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A COLLOCATION METHOD FOR A TWO-POINT BOUNDARY VALUE PROBLEM WITH A RIEMANN-LIOUVILLE-CAPUTO FRACTIONAL DERIVATIVE

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Abstract. Numerical methods for a two-point boundary value problem, where the leading term in the differential operator is a Caputo fractional-order derivative of order $1 < \alpha < 2$, are examined. By reformulating the problem as a Volterra integral equation of the second kind, the problem can be discretized using a collocation method. The performance of this collocation method is compared to a finite difference method applied to the original two-point boundary value problem.

Keywords: Fractional differential equation, Riemann-Liouville-Caputo fractional derivative, two-point boundary value problem, collocation method, weak singularity.

AMS classification: AMS classification codes 34A08, 65L10, 65L60.

§1. Introduction

Judging by the ever-expanding literature on numerical methods for fractional differential equations, this topical area is of interest to many researchers. In numerous publications within this area, the classical numerical method of finite differences has been adapted to deal with the presence of a fractional derivative in the differential equation, which results in the associated system matrix being a relatively dense matrix. This matrix structure has practical implications in terms of the accumulation of rounding errors and in storage issues for problems in higher dimensions, Moreover, the associated numerical analysis of finite difference methods for fractional differential equations can be difficult.

In this paper, we examine an alternative approach to discretizing the following two-point boundary value problem

$$-D^{\alpha}_{RLC}u(x) + b(x)u'(x) + c(x)u(x) = f(x) \text{ for } x \in (0, L), \tag{1a}$$

$$D^{\alpha-1}_{C}u(0) = 0,\ u(L) + \beta_1 u'(L) = \gamma_1. \tag{1b}$$

The leading term in the differential operator is a fractional-order derivative of order α, $\alpha \in (1, 2)$, which is called a Riemann-Liouville-Caputo [7], Patie-Simon [1, 8, 12] or conservative Caputo derivative [15]. It is defined by

$$D^{\alpha}_{RLC}u(x) := \frac{d}{dx}D^{\alpha-1}_{C}u(x) \text{ for } x > 0,$$

where D_C^β denotes the Caputo fractional derivative of order β (see for example [4]) with $n-1<\beta<n$ and n is a positive integer; that is,

$$D_C^\beta v(x) := \frac{1}{\Gamma(n-\beta)} \int_{t=0}^{x} (x-t)^{n-1-\beta}\, v^{(n)}(t)\, dt, \quad \text{where } v^{(n)}(t) := \frac{d^n v(t)}{dt^n}. \tag{2}$$

The constants $\beta_1 \geq 0$ and γ_1 and the functions b, c, f are given and it is assumed that

$$c(x) \geq 0 \text{ for } x \in [0, L].$$

The motivation for using the fractional derivative D_{RLC}^α in problem (1) instead of the more commonly used Riemann-Liouville or Caputo fractional derivatives comes from recent publications modelling physical processes [1, 3, 5, 12]. The use of the Caputo fractional Neumann boundary condition $D_C^{\alpha-1}u(0) = 0$ in combination with the fractional derivative D_{RLC}^α is suggested in [3]. In [6] it is proved that, in the case of problem (1), $D_C^{\alpha-1}u(0) = 0$ is equivalent to $u'(0) = 0$. We shall consider this commonly used boundary condition in the present paper.

The solution of problem (1) has a weak singularity at $x = 0$ so its numerical approximation is troublesome. To deal with it, the problem (1) is first reformulated as a Volterra integral equation of the second kind, which is then discretized using a collocation method on a graded mesh. Relative to the finite difference method, this approach is easier to implement and, moreover, the associated numerical analysis is more natural for such problems involving fractional derivatives. The numerical analysis follows the classical approach for collocation methods for Volterra integral equations [2].

This reformulation for fractional-derivative problems was first presented in [9] for certain types of boundary conditions. The convergence result established in [9] is a significant improvement on the corresponding convergence result in [14] where a finite difference scheme was considered. Reformulation was also applied successfully in [10] to a two-point boundary value problem where the highest-order derivative is of Riemann-Liouville type.

In this current paper, we demonstrate that both the method and analysis of [9] extend easily to problem (1), which was not covered in [9]. In [6], we examined the same problem using a finite difference method; there, to prove first-order convergence, we needed to impose the constraint $b \leq 0$ on the data. In the present paper, using the collocation approach, we derive a convergence result without imposing this constraint on the sign of b.

The paper is structured as follows: In Section 2 the two-point boundary value problem (1) is first shown to be equivalent to another boundary value problem whose highest-order derivative is of Caputo type and whose boundary condition at $x = 0$ is $u'(0) = 0$. This new problem is reformulated as a Volterra integral equation of the second kind. In Section 3 the collocation method for this integral equation is presented on a graded mesh condensing at the endpoint $x = 0$. Error estimates are obtained showing the convergence of the collocation method and the dependence of the order of convergence on the choice of the collocation points and on the grading exponent of the mesh. In Section 4 two examples are used in order to compare our collocation method and the finite difference scheme of [6]. They illustrate that the collocation method is more efficient with both a lower computational cost and a higher order of convergence.

Notation: In this paper C denotes a generic constant that can depend on the data of the boundary value problem (1) and possibly on the mesh grading but is independent of the mesh diameter. Note that C can take different values in different places. For each $g \in C[0,1]$, set $\|g\|_\infty = \max_{0\le x\le L} |g(x)|$.

§2. Reformulations of the problem

In [6] it is proved that problem (1) is equivalent to the Caputo two-point boundary value problem

$$-D_C^\alpha u(x) + b(x)u'(x) + c(x)u(x) = f(x) \quad \text{for } x \in (0, L), \tag{3a}$$

$$u'(0) = 0, \;\; u(L) + \beta_1 u'(L) = \gamma_1. \tag{3b}$$

This follows because

(i) the condition $D_C^{\alpha-1} u(0) = 0$ implies $u'(0) = 0$ (which is proved in [6]);

(ii) using integration by parts, one can deriver the relationship

$$D_C^\alpha u(x) = D_{RLC}^\alpha u(x) - \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} u'(0) \tag{4}$$

between the Caputo and Riemann-Liouville-Caputo fractional derivatives, provided that they exist;

(iii) in [4, Lemma 3.11] it is proved that $D_C^{\alpha-1} u(0) = 0$ if u' is absolutely continuous.

Hence, we shall approximate problem (3). Before considering any numerical method for its numerical approximation, some information about the behaviour of the solution is required. Assuming appropriate regularity conditions on the data problem, in [6] it is proved that the solution of (3) satisfies the bounds

$$|u^{(i)}(x)| \le \begin{cases} C & \text{if } \; i = 0, \\ C x^{\alpha-i} & \text{if } \; i = 1, 2, 3, \dots, \end{cases} \tag{5}$$

showing that a typical solution u of problem (1) has a weak singularity at $x = 0$.

In [6], the standard L2 discretization on a uniform mesh is used to approximate (3). In the present paper, similarly to [9], a collocation method is used instead. To this end we reformulate (3) as a Volterra integral equation of the second kind. Recall the definition of the Riemann-Liouville fractional integral operator of order r, which is

$$(J^r g)(x) := \frac{1}{\Gamma(r)} \int_{t=0}^{x} (x-t)^{r-1} g(t)\, dt. \tag{6}$$

Applying $J^{\alpha-1}$ to (3a) and using the fact that

$$J^{\alpha-1} D_C^\alpha g(x) = J^{\alpha-1} J^{2-\alpha} g''(x) = J g''(x) = g'(x) - g'(0),$$

and $u'(0) = 0$, one has

$$-u'(x) + J^{\alpha-1}(b(x)u'(x) + c(x)u(x)) = J^{\alpha-1} f(x).$$

Noting that $u(x) = \int_{s=0}^{x} u'(s)\,ds + u(0)$ and denoting $y(x) = u'(x)$ and $Y(x) = \int_{s=0}^{x} y(s)\,ds$, then (3a) is rewritten as

$$y(x) - J^{\alpha-1}(b(x)y(x) + c(x)Y(x)) = -J^{\alpha-1}f(x) + u(0)J^{\alpha-1}c(x). \tag{7}$$

Consider the decomposition

$$y(x) := v(x) + u(0)w(x),$$

where $v(x)$ is the solution of

$$v(x) - J^{\alpha-1}(b(x)v(x) + c(x)V(x)) = -J^{\alpha-1}f(x), \quad \text{for } x \in (0, L], \quad v(0) = 0, \tag{8}$$

with $V(x) = \int_{s=0}^{x} v(s)\,ds$, and $w(x)$ is the solution of

$$w(x) - J^{\alpha-1}(b(x)w(x) + c(x)W(x)) = J^{\alpha-1}c(x), \quad \text{for } x \in (0, L], \quad w(0) = 0, \tag{9}$$

with $W(x) = \int_{s=0}^{x} w(s)\,ds$. Hence, both v and w satisfy weakly singular Volterra integral equations of the second kind. From [9, Lemma 2.1], the problems (8) and (9) each have a unique solution.

Once v and w are obtained, then we calculate $u(0)$ from

$$u(x) = \int_{s=0}^{x} u'(s)\,ds + u(0) = \int_{s=0}^{x} v(s)\,ds + u(0)\left(1 + \int_{s=0}^{x} w(s)\,ds\right) \tag{10}$$

and imposing the boundary condition (1b) at $x = L$:

$$\begin{aligned}\int_{s=0}^{L} v(s)\,ds + u(0)\left(1 + \int_{s=0}^{L} w(s)\,ds\right) &= u(L)\\ &= \gamma_1 - \beta_1 u'(L)\\ &= \gamma_1 - \beta_1\left[v(L) + u(0)w(L)\right].\end{aligned}$$

Thus, one has

$$u(0) = \frac{\gamma_1 - \beta_1 v(L) - \int_{s=0}^{L} v(s)\,ds}{1 + \beta_1 w(L) + \int_{s=0}^{L} w(s)\,ds}. \tag{11}$$

The denominator in (11) must not be zero, i.e.,

$$1 + \beta_1 w(L) + \int_{s=0}^{L} w(s)\,ds \neq 0. \tag{12}$$

Since $c(x) \geq 0$, a proof by contradiction argument, as in the proof of [9, Lemma 4.1], can be used to prove that $w(x) \geq 0$ for $x \in [0, L]$. Hence (12) is satisfied. Moreover, from [9, Lemma 2.1], we have $\|v\|_\infty \leq C$ and $\|w\|_\infty \leq C$. Thus $|u(0)| \leq C$ from (11).

§3. The collocation method

Consider the problem

$$z(x) - J^{\alpha-1}(b(x)z(x) + c(x)Z(x)) = J^{\alpha-1}g(x), \quad \text{for } x \in (0, L], \quad z(0) = 0 \tag{13}$$

with $Z(x) := \int_{s=0}^{x} z(s)\,ds$. Observe that if $g(x) = -f(x)$ or $g(x) = c(x)$, then we have the problems (8) and (9) associated with the components v and w of u. From the definition (6), one can write (13) as: Find z such that

$$\begin{aligned} z(x) - \frac{1}{\Gamma(\alpha-1)} \int_{t=0}^{x} (x-t)^{\alpha-2} \left[b(t)z(t) + c(t) \int_{s=0}^{t} z(s)\,ds \right] dt \\ = \frac{1}{\Gamma(\alpha-1)} \int_{t=0}^{x} (x-t)^{\alpha-2} g(t)\,dt, \quad \text{for } x \in (0, L], \end{aligned} \tag{14}$$

which will be approximated using the collocation method.

Let N be a positive integer. Consider the graded mesh

$$x_i = L(i/N)^r \quad \text{for } i = 0, 1, \ldots, N, \quad h_i = x_{i+1} - x_i, \quad \text{for } i = 0, 1, \ldots, N-1, \tag{15}$$

where $r \geq 1$ is the grading exponent. If $r = 1$ the mesh is uniform, while the larger r is, the more the grid condenses near $x = 0$. We set $h = \max_{0 \leq i \leq N-1} h_i$ and $h_N = 0$.

The computed solution $z_h \in S_{m-1}^{-1}$, where

$$S_{m-1}^{-1} := \{v : \ v|_{(x_i, x_{i+1})} \in \pi_{m-1}, \ i = 0, 1, \ldots, N-1\}$$

and π_{m-1} denotes the space of polynomials of degree at most $m-1$. Thus, the elements of S_{m-1}^{-1} are piecewise polynomials of degree at most $m-1$ that may be discontinuous at the points x_i. The set of collocation points is

$$X_h = \{x_i + c_j h_i : \ 0 \leq c_1 < c_2 < \cdots < c_m \leq L, \ i = 0, 1, \ldots, N-1\},$$

where $\{c_i\}$ are chosen by the user. If $c_1 = 0$ and $c_m = 1$, then $z_h \in S_{m-1}^{-1} \cap C[0, L]$. The collocation solution $z_h \in S_{m-1}^{-1}$ is computed by imposing

$$\begin{aligned} z_h(x) - \frac{1}{\Gamma(\alpha-1)} \int_{t=0}^{x} (x-t)^{\alpha-2} \left[b(t)z_h(t) + c(t) \int_{s=0}^{t} z_h(s)\,ds \right] dt \\ = \frac{1}{\Gamma(\alpha-1)} \int_{t=0}^{x} (x-t)^{\alpha-2} g(t)\,dt \quad \text{for all } x \in X_h \cup \{L\}. \end{aligned} \tag{16}$$

Note that the collocation method solves mesh interval by mesh interval; thus on each interval one solves a system of m equations (or $m-1$ equations if $c_1 = 0$ and $c_m = 1$) where the unknowns are located at the collocation points. Therefore, collocation methods are more efficient than finite difference methods where one has to solve a single large linear system (see [6] for a comparison).

In practice, the integrals in (16) are evaluated using quadrature formulas with the collocation points as nodes and the functions b, c and g are replaced by polynomials of degree $m-1$ that interpolate to these functions at the collocation points. The computed solution is denoted by $\hat{z}_h$ and satisfies the following result.

Lemma 1. *Assume that $b, c, g \in C^m[0,1]$. Then the collocation solution $\hat{z}_h$ satisfies*

$$\max_{0\le i\le N} |(\hat{z}_h - z)(x_i)| \le Ch^{\min\{r(\alpha-1),m\}}.$$

If in addition the collocation points $\{c_j\}$ are such that

$$\int_{s=0}^{1} \prod_{j=1}^{m} (s - c_j)\, ds = 0, \tag{17}$$

and $b, c, g \in C^{m+1}[0,1]$, then if $r(\alpha-1) \ge m$,

$$\max_{0\le i\le N} \max_{1\le j\le m} |(\hat{z}_h - z)(x_i + c_j h_i)| \le Ch^{m+\alpha-1}.$$

Proof. See [2, Theorem 6.2.14] and [9, Corollary 3.1 and Corollary 3.2]. □

Using the quadrature formulas, one computes the approximations $\hat{v}_h$ and $\hat{w}_h$ to v_h and w_h, respectively. Assuming that h is sufficiently small so that

$$1 + \beta_1 \hat{w}_h(L) + \int_{s=0}^{L} \hat{w}_h(s)\, ds \ne 0,$$

by imitating (10) and (11) one constructs the following approximations of u and u':

$$\hat{u}_h(x) = \int_{s=0}^{x} \hat{v}_h(s)\, ds + \hat{u}_h(0)\left(1 + \int_{s=0}^{x} \hat{w}_h(s)\, ds\right), \tag{18}$$

$$\hat{u}_h'(x) = \hat{v}_h(x) + \hat{u}_h(0)\hat{w}_h(x), \tag{19}$$

with

$$\hat{u}_h(0) = \frac{\gamma_1 - \beta_1 \hat{v}_h(L) - \int_{s=0}^{L} \hat{v}_h(s)\, ds}{1 + \beta_1 \hat{w}_h(L) + \int_{s=0}^{L} \hat{w}_h(s)\, ds}.$$

Recalling (11), we see that

$$|(u - \hat{u}_h)(0)| \le C\left(|(v - \hat{v}_h)(L)| + \int_{s=0}^{L} |(v - \hat{v}_h)(s)|\, ds + |(w - \hat{w}_h)(L)| + \int_{s=0}^{L} |(w - \hat{w}_h)(s)|\, ds\right)$$

and from (10) and (18) we have

$$(u - \hat{u}_h)(x) = \int_{s=0}^{x} (v - \hat{v}_h)(s)\, ds + \hat{u}_h(0) \int_{s=0}^{x} (w - \hat{w}_h)(s)\, ds + (u - \hat{u}_h)(0)\left(1 + \int_{s=0}^{x} w(s)\, ds\right).$$

Using Lemma 1 to bound $\|v - \hat{v}_h\|_\infty$ and $\|w - \hat{w}_h\|_\infty$ (as in [9, Theorem 4.2]), one can establish the following error bound for the collocation method (16).

Theorem 2. *Assume that $b, c, f \in C^m[0,1]$. Let h be sufficiently small. Then the collocation solution u_h of* (1) *and its equivalent problem* (1)*, when product quadrature with collocation points as nodes is used, satisfies the error bound*

$$\max_{0\le i\le N} |(\hat{u}_h - u)(x_i)| + \max_{0\le i\le N} |(\hat{u}_h' - u')(x_i)| \le Ch^{\min\{r(\alpha-1),m\}}, \tag{20}$$

where r is the mesh grading exponent (15). *If in addition* (17) *is satisfied and $b, c, f \in C^{m+1}[0,1]$, then for $r(\alpha - 1) \geq m$ one obtains*

$$\max_{0\leq i\leq N} |(\hat{u}_h - u)(x_i)| + \max_{0\leq i\leq N} \max_{1\leq j\leq m} |(\hat{u}_h' - u')(x_i + c_j h_i)| \leq Ch^{m+\alpha-1}. \tag{21}$$

Remark 1. The error estimate of Theorem 2 for our collocation method does not place any constraint on the sign of b. In contrast, the convergence analysis for the finite difference method in [6] is valid only when $b \leq 0$; its analysis without the restriction $b \leq 0$ is still open.

§4. Numerical experiments

Numerical results are given in this section for two examples with b and f constants and $c \equiv 0$. In the first example $b < 0$ while $b > 0$ in the second example. The exact solution of both examples can be obtained using Laplace transforms. The maximum error and maximum derivative errors in the computed solution $\{\hat{u}_h\}$ are denoted by

$$E_N := \max_{0\leq i\leq N} |(\hat{u}_h - u)(x_i)|, \quad D_N := \max_{0\leq i\leq N} \max_{1\leq j\leq m} |(\hat{u}_h' - u')(x_i + c_j h_j)|.$$

Note that the errors in the approximate solutions are computed at the mesh points and the errors in the approximate first derivative of the solution are computed at the collocation points. The orders of convergence are computed from these values in a standard way:

$$p_N := \log_2\left(\frac{E_N}{E_{2N}}\right), \quad q_N := \log_2\left(\frac{D_N}{D_{2N}}\right).$$

The solutions of both examples are approximated using our collocation method and the finite difference scheme considered in [6]. In the former, for the sake of brevity, we only consider a specific collocation method with $m = 1$ and $c_1 = 1/2$ using both a uniform and a graded mesh with $r = 1/(\alpha - 1)$. The collocation point $c_1 = 1/2$ is special because (17) is satisfied, i.e.,

$$\int_{s=0}^{1} \left(s - \frac{1}{2}\right) ds = 0,$$

and, as a result, the collocation method with $c_1 = 1/2$ provides more accurate approximations than for $c_1 \neq 1/2$. Theorem 2 tells us that the optimal graded mesh is obtained when $r = m/(\alpha - 1)$, as this then gives the highest possible rate of convergence $O(h^{m+\alpha-1})$ — any larger value of r would not improve the rate of convergence but would increase the mesh width near $x = L$ and consequently increase the constant multiplier C in the error bound. As we use $m = 1$ in our experiments, we choose $r = 1/(\alpha - 1)$ to get the optimal mesh grading.

A numerical approximation $\hat{z}_h(x)$ (for all $x \in X_h \cup \{L\}$) to the solution of (16) is computed and then the maximum derivative errors D_N can be computed. To approximate the nodal values $u(x_i) = \int_{s=0}^{x_i} y(s)\, ds + u(0)$, all integrals of the form $\int_{s=0}^{x_i} \hat{z}_h(s)\, ds$ are approximated using the composite trapezoidal rule, i.e.,

$$\int_{s=0}^{x_i} \hat{z}_h(s)\, ds \approx \frac{x_1 - x_0}{2}\frac{\hat{z}_h(h_0/2)}{2} + \sum_{l=1}^{i-1} \frac{x_{l+1} - x_{l-1}}{2}\frac{\hat{z}_h(x_{l-1} + h_{l-1}/2) + \hat{z}_h(x_l + h_l/2)}{2}$$
$$+ \frac{x_i - x_{i-1}}{2}\frac{\hat{z}_h(x_{i-1} + h_{i-1}/2) + z_h(x_i + h_i/2)}{2}, \quad 0 < i \leq N,$$

with $\hat{z}_h(x_N + h_N/2) = \hat{z}_h(x_N)$. The maximum errors E_N can then be computed.

The finite difference scheme [6] is defined on a uniform mesh. It is given by

$$\left(-D^{\alpha}_{C,L2}u_h + bD^0u_h + cu_h\right)(jL/N) = f(jL/N) \text{ for } j = 1, 2, \ldots, N-1,$$
$$-D^+u_h(0) = 0, \quad u_h(L) + \beta_1 D^- u_h(L) = \gamma_1,$$

where $D^{\alpha}_{C,L2}$ is the well-known L2 approximation [11] of the Caputo fractional derivative D^{α}_C and D^0, D^- and D^+ are the standard central, backward and forward differences, respectively.

Example I

Consider the problem

$$-D^{\alpha}_{RLC}u - 0.5u' = 1 \text{ on } (0, 1), \quad D^{\alpha-1}u(0) = 0, \; u(1) = 0. \tag{22}$$

Its exact solution can be obtained in closed form using Laplace transforms (see [6]):

$$u(x) = -x^{\alpha}E_{\alpha-1,\alpha+1}(-0.5x^{\alpha-1}) + E_{\alpha-1,\alpha+1}(-0.5) \text{ for } 0 \le x \le 1,$$

where $E_{\beta,\gamma}(\cdot)$ is the two-parameter Mittag-Leffler function defined by

$$E_{\beta,\gamma}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta + \gamma)} \text{ for } \beta, \gamma > 0 \text{ and all real numbers } z.$$

In order to compute the errors, the Mittag-Leffler function $E_{\beta,\gamma}(z)$ is evaluated in our code using the function *mlf* provided at MatLab Central File exchange [13].

The maximum nodal errors and orders of convergence of the finite difference scheme proposed in [6] are given in Table 1. The computed orders of convergence indicate that this method is first-order convergent, in agreement with the convergence result proved in [6].

The solution of Example I is now approximated by the collocation method (16) for $m = 1$ and $c_1 = 1/2$. Numerical results using uniform and graded meshes are given in Tables 2 and 3. We see that the collocation method is more accurate on the graded mesh than on the uniform mesh and that both approaches are more accurate than the finite difference scheme [6] (see Table 1) for all the values of α. The order of convergence of the collocation method on the graded mesh is predicted by (21). Note that this theoretical error bound has not been established in the case of a uniform mesh (as (21) requires $r(\alpha - 1) \ge m$). Nevertheless, superconvergence is still observed in Table 2, when a uniform mesh is used. Unlike [6], the collocation theory also gives error estimates for numerical approximations $\hat{u}'_h$ to u'. In Tables 4 and 5 the maximum derivative errors for the collocation method are given using a uniform and a graded mesh. If N is sufficiently large, the approximation of u' is again more accurate when a graded mesh is used and the computed orders of convergence are in agreement with Theorem 2.

Example II

Consider the following problem

$$-D^{\alpha}_{RLC}u + 0.5u' = 1 \text{ on } (0, 1), \quad D^{\alpha-1}u(0) = 0, \; u(1) = 0, \tag{23}$$

Table 1: Example I: Maximum nodal errors and orders of convergence using the finite difference scheme [6] on a uniform mesh

	N=32	N=64	N=128	N=256	N=512	N=1024	N=2048	N=4096
$\alpha = 1.1$	1.789E-02 1.000	8.946E-03 1.000	4.473E-03 1.000	2.237E-03 1.000	1.118E-03 1.000	5.592E-04 1.000	2.796E-04 1.000	1.398E-04
$\alpha = 1.2$	1.840E-02 0.999	9.207E-03 0.999	4.605E-03 1.000	2.303E-03 1.000	1.152E-03 1.000	5.759E-04 1.000	2.880E-04 1.000	1.440E-04
$\alpha = 1.3$	1.888E-02 0.998	9.457E-03 0.998	4.734E-03 0.999	2.368E-03 0.999	1.185E-03 1.000	5.925E-04 1.000	2.963E-04 1.000	1.482E-04
$\alpha = 1.4$	1.929E-02 0.995	9.681E-03 0.997	4.852E-03 0.998	2.430E-03 0.998	1.216E-03 0.999	6.085E-04 0.999	3.044E-04 1.000	1.522E-04
$\alpha = 1.5$	1.956E-02 0.990	9.848E-03 0.993	4.948E-03 0.995	2.482E-03 0.997	1.244E-03 0.998	6.231E-04 0.998	3.119E-04 0.999	1.561E-04
$\alpha = 1.6$	1.958E-02 0.983	9.902E-03 0.987	4.994E-03 0.991	2.513E-03 0.993	1.263E-03 0.995	6.338E-04 0.996	3.178E-04 0.997	1.592E-04
$\alpha = 1.7$	1.915E-02 0.974	9.748E-03 0.980	4.943E-03 0.984	2.500E-03 0.987	1.261E-03 0.989	6.353E-04 0.991	3.195E-04 0.993	1.605E-04
$\alpha = 1.8$	1.804E-02 0.966	9.236E-03 0.971	4.711E-03 0.975	2.396E-03 0.979	1.216E-03 0.982	6.156E-04 0.984	3.111E-04 0.987	1.570E-04
$\alpha = 1.9$	1.590E-02 0.966	8.142E-03 0.969	4.160E-03 0.971	2.122E-03 0.974	1.080E-03 0.976	5.493E-04 0.978	2.789E-04 0.980	1.414E-04

whose exact solution is (see [6])

$$u(x) = -x^{\alpha} E_{\alpha-1,\alpha+1}(0.5x^{\alpha-1}) + E_{\alpha-1,\alpha+1}(0.5).$$

The error analysis in [6] does not apply to this example because $b > 0$, nevertheless the numerical results from Table 6 show that the finite difference method proposed in that paper on a uniform mesh also converges with first order to the solution u. On the other hand, the error estimates for our collocation method remain valid for this example; the numerical results given in Tables 7 and 8 show that the collocation method converges with order $O(h^{\alpha})$ using either a uniform or a graded mesh. Observe also that the maximum errors for the collocation method on both meshes are similar in this example but smaller than the finite difference errors in Table 6.

Finally, it is shown in Tables 9 and 10 that the numerical approximations $\hat{u}_h'$ generated by the collocation method on a uniform and a graded mesh also converge to u'. Conclusions similar to Example I are reached.

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Table 2: Example I: Maximum errors and orders of convergence using a collocation method for $m = 1$ and $c_1 = 1/2$ on a uniform mesh

	N=32	N=64	N=128	N=256	N=512	N=1024	N=2048	N=4096
$\alpha = 1.1$	3.741E-03 1.072	1.779E-03 1.074	8.451E-04 1.076	4.010E-04 1.077	1.901E-04 1.078	9.001E-05 1.080	4.259E-05 1.081	2.014E-05
$\alpha = 1.2$	2.670E-03 1.157	1.197E-03 1.163	5.345E-04 1.168	2.380E-04 1.171	1.056E-04 1.175	4.679E-05 1.178	2.068E-05 1.181	9.125E-06
$\alpha = 1.3$	1.877E-03 1.253	7.875E-04 1.262	3.283E-04 1.269	1.362E-04 1.275	5.628E-05 1.280	2.318E-05 1.283	9.525E-06 1.286	3.905E-06
$\alpha = 1.4$	1.296E-03 1.355	5.066E-04 1.366	1.965E-04 1.375	7.577E-05 1.381	2.910E-05 1.386	1.114E-05 1.389	4.251E-06 1.392	1.620E-06
$\alpha = 1.5$	8.770E-04 1.459	3.191E-04 1.471	1.151E-04 1.480	4.126E-05 1.486	1.473E-05 1.490	5.245E-06 1.493	1.864E-06 1.495	6.612E-07
$\alpha = 1.6$	6.289E-04 1.525	2.185E-04 1.550	7.460E-05 1.567	2.518E-05 1.578	8.434E-06 1.585	2.811E-06 1.590	9.337E-07 1.593	3.095E-07
$\alpha = 1.7$	5.005E-04 1.639	1.607E-04 1.661	5.082E-05 1.675	1.592E-05 1.683	4.957E-06 1.689	1.538E-06 1.692	4.759E-07 1.694	1.470E-07
$\alpha = 1.8$	3.795E-04 1.750	1.128E-04 1.768	3.311E-05 1.779	9.647E-06 1.785	2.798E-06 1.789	8.095E-07 1.792	2.338E-07 1.794	6.743E-08
$\alpha = 1.9$	2.749E-04 1.861	7.565E-05 1.876	2.062E-05 1.883	5.589E-06 1.888	1.510E-06 1.890	4.075E-07 1.892	1.098E-07 1.893	2.958E-08

Table 3: Example I: Maximum errors and orders of convergence using a collocation method for $m = 1$ and $c_1 = 1/2$ on a graded mesh with $r = 1/(\alpha - 1)$

	N=32	N=64	N=128	N=256	N=512	N=1024	N=2048	N=4096
$\alpha = 1.1$	3.725E-04 1.576	1.249E-04 1.473	4.500E-05 1.375	1.735E-05 1.288	7.108E-06 1.220	3.050E-06 1.173	1.352E-06 1.143	6.122E-07
$\alpha = 1.2$	3.879E-04 1.556	1.320E-04 1.483	4.719E-05 1.409	1.777E-05 1.345	6.994E-06 1.295	2.850E-06 1.260	1.190E-06 1.237	5.047E-07
$\alpha = 1.3$	3.770E-04 1.629	1.219E-04 1.570	4.105E-05 1.507	1.444E-05 1.451	5.283E-06 1.405	1.996E-06 1.370	7.720E-07 1.346	3.037E-07
$\alpha = 1.4$	3.471E-04 1.717	1.055E-04 1.673	3.310E-05 1.622	1.076E-05 1.572	3.617E-06 1.528	1.254E-06 1.492	4.457E-07 1.465	1.615E-07
$\alpha = 1.5$	3.122E-04 1.801	8.958E-05 1.773	2.622E-05 1.736	7.867E-06 1.697	2.426E-06 1.660	7.678E-07 1.625	2.489E-07 1.596	8.233E-08
$\alpha = 1.6$	2.793E-04 1.870	7.639E-05 1.857	2.108E-05 1.837	5.902E-06 1.812	1.681E-06 1.786	4.873E-07 1.759	1.439E-07 1.734	4.327E-08
$\alpha = 1.7$	2.506E-04 1.920	6.622E-05 1.919	1.751E-05 1.912	4.654E-06 1.901	1.246E-06 1.888	3.367E-07 1.874	9.188E-08 1.859	2.533E-08
$\alpha = 1.8$	2.263E-04 1.952	5.847E-05 1.958	1.505E-05 1.958	3.874E-06 1.956	9.985E-07 1.952	2.580E-07 1.948	6.686E-08 1.943	1.739E-08
$\alpha = 1.9$	2.048E-04 1.966	5.242E-05 1.979	1.329E-05 1.982	3.364E-06 1.984	8.506E-07 1.984	2.150E-07 1.984	5.437E-08 1.983	1.375E-08

Table 4: Example I: Maximum derivative errors and orders of convergence using a collocation method for $m = 1$ and $c_1 = 1/2$ on a uniform mesh

	N=32	N=64	N=128	N=256	N=512	N=1024	N=2048	N=4096
$\alpha = 1.1$	1.460E-03 0.128	1.337E-03 0.131	1.221E-03 0.135	1.112E-03 0.138	1.010E-03 0.142	9.156E-04 0.145	8.281E-04 0.148	7.474E-04
$\alpha = 1.2$	3.141E-03 0.300	2.551E-03 0.311	2.056E-03 0.321	1.645E-03 0.330	1.308E-03 0.339	1.035E-03 0.346	8.142E-04 0.352	6.378E-04
$\alpha = 1.3$	3.452E-03 0.502	2.437E-03 0.519	1.701E-03 0.533	1.176E-03 0.545	8.058E-04 0.555	5.486E-04 0.563	3.714E-04 0.570	2.502E-04
$\alpha = 1.4$	2.767E-03 0.717	1.683E-03 0.736	1.010E-03 0.751	6.000E-04 0.763	3.537E-04 0.772	2.072E-04 0.778	1.208E-04 0.783	7.017E-05
$\alpha = 1.5$	1.832E-03 0.936	9.572E-04 0.955	4.939E-04 0.968	2.526E-04 0.977	1.283E-04 0.984	6.490E-05 0.988	3.271E-05 0.992	1.645E-05
$\alpha = 1.6$	1.067E-03 1.154	4.795E-04 1.169	2.132E-04 1.180	9.411E-05 1.187	4.135E-05 1.191	1.811E-05 1.194	7.915E-06 1.196	3.454E-06
$\alpha = 1.7$	5.672E-04 1.368	2.198E-04 1.380	8.443E-05 1.388	3.227E-05 1.392	1.229E-05 1.395	4.673E-06 1.397	1.774E-06 1.398	6.731E-07
$\alpha = 1.8$	2.819E-04 1.578	9.439E-05 1.587	3.141E-05 1.593	1.041E-05 1.596	3.445E-06 1.598	1.138E-06 1.599	3.758E-07 1.599	1.241E-07
$\alpha = 1.9$	1.330E-04 1.786	3.858E-05 1.792	1.114E-05 1.796	3.208E-06 1.798	9.227E-07 1.799	2.652E-07 1.799	7.619E-08 1.800	2.188E-08

Table 5: Example I: Maximum derivative errors and orders of convergence using a collocation method for $m = 1$ and $c_1 = 1/2$ on a graded mesh with $r = 1/(\alpha - 1)$

	N=32	N=64	N=128	N=256	N=512	N=1024	N=2048	N=4096
$\alpha = 1.1$	2.595E-03 1.038	1.264E-03 1.069	6.025E-04 1.084	2.842E-04 1.092	1.333E-04 1.096	6.240E-05 1.098	2.916E-05 1.099	1.361E-05
$\alpha = 1.2$	1.897E-03 1.166	8.453E-04 1.182	3.727E-04 1.190	1.634E-04 1.194	7.139E-05 1.197	3.114E-05 1.198	1.357E-05 1.199	5.912E-06
$\alpha = 1.3$	1.267E-03 1.272	5.246E-04 1.284	2.155E-04 1.290	8.809E-05 1.294	3.592E-05 1.296	1.462E-05 1.298	5.948E-06 1.299	2.418E-06
$\alpha = 1.4$	8.025E-04 1.372	3.100E-04 1.382	1.189E-04 1.388	4.543E-05 1.393	1.730E-05 1.395	6.579E-06 1.397	2.499E-06 1.398	9.482E-07
$\alpha = 1.5$	4.858E-04 1.468	1.756E-04 1.478	6.306E-05 1.484	2.254E-05 1.489	8.028E-06 1.492	2.853E-06 1.495	1.013E-06 1.496	3.590E-07
$\alpha = 1.6$	2.804E-04 1.560	9.509E-05 1.570	3.203E-05 1.577	1.073E-05 1.583	3.582E-06 1.587	1.192E-06 1.590	3.958E-07 1.593	1.312E-07
$\alpha = 1.7$	1.524E-04 1.650	4.857E-05 1.659	1.538E-05 1.667	4.842E-06 1.673	1.518E-06 1.678	4.742E-07 1.683	1.477E-07 1.686	4.592E-08
$\alpha = 1.8$	9.695E-05 1.888	2.619E-05 1.890	7.067E-06 1.819	2.002E-06 1.761	5.908E-07 1.766	1.737E-07 1.771	5.091E-08 1.774	1.488E-08
$\alpha = 1.9$	7.537E-05 1.939	1.966E-05 1.942	5.117E-06 1.943	1.330E-06 1.944	3.457E-07 1.945	8.979E-08 1.945	2.332E-08 1.945	6.056E-09

Table 6: Example II: Maximum errors and orders of convergence using a finite difference scheme on a uniform mesh

	N=32	N=64	N=128	N=256	N=512	N=1024	N=2048	N=4096
$\alpha = 1.1$	1.006E-01	5.061E-02	2.539E-02	1.271E-02	6.362E-03	3.182E-03	1.592E-03	7.959E-04
	0.991	0.995	0.998	0.999	0.999	1.000	1.000	
$\alpha = 1.2$	9.787E-02	4.953E-02	2.492E-02	1.250E-02	6.262E-03	3.134E-03	1.568E-03	7.841E-04
	0.982	0.991	0.995	0.998	0.999	0.999	1.000	
$\alpha = 1.3$	9.051E-02	4.605E-02	2.324E-02	1.168E-02	5.859E-03	2.934E-03	1.468E-03	7.346E-04
	0.975	0.986	0.992	0.996	0.998	0.999	0.999	
$\alpha = 1.4$	8.136E-02	4.162E-02	2.109E-02	1.063E-02	5.339E-03	2.677E-03	1.341E-03	6.713E-04
	0.967	0.981	0.989	0.993	0.996	0.997	0.998	
$\alpha = 1.5$	7.170E-02	3.691E-02	1.880E-02	9.508E-03	4.791E-03	2.407E-03	1.208E-03	6.053E-04
	0.958	0.974	0.983	0.989	0.993	0.995	0.997	
$\alpha = 1.6$	6.197E-02	3.213E-02	1.646E-02	8.373E-03	4.237E-03	2.137E-03	1.075E-03	5.403E-04
	0.948	0.965	0.975	0.983	0.987	0.991	0.993	
$\alpha = 1.7$	5.219E-02	2.725E-02	1.407E-02	7.204E-03	3.669E-03	1.861E-03	9.410E-04	4.747E-04
	0.938	0.954	0.965	0.973	0.979	0.984	0.987	
$\alpha = 1.8$	4.219E-02	2.213E-02	1.150E-02	5.931E-03	3.043E-03	1.555E-03	7.916E-04	4.020E-04
	0.931	0.945	0.955	0.963	0.969	0.974	0.978	
$\alpha = 1.9$	3.166E-02	1.653E-02	8.583E-03	4.436E-03	2.284E-03	1.173E-03	6.007E-04	3.070E-04
	0.937	0.946	0.952	0.957	0.962	0.965	0.968	

Table 7: Example II: Maximum errors and orders of convergence using a collocation method for $m = 1$ and $c_1 = 1/2$ on a uniform mesh

	N=32	N=64	N=128	N=256	N=512	N=1024	N=2048	N=4096
$\alpha = 1.1$	5.498E-03	2.438E-03	1.085E-03	4.844E-04	2.170E-04	9.747E-05	4.391E-05	1.983E-05
	1.173	1.168	1.163	1.159	1.155	1.151	1.147	
$\alpha = 1.2$	1.525E-03	5.963E-04	2.684E-04	1.304E-04	6.160E-05	2.854E-05	1.304E-05	5.905E-06
	1.355	1.152	1.041	1.082	1.110	1.129	1.144	
$\alpha = 1.3$	9.601E-04	4.722E-04	2.177E-04	9.681E-05	4.207E-05	1.799E-05	7.607E-06	3.190E-06
	1.024	1.117	1.169	1.202	1.226	1.242	1.254	
$\alpha = 1.4$	7.204E-04	3.253E-04	1.388E-04	5.725E-05	2.312E-05	9.199E-06	3.622E-06	1.415E-06
	1.147	1.229	1.277	1.308	1.329	1.345	1.356	
$\alpha = 1.5$	4.571E-04	1.935E-04	7.718E-05	2.977E-05	1.124E-05	4.180E-06	1.538E-06	5.615E-07
	1.240	1.326	1.374	1.406	1.427	1.442	1.454	
$\alpha = 1.6$	2.723E-04	1.027E-04	3.867E-05	1.404E-05	4.983E-06	1.740E-06	6.004E-07	2.054E-07
	1.407	1.408	1.461	1.495	1.518	1.535	1.547	
$\alpha = 1.7$	2.189E-04	5.902E-05	1.732E-05	5.980E-06	2.009E-06	6.622E-07	2.154E-07	6.936E-08
	1.891	1.769	1.534	1.574	1.601	1.620	1.635	
$\alpha = 1.8$	1.954E-04	5.001E-05	1.266E-05	3.186E-06	7.994E-07	2.208E-07	6.824E-08	2.083E-08
	1.966	1.982	1.990	1.995	1.856	1.694	1.712	
$\alpha = 1.9$	1.957E-04	4.998E-05	1.263E-05	3.175E-06	7.960E-07	1.993E-07	4.986E-08	1.247E-08
	1.970	1.984	1.992	1.996	1.998	1.999	1.999	

Table 8: Example II: Maximum errors and orders of convergence using a collocation method for $m = 1$ and $c_1 = 1/2$ on a graded mesh with $r = 1/(\alpha - 1)$

	N=32	N=64	N=128	N=256	N=512	N=1024	N=2048	N=4096
$\alpha = 1.1$	4.686E-03	1.347E-03	4.078E-04	2.305E-04	1.169E-04	5.667E-05	2.691E-05	1.266E-05
	1.799	1.724	0.823	0.979	1.045	1.074	1.088	
$\alpha = 1.2$	2.150E-03	1.057E-03	4.924E-04	2.204E-04	9.701E-05	4.238E-05	1.846E-05	8.034E-06
	1.024	1.102	1.159	1.184	1.195	1.199	1.200	
$\alpha = 1.3$	1.908E-03	8.072E-04	3.315E-04	1.345E-04	5.440E-05	2.199E-05	8.895E-06	3.602E-06
	1.241	1.284	1.301	1.306	1.307	1.306	1.304	
$\alpha = 1.4$	1.339E-03	5.069E-04	1.897E-04	7.084E-05	2.649E-05	9.934E-06	3.735E-06	1.407E-06
	1.401	1.418	1.421	1.419	1.415	1.411	1.408	
$\alpha = 1.5$	8.558E-04	2.953E-04	1.015E-04	3.498E-05	1.210E-05	4.208E-06	1.469E-06	5.143E-07
	1.535	1.540	1.537	1.531	1.524	1.519	1.514	
$\alpha = 1.6$	5.309E-04	1.679E-04	5.315E-05	1.690E-05	5.407E-06	1.740E-06	5.625E-07	1.826E-07
	1.661	1.659	1.653	1.644	1.636	1.629	1.623	
$\alpha = 1.7$	3.811E-04	9.938E-05	2.834E-05	8.311E-06	2.453E-06	7.287E-07	2.177E-07	6.535E-08
	1.939	1.810	1.770	1.760	1.751	1.743	1.736	
$\alpha = 1.8$	3.000E-04	7.754E-05	1.972E-05	4.972E-06	1.249E-06	3.289E-07	9.056E-08	2.506E-08
	1.952	1.976	1.987	1.994	1.925	1.861	1.854	
$\alpha = 1.9$	2.395E-04	6.144E-05	1.556E-05	3.916E-06	9.822E-07	2.460E-07	6.155E-08	1.539E-08
	1.963	1.981	1.991	1.995	1.998	1.999	1.999	

Table 9: Example II: Maximum derivative errors and orders of convergence using a collocation method for $m = 1$ and $c_1 = 1/2$ on a uniform mesh

	N=32	N=64	N=128	N=256	N=512	N=1024	N=2048	N=4096
$\alpha = 1.1$	1.130E-02	8.939E-03	7.133E-03	5.737E-03	4.648E-03	3.789E-03	3.106E-03	2.560E-03
	0.339	0.326	0.314	0.304	0.295	0.287	0.279	
$\alpha = 1.2$	1.141E-02	7.808E-03	5.430E-03	3.826E-03	2.725E-03	1.958E-03	1.418E-03	1.034E-03
	0.547	0.524	0.505	0.490	0.477	0.466	0.456	
$\alpha = 1.3$	7.765E-03	4.703E-03	2.899E-03	1.812E-03	1.145E-03	7.297E-04	4.682E-04	3.020E-04
	0.723	0.698	0.678	0.662	0.650	0.640	0.632	
$\alpha = 1.4$	4.590E-03	2.469E-03	1.350E-03	7.476E-04	4.178E-04	2.351E-04	1.329E-04	7.545E-05
	0.895	0.871	0.853	0.840	0.830	0.822	0.817	
$\alpha = 1.5$	2.506E-03	1.195E-03	5.777E-04	2.822E-04	1.388E-04	6.859E-05	3.402E-05	1.691E-05
	1.069	1.048	1.034	1.024	1.017	1.012	1.008	
$\alpha = 1.6$	1.293E-03	5.444E-04	2.318E-04	9.946E-05	4.289E-05	1.855E-05	8.042E-06	3.491E-06
	1.248	1.232	1.221	1.214	1.209	1.206	1.204	
$\alpha = 1.7$	6.380E-04	2.363E-04	8.828E-05	3.316E-05	1.250E-05	4.721E-06	1.785E-06	6.757E-07
	1.433	1.420	1.412	1.408	1.405	1.403	1.402	
$\alpha = 1.8$	3.027E-04	9.834E-05	3.216E-05	1.055E-05	3.472E-06	1.143E-06	3.768E-07	1.242E-07
	1.622	1.613	1.607	1.604	1.602	1.601	1.601	
$\alpha = 1.9$	1.983E-04	5.246E-05	1.385E-05	3.656E-06	9.651E-07	2.657E-07	7.627E-08	2.190E-08
	1.918	1.921	1.922	1.922	1.861	1.801	1.800	

Table 10: Example II: Maximum derivative errors and orders of convergence using a collocation method for $m = 1$ and $c_1 = 1/2$ on a graded mesh with $r = 1/(\alpha - 1)$

	N=32	N=64	N=128	N=256	N=512	N=1024	N=2048	N=4096
$\alpha = 1.1$	2.130E-02 0.998	1.066E-02 1.045	5.165E-03 1.070	2.460E-03 1.084	1.160E-03 1.091	5.447E-04 1.095	2.550E-04 1.097	1.192E-04
$\alpha = 1.2$	1.338E-02 1.116	6.172E-03 1.152	2.778E-03 1.172	1.233E-03 1.184	5.427E-04 1.190	2.378E-04 1.194	1.039E-04 1.197	4.533E-05
$\alpha = 1.3$	7.353E-03 1.214	3.170E-03 1.247	1.335E-03 1.268	5.546E-04 1.280	2.283E-04 1.288	9.352E-05 1.292	3.818E-05 1.295	1.556E-05
$\alpha = 1.4$	3.743E-03 1.303	1.517E-03 1.338	6.000E-04 1.360	2.338E-04 1.374	9.019E-05 1.383	3.458E-05 1.389	1.321E-05 1.393	5.030E-06
$\alpha = 1.5$	1.783E-03 1.385	6.829E-04 1.423	2.547E-04 1.447	9.341E-05 1.464	3.387E-05 1.475	1.218E-05 1.482	4.361E-06 1.488	1.555E-06
$\alpha = 1.6$	1.062E-03 1.687	3.298E-04 1.680	1.030E-04 1.553	3.510E-05 1.546	1.202E-05 1.561	4.074E-06 1.571	1.371E-06 1.578	4.593E-07
$\alpha = 1.7$	6.272E-04 1.783	1.822E-04 1.781	5.304E-05 1.774	1.551E-05 1.765	4.562E-06 1.757	1.350E-06 1.749	4.017E-07 1.667	1.265E-07
$\alpha = 1.8$	3.720E-04 1.867	1.020E-04 1.869	2.793E-05 1.868	7.652E-06 1.865	2.101E-06 1.860	5.787E-07 1.856	1.599E-07 1.851	4.432E-08
$\alpha = 1.9$	2.228E-04 1.933	5.836E-05 1.938	1.523E-05 1.940	3.968E-06 1.941	1.033E-06 1.941	2.691E-07 1.941	7.011E-08 1.940	1.828E-08

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