3/2}}\right)\partial_\tau^k e^{-\tau\mathcal{L}} f(x)d\tau\right| \leq C\, S^W_\alpha[f]\int_0^\infty \frac{ye^{-\frac{y^2}{4\tau}}}{\tau^{3/2}}\tau^{-k+\alpha/2}d\tau \\
&\leq C\, S^W_\alpha[f] y^{-2k+\alpha}.
\end{aligned}
$$

□

Remark 4. It is clear that if f is a function such that $(1+|\cdot|)^\alpha f \in L^\infty(\mathbb{R}^n)$, then $f \in L^\infty(\mathbb{R}^n)$. Therefore, the following Remark 5 establishes that in the definition of $\Lambda^W_{\alpha/2}$, we can consider indistinctly $f \in L^\infty(\mathbb{R}^n)$ or $(1+|x|)^\alpha f \in L^\infty(\mathbb{R}^n)$. The proposition was proved in the case $\mathcal{L}$ in [2].

Remark 5. Let $\alpha > 0$. If f is a bounded function such that $\|\partial_y^m W_y f\|_{L^\infty(\mathbb{R}^n)} \leq C_m y^{-m+\alpha/2}$, $m = [\alpha/2]+1$, then $|x|^\alpha f \in L^\infty(\mathbb{R}^n)$.

Proof. We shall do the proof only in the case $0 < \alpha < 1$, for the other cases see [2]. Now for $|x| > 1$ and $0 < \alpha < 1$ we have

$$
\begin{aligned}
|x|^\alpha |f(x)| &\leq |x|^\alpha \sup_{0<y<\frac{1}{|x|}} |W_y f(x)| \leq |x|^\alpha \sup_{0<y<\frac{1}{|x|}} \Big(|W_y f(x) - W_{\frac{1}{|x|}} f(x)| + |W_{\frac{1}{|x|}} f(x)|\Big) \\
&\leq |x|^\alpha \sup_{0<y<\frac{1}{|x|}} \left|\int_y^{\frac{1}{|x|}} \partial_{z_1} W_{z_1} f(x) dz_1\right| + C\|f\|_{\Lambda^{W_y}_{\alpha/2}} \\
&\leq C|x|^\alpha \sup_{0<y<\frac{1}{|x|}} \left|\int_y^{\frac{1}{|x|}} z_1^{-1+\alpha} dz_1\right| + C\|f\|_{\Lambda^{W_y}_{\alpha/2}} \leq C.
\end{aligned}
$$

□

Now we shall prove that (1) implies (2) in Theorem 1.

Suppose that f is a function that satisfies the conditions in (1). Let $y < 1$. By using that $\int_{\mathbb{R}^n} \partial_y W_y(x,z) f(x+z)dz = \int_{\mathbb{R}^n} \partial_y W_y(x,-z) f(x-z)dz$, we can write

$$
\begin{aligned}
\partial_y W_y f(x) &= \frac{1}{2}\int_{\mathbb{R}^n} \partial_y W_y(x,z)(f(x-z)+f(x+z)-2f(x))dz \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^n} \Big(\partial_y W_y(x,z) - \partial_y W_y(x,-z)\Big) f(x-z)dz + f(x)\partial_y e^{-y\mathcal{H}} 1(x) \\
&= I + II + III.
\end{aligned}
$$

On the one hand, by using Remark 2 and Lemma 4 we have that

$$|I| \leq C \int_{\mathbb{R}^n} \frac{e^{-\frac{|z|^2 \coth y}{c}} e^{-\frac{|2x-z|^2 \tanh y}{c}} |z|^\alpha}{(\sinh(2y))^{n/2} y} dz \leq C y^{-1+\alpha/2}.$$

Regarding II, observe that

$$\begin{aligned}
&\Big| \partial_y W_y(x,z) - \partial_y W_y(x,-z) \Big| = \left| \partial_y \left(\frac{e^{-\frac{|z|^2 \coth y}{4}}}{(2\pi \sinh(2y))^{n/2}} \left[e^{-\frac{|2x-z|^2 \tanh y}{4}} - e^{-\frac{|2x+z|^2 \tanh y}{4}} \right] \right) \right| \\
&\quad = \left| \partial_y \left(\frac{e^{-\frac{|z|^2 \coth y}{4}}}{(2\pi \sinh(2y))^{n/2}} \right) \left[e^{-\frac{|2x-z|^2 \tanh y}{4}} - e^{-\frac{|2x+z|^2 \tanh y}{4}} \right] \right. \\
&\qquad \left. + \frac{e^{-\frac{|z|^2 \coth y}{4}}}{(2\pi \sinh(2y))^{n/2}} \partial_y \left[e^{-\frac{|2x-z|^2 \tanh y}{4}} - e^{-\frac{|2x+z|^2 \tanh y}{4}} \right] \right| \\
&\quad \leq C e^{-\frac{|z|^2 \coth y}{4}} \left(\frac{|z|^2}{(\sinh(y))^2 (\sinh(2y))^{n/2}} + \frac{\coth(2y)}{(\sinh(2y))^{n/2}} \right) \left| e^{-\frac{|2x-z|^2 \tanh y}{4}} - e^{-\frac{|2x+z|^2 \tanh y}{4}} \right| \\
&\qquad + \frac{e^{-\frac{|z|^2 \coth y}{4}}}{(2\pi \sinh(2y))^{n/2}} \left| \int_{-1}^1 \partial_\theta \partial_y \left(e^{-\frac{|2x-\theta z|^2 \tanh y}{4}} \right) d\theta \right| \\
&\quad = II_a + II_b.
\end{aligned}$$

Observe that

$$\begin{aligned}
\left| e^{-\frac{|2x-z|^2 \tanh y}{4}} - e^{-\frac{|2x+z|^2 \tanh y}{4}} \right| &= \left| \int_{-1}^1 \partial_\theta e^{-\frac{|2x-\theta z|^2 \tanh y}{4}} d\theta \right| = \left| \int_{-1}^1 \nabla_z (e^{-\frac{|2x-\theta z|^2 \tanh y}{4}}) \cdot z \, d\theta \right| \\
&= \left| \int_{-1}^1 e^{-\frac{|2x-\theta z|^2 \tanh y}{4}} (\frac{\theta \tanh y}{2} (2x - \theta z) \cdot z) d\theta \right| \\
&\leq C|z| (\tanh y)^{1/2}.
\end{aligned}$$

Therefore, by using Remark 2 we have that

$$\begin{aligned}
|II_a| &\leq C e^{-\frac{|z|^2 \coth y}{4}} \left(\frac{|z|^3 (\tanh y)^{1/2}}{(\sinh(y))^2 (\sinh(2y))^{n/2}} + \frac{\coth(2y) |z| (\tanh y)^{1/2}}{(\sinh(2y))^{n/2}} \right) \\
&\leq C e^{-\frac{|z|^2}{cy}} \left(\frac{|z|^3}{y^{3/2+n/2}} + \frac{|z|}{y^{1/2+n/2}} \right) \leq C \frac{e^{-\frac{|z|^2}{cy}}}{y^{n/2}}.
\end{aligned}$$

On the other hand, since

$$\begin{aligned}
\Big|\int_{-1}^{1}\partial_\theta\partial_y\left(e^{-\frac{|2x-\theta z|^2\tanh y}{4}}\right)d\theta\Big| &= \left|\int_{-1}^{1}\nabla_z\partial_y\left(e^{-\frac{|2x-\theta z|^2\tanh y}{4}}\right)\cdot z\,d\theta\right| \\
&= \left|\int_{-1}^{1}\partial_y\left(e^{-\frac{|2x-\theta z|^2\tanh y}{4}}\frac{\theta\tanh y}{2}(2x-\theta z)\cdot z\right)d\theta\right| \\
&= \left|\int_{-1}^{1}e^{-\frac{|2x-\theta z|^2\tanh y}{4}}\left(-\frac{\theta\tanh y}{2}\frac{|2x-\theta z|^2}{4\cosh^2(y)}(2x-\theta z)\cdot z+\frac{\theta(2x-\theta z)\cdot z}{2\cosh^2 y}\right)d\theta\right| \\
&\le C\frac{|z|}{(\tanh y)^{1/2}\cosh^2 y},
\end{aligned}$$

we have that $|II_b| \le C\frac{e^{-\frac{|z|^2\coth y}{4}}}{(2\pi\sinh(2y))^{n/2}}\frac{|z|}{(\tanh y)^{1/2}\cosh^2 y} \le C\frac{e^{-\frac{|z|^2}{cy}}}{y^{n/2}}$. Estimates II_a and II_b and the fact that $y<1$ allow us to get $|II| \le C\|f\|_\infty y^{-1+\alpha/2}$.

Finally, by using Remark 2 and Lemma 3 (2) we get

$$\begin{aligned}
|III| &\le C|f(x)|(1+|x|^2)\frac{e^{-\frac{\tanh(2y)|x|^2}{2}}}{(\cosh(2y))^{n/2}} \le C|f(x)|(1+|x|^2)e^{-cy|x|^2} \\
&\le C([f]_{M^\alpha}+\|f\|_\infty)y^{-1+\alpha/2}.
\end{aligned}$$

This is the end of the proof of Theorem 1.

§3. Proof of Theorem 2.

Since $\|W_yf\|_\infty \le C\|f\|_\infty$ and $\|\partial_y^\ell W_yf\|_\infty \le C\frac{\|f\|_\infty}{y^\ell}$ for $\ell\in\mathbb{N}$, we can apply Fubini's Theorem and the derivatives and the integral commute.

Let $f\in\Lambda^W_{\alpha/2}$ and $\ell=[\alpha/2+\beta/2]+1$. Then

$$\begin{aligned}
|\partial_y^\ell W_y((Id+\mathcal{H})^{-\beta/2}f(x))| &= \left|\frac{1}{\Gamma(\beta/2)}\int_0^\infty e^{-t}\partial_y^\ell W_y(W_tf)(x)t^{\beta/2}\frac{dt}{t}\right| \\
&\le C\int_0^\infty e^{-t}(\partial_w^\ell W_wf(x)\Big|_{w=y+t})t^{\beta/2}\frac{dt}{t} \\
&\le C\int_0^\infty e^{-t}(y+t)^{-\ell+\alpha/2}t^{\beta/2}\frac{dt}{t} \\
&\overset{\frac{t}{y}=u}{\le} Cy^{\alpha/2+\beta/2-\ell}\int_0^\infty\frac{u^{\beta/2}e^{-yu}}{(1+u)^{\ell-\alpha/2}}\frac{du}{u} \\
&\le Cy^{\alpha/2+\beta/2-\ell}.
\end{aligned}$$

When $f\in L^\infty(\mathbb{R}^n)$ we proceed analogously by using that, for $\ell=[\beta/2]+1$, $\|\partial_y^\ell W_yW_\nu f\|_\infty \le C\frac{\|f\|_\infty}{y^\ell}$.

§4. Remarks about Schrödinger operators

It can be checked that the Harmonic Oscillator, $V(x) = |x|^2$, satisfies condition (1) for all $q < \infty$. On the other hand, one of the fundamental tools in the theory of the operator $\mathcal{L}$ is the so called "critical radius" $\rho(x)$, $x \in \mathbb{R}^n$, defined as

$$\rho(x) := \sup\Big\{r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq 1\Big\}. \tag{4}$$

In the case $\mathcal{H}$, $\rho(x) = \frac{1}{1+|x|}$. The function $\frac{1}{1+|x|}$, appeared before the paper by Shen, [7], in the historical work of Muckenhoupt, [6]. It was related with the Ornstein-Uhlenbeck operator on the line. For the interested reader we refer to the paper [5]. In that paper it is shown that $\frac{1}{1+|x|}$, $n \geq 1$, shares all the properties of $\rho(x)$. Of particular interest is the existence of a covering of the space with balls of type $B(x, \frac{1}{1+|x|})$. All these remarks together say that Theorem 1 could also be proved by changing in an appropriated way the proof given for the operator $\mathcal{L}$ in [2].

Acknowledgements

The authors have been partially supported by MTM2015-66157-C2-1-P, MINECO-FEDER.

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