Monografías Matemáticas *García de Galdeano* **42**, 65–78 (2019)

A PROFILE DECOMPOSITION FOR THE LIMITING SOBOLEV EMBEDDING

Giuseppe Devillanova and Cyril Tintarev

Abstract. For many known non-compact embeddings of two Banach spaces $E \hookrightarrow F$, every bounded sequence in E has a subsequence that takes the form of a *profile decomposition* - a sum of clearly structured terms with asymptotically disjoint supports plus a remainder that vanishes in the norm of F. In this note we construct a profile decomposition for arbitrary sequences in the Sobolev space $H^{1,2}(M)$ of a compact Riemannian manifold, relative to the embedding of $H^{1,2}(M)$ into $L^{2^*}(M)$, generalizing the well-known profile decomposition of Struwe [12, Proposition 2.1] to the case of arbitrary bounded sequences.

Keywords: concentration compactness, profile decompositions, multiscale analysis.

AMS classification: 46E35, 46B50, 58J99, 35B44, 35A25.

§1. Introduction

When the embedding of two Banach spaces $E \hookrightarrow F$ is continuous and not compact, the lack of compactness can be manifested by the (behavior in F of the) difference $u_k - u$ between the elements of a weakly convergent sequence $(u_k)_{k\in\mathbb{N}} \subset E$ and its weak limit u. Therefore one may call *defect of compactness* of $(u_k)_{k\in\mathbb{N}}$ the (sequences of) differences $u_k - u$ taken up to a suitable remainder that vanishes in the norm of F. (Note that, if the embedding is compact and E is reflexive, the defect of compactness is itself infinitesimal and so it can be identified with zero). For many embeddings there exist well-structured representations of the defect of compactness, known as *profile decompositions*. Best studied are profile decompositions relative to Sobolev embeddings, which are sums of terms with asymptotically disjoint supports, called *elementary concentrations* or *bubbles*. Profile decompositions were originally motivated by studies of concentration phenomena in PDE in the early 1980's by Uhlenbeck, Brezis, Coron, Nirenberg, Aubin and Lions, and they play a significant role in the verification process of the convergence of sequences of functions in applied analysis, particularly when the information available via the classical concentration-compactness method is not enough detailed.

Profile decompositions are known to exist when the embedding $E \hookrightarrow F$ is *cocompact* relative to some group $\mathcal{G}$ of isometries on E, see [11]. We recall that an embedding $E \hookrightarrow F$ is called cocompact relative to a group $\mathcal{G}$ of isometries ($\mathcal{G}$-cocompact for short) if any sequence $(u_k)_{k\in\mathbb{N}} \subset E$ such that $g_k(u_k) \rightharpoonup 0$ for any sequence of operators $(g_k)_{k\in\mathbb{N}} \subset \mathcal{G}$ turns out to be infinitesimal in the norm of F. (An elementary example due to Jaffard [7], which is easy to verify, is cocompactness of the embedding of $\ell^\infty(\mathbb{Z})$ into itself relative to the group of shifts $\mathcal{G} := \{g_m := (a_n)_{n\in\mathbb{N}} \mapsto (a_{n+m})_{n\in\mathbb{N}} \mid m \in \mathbb{Z}\}$.) Up to the authors knowledge the first cocompactness result for functional spaces is [8, Lemma 6] by E. Lieb which expresses (using different terminology) that the nonhomogeneous Sobolev space $H^{1,p}(\mathbb{R}^N)$ is cocompactly embedded into $L^q(\mathbb{R}^N)$, when $N > p$ and $q \in (p, p^*)$ (where $p^* = \frac{Np}{N-p}$), relative to the group of shifts $u \mapsto u(\cdot - y)$, $y \in \mathbb{R}^N$. A profile decomposition relative to a group $\mathcal{G}$ of bijective isometries on a Banach space E represents defect of compactness $u_k - u$ as a sum of *elementary concentrations*, or *bubbles*, namely $\sum_{n\in\mathbb{N}\setminus\{0\}} g_k^{(n)} w^{(n)}$ with some $g_k^{(n)} \in \mathcal{G}$ and $w^{(n)} \in E$. Note that in the above sum the index $n = 0$ is not allowed since, in the existing literature, usually $w^{(0)}$ represents the weak-limit u of the sequence and $(g_k^{(0)})_{k\in\mathbb{N}}$ is the constant sequence of constant value the identity map of the space. So, by using this convention, we can use defect of compactness to represent the sequence $(u_k)_{k\in\mathbb{N}}$ as a sum of $\sum_{n\in\mathbb{N}} g_k^{(n)} w^{(n)}$ and a remainder vanishing in F. In the above sums each of the elements $w^{(n)}$ (for $n \geq 1$), called *concentration profiles*, is obtained as the weak-limit (as $k \to \infty$) of the "deflated" sequence $((g_k^{(n)})^{-1}(u_k))_{k\in\mathbb{N}}$.

Typical examples of isometry groups $\mathcal{G}$, involved in profile decompositions, are the above mentioned group of shifts $u \mapsto u(\cdot - y)$ and the rescaling group, which is a product group of shifts and dilations $u \mapsto t^r u(t\cdot)$, $t > 0$, where, for instance, when u belongs to the homogeneous Sobolev space $\dot{H}^{s,p}(\mathbb{R}^N)$ $(N/s > p \geq 1,\ s > 0)$, $r = r(p, s) = \frac{N-ps}{p}$.

Existence of profile decompositions for general bounded sequences in $\dot{H}^{1,p}(\mathbb{R}^N)$ (relative to the rescaling group) was proved by Solimini, see [10, Theorem 2], and independently, but with a weaker form of remainder, by Gérard in [6], with an extension to the case of fractional Sobolev spaces by Jaffard in [7]. Only in [9], for the first time, the authors observed that profile decomposition (and thus concentration phenomena in general) can be understood in functional-analytic terms, rather than in specific function spaces. Actually the results in [9] where extended in [11] to uniformly convex Banach spaces with the Opial condition (without the Opial condition profile decomposition still exists but weak convergence must be replaced by (a less-known) Delta convergence, see [4]). Finally the result has been extended up to a suitable class of metric spaces, see [5] and [3]. Despite the character of the statement in [11] is rather general, profile decompositions are still true, for instance, when the space E is not reflexive (e.g. [2]), or when one only has a semigroup of isometries (e.g. [1]), or when the profile decomposition can be expressed without the explicit use of a group (e.g. Struwe [12]) and so when [11, Theorem 2.10] does not apply.

The present paper generalizes, in the spirit of [10, Theorem 2], Struwe's result [12, Proposition 2.1] (which provides a profile decomposition for Palais-Smale sequences of particular functionals) to the case of general bounded sequences in $\dot{H}^{1,2}(M)$, where M is a smooth compact manifold in dimension $N \geq 3$.

The paper is organized as follows. In Section 2 we introduce some notation and state the main theorem of the paper and the result on which the related proof is based. In Section 3 we prove that the embedding $H^{1,2}(M) \hookrightarrow L^{2^*}(M)$ is cocompact with respect to a group of suitable transformations which are depending on the atlas associated to the manifold. Section 4 is devoted to the proof of (the main) Theorem 1.

§2. Statement of the main result

Let $N \geq 3$ and let (M, g) be a compact smooth Riemannian N-dimensional manifold. We consider the Sobolev space $H^{1,2}(M)$ equipped with the norm defined by the quadratic form of the Laplace-Beltrami operator,

$$\|u\|^2 = \int_M (|\mathrm{d}u|^2 + u^2)\mathrm{d}v_g, \tag{1}$$

(v_g denotes the Riemannian measure of the manifold). For every $y \in M$ we shall denote by $T_y(M)$ the tangent space in y to M, and by $\exp_y$ the exponential (local) map at the point y (defined on a suitable set $U_y \subset T_y(M)$ by setting, for all $v \in U_y$, $\exp_y(v) := \gamma_v(1)$ where γ_v is the unique geodesic, contained in M, such that $\gamma_v(0) = y$ and $\gamma_v'(0) = v$ and extended to the case $v = 0$ by setting $\exp_y(0) = y$). Since we will not use here any property of tangent bundles we will identify tangent spaces of M at different points with $\mathbb{R}^N$ and, for any $\rho > 0$, we shall denote by $B_\rho(0)$ the Euclidean N-dimensional ball centered at the origin with radius ρ. On the other hand, we shall denote by $\mathcal{B}_\rho(y)$ the open coordinate ball (i.e. the subset in M such that $\exp_y^{-1}(\mathcal{B}_\rho(y)) = B_\rho(0)$) with center y and radius $\rho > 0$. For the reader's convenience we recall that the injectivity radius ρ_y of a point $y \in M$ is the radius of the largest ball about the origin in $T_y(M)$ that can be mapped diffeomorfically via the map $\exp_y$, and that, the injectivity radius of the mainfold M, $\rho_M := \inf_{y\in M} \rho_y$. Since M is compact, ρ_M is strictly positive, so we can fix $0 < \rho < \frac{\rho_M}{3}$, moreover, there exists a finite set of points $(z_i)_{i\in I} \subset M$ such that $\left(\mathcal{B}_\rho(z_i), \exp_{z_i}^{-1}\right)_{i\in I}$ is a finite smooth atlas of M.

In what follows we shall fix $\chi \in C_0^\infty(B_\rho(0))$ equal 1 in a neighborhood of 0, so that, setting for $i \in I$

$$\hat{\chi}_i := \hat{\chi}_{z_i} = \chi \circ \exp_{z_i}^{-1} \qquad \text{and} \qquad \chi_i := \frac{\hat{\chi}_i}{\sum_{j\in I} \hat{\chi}_j}, \tag{2}$$

$(\chi_i)_{i\in I}$ is a smooth partition of unity on M subordinated to the covering $(\mathcal{B}_\rho(z_i))_{i\in I}$. Then, since $\|u \circ \exp_{z_i}\|_{L^{2^*}(B_\rho(0))}$ is bounded by the $H^{1,2}(B_\rho(0))$-norm of $u \circ \exp_{z_i}$, the Sobolev embedding $H^{1,2}(M) \hookrightarrow L^{2^*}(M)$ can be deduced from the corresponding one on the Euclidean space (by the use of the fixed partition of unity $(\chi_i)_{i\in I}$). In fact, Theorem 1 below will provide a profile decomposition for bounded sequences in $H^{1,2}(M)$.

Finally we recall that the scalar product associated with (1) can be written with help of the partition of unity $(\chi_s)_{s\in I}$ in the following coordinate form:

$$\begin{aligned}\langle \Phi, \Psi \rangle := &\sum_{s\in I} \int_{B_\rho(0)} \sum_{i,j=1}^{N} g_{ij}^{z_s} \partial_i((\chi_s \Phi)(\exp_{z_s}(\xi)))\partial_j(\Psi(\exp_{z_s}(\xi))) \sqrt{\det(g_{ij}^{z_s})}\mathrm{d}\xi \\ &+ \sum_{s\in I} \int_{B_\rho(0)} (\chi_s \Phi)(\exp_{z_s}(\xi))\Psi(\exp_{z_s}(\xi)) \sqrt{\det(g_{ij}^{z_s})}\mathrm{d}\xi.\end{aligned} \tag{3}$$

Before stating the theorem, we warn the reader that, given a bounded sequence $(v_k)_{k\in\mathbb{N}} \subset H^{1,2}(B_\rho(0))$ and a vanishing sequence of positive numbers $(t_k)_{k\in\mathbb{N}}$, and setting $r = r(2) = \frac{N}{2^*} = \frac{N-2}{2}$, we will say (with a slight abuse on the definition of weak convergence) that the sequence $(t_k^r v_k(t_k \cdot))_{k\in\mathbb{N}}$ weakly converges to $v \in \dot{H}^{1,2}(\mathbb{R}^N)$ if, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}\varphi \subset B_\rho(0)$,

$$\int \varphi(x)\, t_k^r v_k(t_k x)\,\mathrm{d}x \longrightarrow \int \varphi(x)\, v(x)\,\mathrm{d}x \text{ as } k \to \infty.$$

Theorem 1. *Let M be a compact smooth Riemannian N-dimensional manifold ($N \geq 3$). Let $r = \frac{N}{2^*} = \frac{N-2}{2}$, let $\rho \in (0, \frac{\rho_M}{3})$, let $\chi \in C_0^\infty(B_\rho(0))$, $\chi = 1$ on $B_{\frac{\rho}{2}}(0)$, and let $(\chi_i)_{i\in I}$, defined by (2), be a smooth partition of unity on M subordinated to the covering $(\mathcal{B}_\rho(z_i))_{i\in I}$. Then, any bounded sequence $(u_k)_{k\in\mathbb{N}}$ in $H^{1,2}(M)$ has a renamed subsequence for which there exist:*

- *a sequence $\left(Y^{(n)}\right)_{n\in\mathbb{N}\setminus\{0\}}$ of sequences $Y^{(n)} := \left(y_k^{(n)}\right)_{k\in\mathbb{N}} \subset M$, $y_k^{(n)} \to \bar{y}^{(n)} \in M$,*
- *a sequence $\left(J^{(n)}\right)_{n\in\mathbb{N}\setminus\{0\}}$ of sequences $J^{(n)} := \left(j_k^{(n)}\right)_{k\in\mathbb{N}} \subset \mathbb{R}_+$,*
- *a sequence $\left(w^{(n)}\right)_{n\in\mathbb{N}\setminus\{0\}}$ of functions (profiles) $w^{(n)} \in \dot{H}^{1,2}(\mathbb{R}^N)$,*

such that,

$$j_k^{(n)} \longrightarrow \infty \text{ as } k \to \infty \quad \forall n \in \mathbb{N} \setminus \{0\}, \tag{4}$$

$$|j_k^{(n)} - j_k^{(m)}| + 2^{j_k^{(n)}} d(y_k^{(n)}, y_k^{(m)}) \to \infty \text{ whenever } m \neq n, \tag{5}$$

$$2^{-j_k^{(n)} r} u_k \circ \exp_{y_k^{(n)}}(2^{-j_k^{(n)}} \cdot) \rightharpoonup w^{(n)} \text{ in } \dot{H}^{1,2}(\mathbb{R}^N) \text{ as } k \to \infty. \tag{6}$$

Moreover, setting for all $k \in \mathbb{N}$

$$\mathcal{S}_k(x) := \sum_{n\in\mathbb{N}\setminus\{0\}} 2^{j_k^{(n)} r}\, \chi \circ \exp^{-1}_{y_k^{(n)}}(x)\, w^{(n)} \left(2^{j_k^{(n)}} \exp^{-1}_{y_k^{(n)}}(x)\right), \; x \in M, \tag{7}$$

for each $k \in \mathbb{N}$ the series $\mathcal{S}_k$ is unconditionally convergent (with respect to n) in $\dot{H}^{1,2}(M)$ and the sequence $(\mathcal{S}_k)_{k\in\mathbb{N}}$ is uniformly convergent (with respect to k) in $\dot{H}^{1,2}(M)$, and in addition

$$u_k - u - \mathcal{S}_k \to 0 \text{ in } L^{2^*}(M)\,. \tag{8}$$

Finally the following energy bound holds

$$\sum_{n\in\mathbb{N}\setminus\{0\}} \left\|\nabla w^{(n)}\right\|^2_{L^2(\mathbb{R}^N)} + \|u\|^2_{H^{1,2}(M)} \leq \liminf_{k\to\infty} \|u_k\|^2_{H^{1,2}(M)}\,. \tag{9}$$

We want to emphasize that (8) states that, modulo subsequence, the defect of compactness $u_k - u$ of the bounded sequence $(u_k)_{k\in\mathbb{N}}$ (which, modulo subsequence, weakly converges to u) has a representation given (up to a remainder which vanishes in the norm of $L^{2^*}(M)$) by the clearly structured terms in $\mathcal{S}_k$.

The proof of this theorem is based on the following easy corollary to Solimini's profile decomposition [10, Theorem 2].

Theorem 2. *Given $m \in \mathbb{N} \setminus \{0\}$ and $1 < p < \frac{N}{m}$ let $r = \frac{N}{p^*(m)} = \frac{N-mp}{p}$. Let $(v_k)_{k\in\mathbb{N}}$ be a bounded sequence in the homogeneous Sobolev space $\dot{H}^{m,p}(\mathbb{R}^N)$ supported on a compact set $K \subset \mathbb{R}^N$. Then, there exists a (renamed) subsequence (s.t. $v_k \rightharpoonup v$) whose defect of compactness $v_k - v$ has the form*

$$S_k = \sum_{n\in\mathbb{N}\setminus\{0\}} 2^{j_k^{(n)} r} w^{(n)}(2^{j_k^{(n)}}(\cdot - \xi_k^{(n)})), \tag{10}$$

where, for any $n \in \mathbb{N} \setminus \{0\}$, $\Xi^{(n)} := (\xi_k^{(n)})_{k\in\mathbb{N}}$, $\xi_k^{(n)} \to \xi_n \in K$, and $J^{(n)} := (j_k^{(n)})_{k\in\mathbb{N}} \subset \mathbb{R}$ are such that $j_k^{(n)} \to +\infty$ as $k \to \infty$ and $w^{(n)}$ is the weak limit of the sequence $\left(2^{-j_k^{(n)} r} v_k(2^{-j_k^{(n)}} \cdot + \xi_k^{(n)})\right)_{k\in\mathbb{N}}$. Moreover the addenda are asymptotically mutually orthogonal, i.e.

$$|j_k^{(n)} - j_k^{(m)}| + 2^{j_k^{(n)}} |\xi_k^{(n)} - \xi_k^{(m)}| \to \infty \text{ whenever } m \neq n. \tag{11}$$

Proof. We shall assume, without restrictions, that $u_k \rightharpoonup 0$. According to the profile decomposition result [10, Theorem 2], modulo the extraction of a subsequence, each term v_k has concentration terms (depending on n) of the following shape

$$c_k^n := 2^{j_k^{(n)} r} w^{(n)}(2^{j_k^{(n)}}(\cdot - \xi_k^{(n)})) \tag{12}$$

for some $\xi_k^{(n)} \in \mathbb{R}^N$, $j_k^{(n)} \in \mathbb{R}$ where $w^{(n)}$ is obtained as the weak limit of the sequence $\left(2^{-j_k^{(n)} r} v_k(2^{-j_k^{(n)}} \cdot + \xi_k^{(n)})\right)_{k\in\mathbb{N}}$. We claim that the sequence $J^{(n)} := (j_k^{(n)})_{k\in\mathbb{N}}$ is bounded from below. Indeed, on the contrary, the assumption $j_k^{(n)} \to -\infty$ as $k \to \infty$ would imply, since v_k has a bounded support, that

$$\left\| 2^{-j_k^{(n)} r} v_k \left(2^{-j_k^{(n)}} \cdot + \xi_k^{(n)}\right) \right\|_p \to 0 \text{ as } k \to \infty,$$

and so that $w^{(n)} = 0$.

Note that $J^{(n)}$ cannot have any bounded subsequence, since otherwise $(v_k)_{k\in\mathbb{N}}$ should have a nonzero weak limit, in contradiction to our assumptions. By passing to convergent subsequences and subsequent diagonalization we easily get $\xi_k^{(n)} \to \xi_n \in K$.

Finally, condition (11) is the condition of asymptotic orthogonality (decoupling) of bubbles from [10]. □

§3. Cocompactness in Sobolev spaces of compact manifolds

The Sobolev embedding $H^{1,2}(M) \hookrightarrow L^{2^*}(M)$ has the following property of cocompactness type.

Theorem 3. *Let M be a compact smooth Riemannian N-dimensional manifold ($N \geq 3$), and $0 < \rho < \frac{\rho_M}{3}$. Let $\left(\mathcal{B}_\rho(z_i), \exp_{z_i}^{-1}\right)_{i\in I}$ be a finite smooth atlas of M and let $\chi \in C_0^\infty(B_\rho(0))$ so that $(\chi_i)_{i\in I}$, defined by (2), is a smooth partition of unity on M subordinated to the covering $(\mathcal{B}_\rho(z_i))_{i\in I}$. Set $r = r(2) = \frac{N}{2^*} = \frac{N-2}{2}$. If $(u_k)_{k\in\mathbb{N}}$ is any bounded sequence in $H^{1,2}(M)$ such that for every $i \in I$, $(y_k)_{k\in\mathbb{N}} \subset \mathcal{B}_\rho(z_i)$, and for every $(j_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $j_k \to +\infty$*

$$2^{-j_k r}(\chi_i u_k) \circ \exp_{y_k}(2^{-j_k} \cdot) \rightharpoonup 0 \text{ as } k \to \infty, \tag{13}$$

then $u_k \to 0$ in $L^{2^*}(M)$.

Proof. We claim that for all sequences $(\xi_k)_{k\in\mathbb{N}} \subset \mathbb{R}^N$ and $(j_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $j_k \to +\infty$ and for every $i \in I$ we have

$$2^{-j_k r}(\chi_i u_k) \circ \exp_{z_i}(2^{-j_k} \cdot + \xi_k) \rightharpoonup 0 \quad \text{as } k \to \infty. \tag{14}$$

Since (14) is obviously true when $|\xi_k| \geq \rho$, (indeed the terms in (14) are identically zero for k large enough), we shall assume $\xi_k \in B_\rho(0)$ for all $k \in \mathbb{N}$. Given $i \in I$, we set $y_k := \exp_{z_i}(\xi_k) \in M$ and denote by ψ_k the transition map between the charts $(\mathcal{B}_\rho(z_i), \exp_{z_i}^{-1})$ and $(\mathcal{B}_\rho(y_k), \exp_{y_k}^{-1})$ i.e. we set $\psi_k := \exp_{y_k}^{-1} \circ \exp_{z_i}$ (so that $\exp_{z_i} = \exp_{y_k} \circ \psi_k$ and $\psi_k(\xi_k) = 0$). Therefore, for k large enough, by using Taylor expansion of the first order at ξ_k (where, for a lighter notation, we denote by $\psi_k'(\xi_k)$ the Jacobi matrix of ψ_k at ξ_k $(\psi_k'(\xi_k))^{-1}$ its inverse and by $|(\psi_k'(\xi_k))^{-1}|$ the corresponding Jacobian, and drop the dot symbol for the rows-by-columns product) we get, since $j_k \to +\infty$, that

$$\begin{aligned} 2^{-j_k r}(\chi_i u_k)(\exp_{z_i}(2^{-j_k}\xi + \xi_k)) &= 2^{-j_k r}(\chi_i u_k)(\exp_{y_k} \circ \psi_k)(2^{-j_k}\xi + \xi_k) \\ &= 2^{-j_k r}(\chi_i u_k)(\exp_{y_k}(2^{-j_k}(\psi_k'(\xi_k) + o(1))\xi)). \end{aligned} \tag{15}$$

(we are using the Landau symbol $o(1)$ to denote any (matrix valued) function uniformly convergent to zero). In correspondence to any test function $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\begin{aligned} &\int_{B_{2\rho}(0)} \varphi(\xi) 2^{-j_k r}\left[(\chi_i u_k) \circ \exp_{z_i}(2^{-j_k}\xi + \xi_k) - (\chi_i u_k) \circ \exp_{y_k}(2^{-j_k}\psi_k'(\xi_k)\xi)\right] \mathrm{d}\xi \\ &= \int_{B_{2\rho}(0)} \varphi(\xi) 2^{-j_k r}\left[(\chi_i u_k) \circ \exp_{y_k} \circ \psi_k(2^{-j_k}\xi + \xi_k) - (\chi_i u_k) \circ \exp_{y_k}(2^{-j_k}\psi_k'(\xi_k)\xi)\right] \mathrm{d}\xi \\ &= |(\psi_k'(\xi_k))^{-1}| 2^{j_k \frac{N+2}{2}} \int_{|\eta|<C2^{-j_k}} \varphi(2^{j_k}(\psi_k'(\xi_k))^{-1}\eta) \\ &\qquad \times \left[(\chi_i u_k) \circ \exp_{y_k}\left(\psi_k((\psi_k'(\xi_k))^{-1}\eta + \xi_k\right) - (\chi_i u_k) \circ \exp_y(\eta)\right] \mathrm{d}\eta \\ &= |(\psi_k'(\xi_k))^{-1}| 2^{j_k \frac{N+2}{2}} \int_0^1 \mathrm{d}s \int_{|\eta|<C2^{-j_k}} \varphi(2^{j_k}(\psi_k'(\xi_k))^{-1}\eta) \\ &\quad \times \nabla\Big((\chi_i u_k) \circ \exp_{y_k}(s\psi_k\left((\psi_k'(\xi_k))^{-1}\eta + \xi_k\right) + (1-s)\eta)\Big) \cdot (\psi_k((\psi_k'(\xi_k))^{-1}\eta + \xi_k) - \eta)\mathrm{d}\eta, \end{aligned}$$

(the second equality holds by integrating with respect to the variable $\eta = 2^{-j_k}\psi_k'(\xi_k)\xi$). Set, for each $s \in [0,1]$, $\zeta := s\psi_k\left((\psi_k'(\xi_k))^{-1}\eta + \xi_k\right) + (1-s)\eta$, since for $\eta \to 0$, $\zeta = \eta + O(|\eta|^2)$ and since the Jacobian of the transformation is close to 1 in the domain of integration, the modulus of the last expression is bounded by the following one, which, in turn, can be estimated by Cauchy inequality, so we have

$$\begin{aligned} &C 2^{j_k \frac{N+2}{2}} \int_{|\zeta|<C2^{-j_k}} \varphi(2^{j_k}(\psi_k'(\xi_k))^{-1}\eta(\zeta)) |\nabla(\chi_i u_k) \circ \exp_{y_k}(\zeta)| |\zeta|^2 \mathrm{d}\zeta \\ &\le C 2^{j_k \frac{N+2}{2}} \|\nabla(\chi_i u_k) \circ \exp_{y_k}\|_2 \left(\int_{|\zeta|<C2^{-j_k}} |\varphi(2^{j_k}(\psi_k'(\xi_k))^{-1}\eta(\zeta))|^2 |\zeta|^4 \mathrm{d}\zeta\right)^{\frac{1}{2}} \\ &\le C 2^{j_k \frac{N+2}{2}} \|u_k\|_{H^{1,2}(M)} \left(\int_{|\xi|<C} |\varphi(\xi)|^2 2^{-4j_k} |\xi|^4 2^{-j_k N} \mathrm{d}\xi\right)^{\frac{1}{2}} \le C 2^{-j_k} \longrightarrow 0. \end{aligned}$$

Therefore, by taking into account (15), we deduce that both sequences $\left(2^{-j_k r}(\chi_i u_k)(\exp_{y_k}(2^{-j_k}\cdot))\right)_{k\in\mathbb{N}}$ and $\left(2^{-j_k r}(\chi_i u_k)(\exp_{z_i}(2^{-j_k}\ \cdot + \xi_k))\right)_{k\in\mathbb{N}}$ have the same weak limit and, since (13) holds true, (14) holds too.

Consequently, from the cocompactness of the embedding $\dot{H}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ ([10, Theorem 1]), it follows that, for every $i \in I$,

$$(\chi_i u_k) \circ \exp_{z_i} \to 0 \quad \text{in } L^{2^*}(\mathbb{R}^N) \text{ as } k \to \infty, \tag{16}$$

and therefore, since $(\chi_i)_{i\in I}$ is a partition of unity subordinated to the atlas $\left(\mathcal{B}_\rho(z_i), \exp_{z_i}^{-1}\right)_{i\in I}$, we deduce that

$$\int_M |u_k|^{2^*} \mathrm{d}v_g = \int_M \left|\sum_{i\in I} \chi_i u_k\right|^{2^*} \mathrm{d}v_g \le C \sum_{i\in I} \int_{\mathcal{B}_\rho(z_i)} |\chi_i u_k|^{2^*} \mathrm{d}v_g$$
$$\le C \sum_{i\in I} \int_{B_\rho(0)} |u_k \circ \exp_{z_i}(\xi)|^{2^*} \mathrm{d}\xi \to 0,$$

which proves the statement of the theorem. □

§4. Proof of Theorem 1 (profile decomposition)

1. Without loss of generality we may assume (by replacing u_k with $u_k - u$) that $u_k \rightharpoonup 0$.

Then, setting for all $i \in I$

$$v_{k,i} := (\chi_i u_k) \circ \exp_{z_i} \tag{17}$$

we get that the sequence $(v_{k,i})_{k\in\mathbb{N}}$ is bounded in $H_0^{1,2}(B_\rho(0))$ (and weakly converges to zero), and so we can consider a profile decomposition of $(v_{k,i})_{k\in\mathbb{N}}$ given by Theorem 2 when $m = 1$ and $r = \frac{N-2}{2}$. An iterated extraction allows to find a subsequence which has a profile decomposition for every $i \in I$ i.e. such that for all $i \in I$ the defect of compactness of $v_{k,i}$ has the following form

$$S_{k,i} = \sum_{n\in\mathbb{N}\setminus\{0\}} 2^{j_{k,i}^{(n)} r} w_i^{(n)} \left(2^{j_{k,i}^{(n)}} \left(\cdot - \xi_{k,i}^{(n)}\right)\right) =: \sum_{n\in\mathbb{N}\setminus\{0\}} c_{k,i}^{(n)}. \tag{18}$$

By taking into account (17) we will be able to get concentration terms of $\chi_i u_k$ by composing each concentration term $c_{k,i}^{(n)}$ of $v_{k,i}$ with $\exp_{z_i}^{-1}$. In more detail, we consider for all $i \in I$ the term, defined on $\mathcal{B}_\rho(z_i)$,

$$C_{k,i}^{(n)} := c_{k,i}^{(n)} \circ \exp_{z_i}^{-1} = 2^{j_{k,i}^{(n)} r} w_i^{(n)} \left(2^{j_{k,i}^{(n)}} \left(\exp_{z_i}^{-1}(\cdot) - \xi_{k,i}^{(n)}\right)\right). \tag{19}$$

Setting

$$y_{k,i}^{(n)} := \exp_{z_i}(\xi_{k,i}^{(n)}), \tag{20}$$

we have that

$$C_{k,i}^{(n)} = 2^{j_{k,i}^{(n)} r} w_i^{(n)} \left(2^{j_{k,i}^{(n)}} \left(\exp_{z_i}^{-1}(\cdot) - \exp_{z_i}^{-1}(y_{k,i}^{(n)})\right)\right). \tag{21}$$

Since for all $i \in I$ and $n \in \mathbb{N} \setminus \{0\}$

$$w_i^{(n)} := \operatorname*{w-lim}_{k\to\infty} 2^{-j_{k,i}^{(n)} r} (\chi_i u_k) \circ \exp_{z_i} \left(2^{-j_{k,i}^{(n)}} \cdot + \xi_{k,i}^{(n)}\right), \tag{22}$$

we can see that $w_i^{(n)}$ "evaluates" $\chi_i u_k$ on points belonging to $\mathcal{B}_\rho(z_i)$ which are mapped by $\exp_{z_i}^{-1}$ in subsets of $B_\rho(0)$ which are (for large k) concentrated around the points $\xi_{k,i}^{(n)}$. So, due to (20), it is sufficient to evaluate $w_i^{(n)}$ on points which belong also to $\mathcal{B}_\rho(y_{k,i}^{(n)})$. So, setting

$$B_{i,k,n} := \exp_{y_{k,i}^{(n)}}^{-1}(\mathcal{B}_\rho(y_{k,i}^{(n)}) \cap \mathcal{B}_\rho(z_i)) \subset B_\rho(0), \tag{23}$$

we shall consider the transition map between the charts $(\mathcal{B}_\rho(y_{k,i}^{(n)}), \exp_{y_{k,i}^{(n)}}^{-1})$ and $(\mathcal{B}_\rho(z_i), \exp_{z_i}^{-1})$, i.e. the map

$$\psi_{i,k,n} := \exp_{z_i}^{-1} \circ \exp_{y_{k,i}^{(n)}} \tag{24}$$

defined on $B_{i,k,n}$. Note that $\psi_{i,k,n}(0) = \xi_{k,i}^{(n)}$, moreover, by setting for any $x \in B_{i,k,n}$

$$\eta := 2^{j_{k,i}^{(n)}} \exp_{y_{k,i}^{(n)}}^{-1}(x), \tag{25}$$

we have $\exp_{z_i}^{-1}(x) = \psi_{i,k,n}(2^{-j_{k,i}^{(n)}}\eta)$ for all $x \in B_{i,k,n}$. Therefore (by using Taylor expansion of the first order of the transition map $\psi_{i,k,n}$ at 0, where, to use a lighter notation we denote by $\psi'_{i,k,n}(0)$ the Jacobi matrix of $\psi_{i,k,n}$ at zero, $(\psi'_{i,k,n}(0))^{-1}$ its inverse and omit the dot symbol for the rows-by-columns product) we deduce

$$\begin{aligned} 2^{j_{k,i}^{(n)}}\left(\exp_{z_i}^{-1}(x) - \xi_{k,i}^{(n)}\right) &= 2^{j_{k,i}^{(n)}}\left(\psi_{i,k,n}(2^{-j_{k,i}^{(n)}}\eta) - \xi_{k,i}^{(n)}\right) = 2^{j_{k,i}^{(n)}}\left(\psi_{i,k,n}(2^{-j_{k,i}^{(n)}}\eta) - \psi_{i,k,n}(0)\right) \\ &= \psi'_{i,k,n}(0)\eta + O(2^{-j_{k,i}^{(n)}}\eta^2) = 2^{j_{k,i}^{(n)}}\psi'_{i,k,n}(0)\exp_{y_{k,i}^{(n)}}^{-1}(x) + O\left(2^{j_{k,i}^{(n)}}\left(\exp_{y_{k,i}^{(n)}}^{-1}(x)\right)^2\right). \end{aligned} \tag{26}$$

Without loss of generality, applying Arzelà-Ascoli theorem and passing to a suitable subsequence, we can assume that $(\psi_{i,k,n})_{k\in\mathbb{N}}$ converges in the norm of $C^1(\mathbb{R}^N)$ as $k \to \infty$ to some function $\psi_{i,n}$. We claim that, under a suitable renaming of the profile $w_i^{(n)}$, namely by renaming $w_i^{(n)}(\psi'_{i,n}(0)\ \cdot)$ as $w_i^{(n)}$, concentration terms $C_{k,i}^{(n)}$ (of $\chi_i u_k$) in (19) (which we now extend to functions on the whole manifold by multiplying with a cut-off function, cf. (2)) take the following form:

$$\widetilde{C}_{k,i}^{(n)} := 2^{j_{k,i}^{(n)} r}\chi \circ \exp_{y_{k,i}^{(n)}} w_i^{(n)}\left(2^{j_{k,i}^{(n)}}\exp_{y_{k,i}^{(n)}}^{-1}(\cdot)\right). \tag{27}$$

For this purpose we show that, as $k \to \infty$,

$$\int_{\mathcal{B}_\rho(y_{k,i}^{(n)})\cap\mathcal{B}_\rho(z_i)} \left|2^{j_{k,i}^{(n)} r}\mathrm{d}\left(w_i^{(n)}\left(2^{j_{k,i}^{(n)}}(\exp_{z_i}^{-1}(x) - \xi_{k,i}^{(n)})\right) - w_i^{(n)}\left(2^{j_{k,i}^{(n)}}\psi'_{i,n}(0)\exp_{y_{k,i}^{(n)}}^{-1}(x)\right)\right)\right|^2 \mathrm{d}v_g \to 0. \tag{28}$$

Indeed, the previous relation written under the coordinate map $\exp_{y_{k,i}^{(n)}}$, i.e. by setting $\xi = \exp_{y_{k,i}^{(n)}}^{-1}(x)$ becomes (by taking into account (24) and (23))

$$\int_{B_{i,k,n}} \left|2^{j_{k,i}^{(n)} r}\nabla\left(w_i^{(n)}\left(2^{j_{k,i}^{(n)}}(\psi_{i,k,n}(\xi) - \xi_{k,i}^{(n)})\right) - w_i^{(n)}\left(2^{j_{k,i}^{(n)}}\psi'_{i,n}(0)\xi\right)\right)\right|^2 \mathrm{d}\xi \to 0 \text{ as } k \to \infty,$$

and, by taking into account (25) (and by a null extension to whole of $\mathbb{R}^N$ of the involved functions), the claim will follow if, as $k \to \infty$,

$$2^{-j_{k,i}^{(n)}\frac{N+2}{2}} \int_{\mathbb{R}^N} \left|\psi'_{i,k,n}(2^{-j_{k,i}^{(n)}}\eta)\nabla w_i^{(n)}\left(2^{j_{k,i}^{(n)}}(\psi_{i,k,n}(2^{-j_{k,i}^{(n)}}\eta) - \xi_{k,i}^{(n)})\right) - \psi'_{i,n}(0)\nabla w_i^{(n)}(\psi'_{i,n}(0)\eta)\right|^2 \mathrm{d}\eta \to 0.$$

This last convergence easily follows by Lebesgue dominated convergence theorem, indeed (for all n and for all i) $\nabla w_i^{(n)} \in L^2(\mathbb{R}^N)$, and when $k \to \infty$, we have $j_{k,i}^{(n)} \to +\infty$, and (by taking into account that convergence of $(\psi_{i,k,n})_{k\in\mathbb{N}}$ and $\left(\psi'_{i,k,n}\right)_{k\in\mathbb{N}}$ to $\psi_{i,n}$ and $\psi'_{i,n}$ respectively is uniform) the pointwise convergence of $\psi'_{i,k,n}(2^{-j^{(n)}_{k,i}}\eta)\to\psi'_{i,n}(0)$, $2^{j^{(n)}_{k,i}}\left(\psi_{i,k,n}(2^{-j^{(n)}_{k,i}}\eta)-\xi^{(n)}_i\right)\to\psi'_{i,n}(0)\eta$ (as easily follows by (26) and (25)).

It is easy to see now that the renamed profiles $w^{(n)}_i$ are obtained as pointwise limits (and thus also as weak limits)

$$w^{(n)}_i(\xi)=\lim_{k\to\infty}2^{-j^{(n)}_{k,i}r}(\chi_i u_k)\circ\exp_{y^{(n)}_{k,i}}\left(2^{-j^{(n)}_{k,i}}\xi\right),\ \text{for a.e. }\xi\in\mathbb{R}^N. \tag{29}$$

2. Since each $\overline{\mathcal{B}}_\rho(z_i)\subset\mathcal{B}_{2\rho}(z_i)\subset M$ and M is compact, we may assume that for all $n\in\mathbb{N}\setminus\{0\}$ and for all $i\in I$, there exist, up to subsequences, points of concentration

$$\bar{y}^{(n)}_i:=\lim_{k\to\infty}y^{(n)}_{k,i}. \tag{30}$$

In order to achieve the orthogonality relation (5) we shall introduce the following equivalence relation on the set of sequences in $M\times\mathbb{R}$. Namely given $(y_k,j_k)_{k\in\mathbb{N}}$ and $\left(y'_k,j'_k\right)_{k\in\mathbb{N}}$ in $M\times\mathbb{Z}$ we shall write

$$(y_k,j_k)_{k\in\mathbb{N}}\simeq\left(y'_k,j'_k\right)_{k\in\mathbb{N}}\quad\text{when}\quad\left(|j_k-j'_k|+2^{j_k}d(y_k,y'_k)\right)_{k\in\mathbb{N}}\ \text{is a bounded sequence.} \tag{$\mathcal{R}$}$$

Since the set I is a finite set, the number of sequences $\left(y^{(n)}_{k,i},j^{(n)}_{k,i}\right)_{k\in\mathbb{N}}$ which can be equivalent to a fixed sequence $\left(y^{(\bar{n})}_{k,\bar{\imath}},j^{(\bar{n})}_{k,\bar{\imath}}\right)_{k\in\mathbb{N}}$ is finite. Therefore we can exploit the unconditional convergence with respect to the indexes (n) of the series $S_{k,i}$ and synchronize them by replacing $\bar{n}$ and all the indexes m in the finite set

$$\mathcal{N}_{\bar{n}}:=\left\{m\in\mathbb{N}\setminus\{0\}\mid\exists i\in I\text{ s.t. }\left(y^{(n)}_{k,i},j^{(n)}_{k,i}\right)_{k\in\mathbb{N}}\simeq\left(y^{(\bar{n})}_{k,\bar{\imath}},j^{(\bar{n})}_{k,\bar{\imath}}\right)_{k\in\mathbb{N}}\right\} \tag{31}$$

with, say, the smallest integer in $\mathcal{N}_{\bar{n}}$.

Thanks to this synchronization procedure the following property

$$\left(y^{(n)}_{k,i_1},j^{(n)}_{k,i_1}\right)_{k\in\mathbb{N}}\simeq\left(y^{(m)}_{k,i_2},j^{(m)}_{k,i_2}\right)_{k\in\mathbb{N}}\quad\Longleftrightarrow\quad m=n, \tag{32}$$

holds true for all $i_1,i_2\in I$ and $m,n\in\mathbb{N}\setminus\{0\}$.

Note also that when $\left(y^{(n)}_{k,i_1},j^{(n)}_{k,i_1}\right)_{k\in\mathbb{N}}\simeq\left(y^{(n)}_{k,i_2},j^{(n)}_{k,i_2}\right)$, since $\left(\left|j^{(n)}_{k,i_2}-j^{(n)}_{k,i_1}\right|\right)_{k\in\mathbb{N}}$ is bounded, we can set, modulo subsequences

$$j(i_1,i_2,n):=\lim_{k\to+\infty}j^{(n)}_{k,i_2}-j^{(n)}_{k,i_1}\in\mathbb{R}, \tag{33}$$

so that, by redefining $w^{(n)}_{i_2}(2^{-j(i_1,i_2,n)}\cdot)$ as (the corresponding profile) $w^{(n)}_{i_2}$, we can assume that $\left(j^{(n)}_{k,i_2}\right)_{k\in\mathbb{N}}=\left(j^{(n)}_{k,i_1}\right)_{k\in\mathbb{N}}$. Moreover, since also $\left(2^{j^{(n)}_{k,i_1}}d\left(y^{(n)}_{k,i_1},y^{(n)}_{k,i_2}\right)\right)_{k\in\mathbb{N}}$ is bounded, we get (by (4)) that (see (30))

$$\bar{y}^{(n)}_{i_1}=\bar{y}^{(n)}_{i_2}\qquad\text{for all }\left(y^{(n)}_{k,i_1},j^{(n)}_{k,i_1}\right)_{k\in\mathbb{N}}\simeq\left(y^{(n)}_{k,i_2},j^{(n)}_{k,i_2}\right)_{k\in\mathbb{N}}. \tag{34}$$

Finally, we show that the elementary concentrations terms $C^{(n)}_{k,i}$ do not change (up to a vanishing term) by varying $\left(y^{(n)}_{k,i}, j^{(n)}_{k,i}\right)_{k\in\mathbb{N}}$ in the same equivalence class. Namely the following property holds true

$$\left(y^{(n)}_{k,i_1}, j^{(n)}_{k,i_1}\right)_{k\in\mathbb{N}} \simeq \left(y^{(n)}_{k,i_2}, j^{(n)}_{k,i_2}\right)_{k\in\mathbb{N}} \quad \Rightarrow \quad \left\|C^{(n)}_{k,i_1} - C^{(n)}_{k,i_2}\right\| \to 0, \tag{35}$$

for all $i_1, i_2 \in I$. Since, as shown above, we can assume, without restrictions, that $\left(j^{(n)}_{k,i_1}\right)_{k\in\mathbb{N}} = \left(j^{(n)}_{k,i_2}\right)_{k\in\mathbb{N}}$ (and we shall denote, to shorten notation, their common value as $\left(j^{(n)}_{k}\right)_{k\in\mathbb{N}}$) it will suffice to prove that, set $\bar{\xi}^{(n)}_{k,i_1} = \exp^{-1}_{z_{i_1}} y^{(n)}_{k,i_1}$ and $\bar{\xi}^{(n)}_{k,i_2} = \exp^{-1}_{z_{i_1}} y^{(n)}_{k,i_2}$, we have

$$\int_{\mathcal{B}_\rho(z_{i_1})} \left|2^{j^{(n)}_k r}\mathrm{d}\left(w^{(n)}_{i_1}\left(2^{j^{(n)}_k}(\exp^{-1}_{z_{i_1}}(x) - \bar{\xi}^{(n)}_{k,i_2})\right) - w^{(n)}_{i_1}\left(2^{j^{(n)}_k}(\exp^{-1}_{z_{i_1}}(x) - \bar{\xi}^{(n)}_{k,i_1})\right)\right)\right|^2 \mathrm{d}v_g \to 0 \text{ as } k \to \infty. \tag{36}$$

Indeed, by (20), we get, modulo subsequences, that

$$\begin{aligned} 2^{j^{(n)}_k}|\bar{\xi}^{(n)}_{k,i_2} - \bar{\xi}^{(n)}_{k,i_1}| &= 2^{j^{(n)}_k}|\exp^{-1}_{z_{i_1}} y^{(n)}_{k,i_2} - \exp^{-1}_{z_{i_1}} y^{(n)}_{k,i_1}| \\ &= 2^{j^{(n)}_k}|d(y^{(n)}_{k,i_2}, z_{i_1}) - d(y^{(n)}_{k,i_1}, z_{i_1})| \le 2^{j^{(n)}_k} d(y^{(n)}_{k,i_2}, y^{(n)}_{k,i_1}) \to 0. \end{aligned}$$

Then, (5) follows directly from (34).

3. Consider now the sum $\sum_{n\in\mathbb{N}\setminus\{0\}}\sum_{i\in I}\widetilde{C}^{(n)}_{k,i}$, with the sequences $(y^{(n)}_{k,i})_{k\in\mathbb{N}}$ and $(j^{(n)}_{k,i})_{k\in\mathbb{N}}$, which are synchronized at the Step 2 as $(y^{(n)}_{k})_{k\in\mathbb{N}}$ and $(j^{(n)}_{k})_{k\in\mathbb{N}}$ with $y^{(n)}_k \to \bar{y}^{(n)}$, and (29) takes form

$$w^{(n)}_i(\xi) = \lim_{k\to\infty} 2^{-j^{(n)}_k r}(\chi_i u_k) \circ \exp_{y^{(n)}_k}\left(2^{-j^{(n)}_k}\xi\right), \text{ for a.e. } \xi \in \mathbb{R}^N. \tag{37}$$

Since $j^{(n)}_k \to \infty$ implies $\exp_{y^{(n)}_k}\left(2^{-j^{(n)}_k}\xi\right) \to \bar{y}^{(n)}$ in M, the latter equality yields

$$w^{(n)}_i(\xi) = \chi_i(\bar{y}^{(n)}) \lim_{k\to\infty} 2^{-j^{(n)}_k r} u_k \circ \exp_{y^{(n)}_k}\left(2^{-j^{(n)}_k}\xi\right), \text{ for a.e. } \xi \in \mathbb{R}^N, \tag{38}$$

Then, setting

$$w^{(n)} := \sum_{i\in I} w^{(n)}_i. \tag{39}$$

we get relation (6) immediately from (38), $w^{(n)}_i = \chi_i(\bar{y}^{(n)})w^{(n)}$, and since, by Step 1, defect of compactness of $\chi_i u_k$ is an unconditionally convergent series, we have

$$\begin{aligned} \sum_{i\in I}\sum_{n\in\mathbb{N}\setminus\{0\}}\widetilde{C}^{(n)}_{k,i}(x) &= \sum_{n\in\mathbb{N}\setminus\{0\}}\sum_{i\in I}\widetilde{C}^{(n)}_{k,i}(x) = \sum_{n\in\mathbb{N}\setminus\{0\}}\sum_{i\in I} 2^{j^{(n)}_k r}\chi \circ \exp_{y^{(n)}_k} w^{(n)}_i\left(2^{j^{(n)}_k}\exp^{-1}_{y^{(n)}_k}(x)\right) \\ &= \sum_{n\in\mathbb{N}\setminus\{0\}} 2^{j^{(n)}_k r}\chi \circ \exp_{y^{(n)}_k} w^{(n)}\left(2^{j^{(n)}_k}\exp^{-1}_{y^{(n)}_k}(x)\right), \; x \in M, \end{aligned}$$

which gives (7).

4. In order to prove the "energy" estimate (9), assume, without loss of generality, that the sum in (7) is finite and that all $w^{(n)}$ have compact support, and expand by bilinearity the trivial inequality $\|u - u_k + \mathcal{S}_k\|^2_{H^{1,2}(M)} \geq 0$. Then, by using the norm (1) and the representation (3) of the scalar product in $H^{1,2}(M)$, we have

$$\begin{aligned} 0 \leq\ & \|u_k\|^2 + \|u\|^2 - 2\langle u_k, u\rangle + 2\langle u - u_k, \mathcal{S}_k\rangle \\ & + \sum_n \|2^{j_k^{(n)} r} \chi \circ \exp^{-1}_{y_k^{(n)}} w^{(n)}\left(2^{j_k^{(n)}} \exp^{-1}_{y_k^{(n)}}(\cdot)\right)\|^2 \\ & - \sum_{m\neq n} \left\langle 2^{j_k^{(m)} r} \chi \circ \exp^{-1}_{y_k^{(m)}} w^{(m)}\left(2^{j_k^{(m)}} \exp^{-1}_{y_k^{(m)}}(\cdot)\right), 2^{j_k^{(n)} r} \chi \circ \exp^{-1}_{y_k^{(n)}} w^{(n)}\left(2^{j_k^{(n)}} \exp^{-1}_{y_k^{(n)}}(\cdot)\right)\right\rangle. \end{aligned} \tag{40}$$

The first line of (40) can be evaluated taking into account that $u_k \rightharpoonup u$, $\mathcal{S}_k \rightharpoonup 0$, that (6) defines profiles $w^{(n)}$, and that $r = \frac{N-2}{2}$:

$$\begin{aligned} & \|u_k\|^2 + \|u\|^2 - 2\langle u_k, u\rangle + 2\langle u - u_k, \mathcal{S}_k\rangle \\ & = \|u_k^2\| + \|u^2\| - 2\|u\|^2 + o(1) - 2\sum_n \left\langle u_k, 2^{j_k^{(n)} r} \chi \circ \exp^{-1}_{y_k^{(n)}} w^{(n)}\left(2^{j_k^{(n)}} \exp^{-1}_{y_k^{(n)}}(\cdot)\right)\right\rangle \\ & = \|u_k\|^2 - \|u\|^2 + o(1) \\ & \quad - 2\sum_n 2^{j_k^{(n)} r} \int_{|\xi|<\rho} \sum_{i,j=1}^N g_{ij}^{y_k^{(n)}} \partial_i \left(u_k(\exp_{y_k^{(n)}}(\xi))\right) \partial_j \left(\chi(\xi)\, w^{(n)}(2^{j_k^{(n)}}\xi)\right) \sqrt{\det g_{ij}^{y_k^{(n)}}(\xi)} \mathrm{d}\xi \\ & \quad - 2\sum_n 2^{j_k^{(n)} r} \int_{|\xi|<\rho} u_k(\exp_{y_k^{(n)}}(\xi)) \chi(\xi)\, w^{(n)}(2^{j_k^{(n)}}\xi) \sqrt{\det g_{ij}^{y_k^{(n)}}(\xi)} \mathrm{d}\xi \\ & = \|u_k\|^2 - \|u\|^2 + o(1) \\ & \quad - 2\sum_n \int_{|\eta|<\rho 2^{j_k^{(n)}}} \sum_{i,j=1}^N g_{ij}^{y_k^{(n)}} \partial_i \left(2^{-j_k^{(n)} r} u_k \circ \exp_{y_k^{(n)}}(2^{-j_k^{(n)}}\eta)\right) \partial_j \left(\chi(2^{-j_k^{(n)}}\eta)\, w^{(n)}(\eta)\right) \\ & \qquad\qquad \cdot \sqrt{\det g_{ij}^{y_k^{(n)}}(2^{-j_k^{(n)}}\eta)}\, \mathrm{d}\eta \\ & \quad - 2\sum_n 2^{-2j_k^{(n)}} \int_{|\eta|<\rho 2^{j_k^{(n)}}} 2^{-j_k^{(n)} r} u_k \circ \exp_{y_k^{(n)}}(2^{-j_k^{(n)}}\eta) \chi(2^{-j_k^{(n)}}\eta) w^{(n)}(\eta) \sqrt{\det g_{ij}^{y_k^{(n)}}(2^{-j_k^{(n)}}\eta)} \mathrm{d}\eta \\ & = \|u_k\|^2 - \|u\|^2 + o(1) - 2\sum_n \int_{\mathbb{R}^N} \sum_i^N |\partial_i w^{(n)}(\eta)|^2 \mathrm{d}\eta - 2\sum_n 2^{-2j_k^{(n)}} \int_{\mathbb{R}^N} |w^{(n)}(\eta)|^2 \mathrm{d}\eta \\ & = \|u_k\|^2 - \|u\|^2 - 2\sum_n \|\nabla w^{(n)}\|_2^2 + o(1). \end{aligned}$$

(In the third equality we have set $\eta = 2^{j_k^{(n)}}\xi$, while in the fourth we have used the fact, due to (6) that $2^{-j_k^{(n)}} \chi(2^{-j_k^{(n)}}\cdot)(u_k \circ \exp_{y_k^{(n)}})(2^{-j_k^{(n)}}\cdot) \rightharpoonup \chi(0) w^{(n)} = w^{(n)}$ as $k \to \infty$ (in our slightly modified sense of weak convergence). Note that we have still denoted by ∂_i (resp. ∂_j) the derivative with respect to the i^{th} (resp j^{th}) component of $\eta = 2^{j_k^{(n)}}\xi$. Finally, in the last equality, we have used (1)).

In order to estimate the second line of (40) we shall split (according to (1)) the $H^{1,2}(M)$-norm into the L^2-norm of the gradient (gradient part) and the L^2-norm of the function (L^2 part) and consider first the latter. Since

$$
\begin{aligned}
&\sum_n \left\| 2^{j_k^{(n)} \frac{N-2}{2}} \chi \circ \exp^{-1}_{y_k^{(n)}} w^{(n)}(2^{j_k^{(n)}} \exp^{-1}_{y_k^{(n)}}(\cdot)) \right\|_2^2 \\
&\quad = \sum_n 2^{j_k^{(n)}(N-2)} \int_{\mathcal{B}_\rho(y_n)} |\chi \circ \exp^{-1}_{y_k^{(n)}}(x)\, w^{(n)}(2^{j_k^{(n)}} \exp^{-1}_{y_k^{(n)}}(x))|^2 \mathrm{d}v_g \\
&\quad = \sum_n 2^{j_k^{(n)}(N-2)} \int_{|\xi|<\rho} |\chi(\xi)(w^{(n)}(2^{j_k^{(n)}} \xi)|^2 \sqrt{\det g_{ij}^{y_k^{(n)}}(\xi)}\mathrm{d}\xi \\
&\quad = \sum_n 2^{-2j_k^{(n)}} \int_{|\eta|<\rho 2^{j_k^{(n)}}} |\chi(2^{-j_k^{(n)}}\eta) w^{(n)}(\eta)|^2 \sqrt{\det g_{ij}^{y_k^{(n)}}(2^{-j_k^{(n)}}\eta)}\mathrm{d}\eta \to 0 \text{ as } k \to \infty,
\end{aligned}
$$

(since $j_k^{(n)} \to \infty$) as $k \to \infty$, the second line of (40) is evaluated in the limit by the sum of the gradient terms as follows:

$$
\begin{aligned}
&\sum_n 2^{j_k^{(n)}(N-2)} \int_{\mathcal{B}_\rho(y_k^{(n)})} \left| \mathrm{d}\left(\chi \circ \exp^{-1}_{y_k^{(n)}}(x)\, w^{(n)}(2^{j_k^{(n)}} \exp^{-1}_{y_k^{(n)}}(x))\right)\right|^2 \mathrm{d}v_g \\
&\quad = \sum_n 2^{j_k^{(n)}(N-2)} \int_{|\xi|<\rho} \sum_{i,j=1}^N g_{ij}^{y_k^{(n)}}(\xi) \partial_i \left(\chi(\xi)\, w^{(n)}(2^{j_k^{(n)}}\xi)\right) \partial_j \left(\chi(\xi)\, w^{(n)}(2^{j_k^{(n)}}\xi)\right) \sqrt{\det g_{ij}^{y_k^{(n)}}(\xi)}\mathrm{d}\xi \\
&\quad = \sum_n \int_{|\eta|<\rho 2^{j_k^{(n)}}} \sum_{i,j=1}^N g_{ij}^{y_k^{(n)}} \partial_i \left(\chi(2^{-j_k^{(n)}}\eta)\, w^{(n)}(\eta)\right) \partial_j \left(\chi(2^{-j_k^{(n)}}\eta)\, w^{(n)}(\eta)\right) \sqrt{\det g_{ij}^{y_k^{(n)}}(2^{-j_k^{(n)}}\eta)}\, \mathrm{d}\eta \\
&\quad \longrightarrow \sum_n \int_{\mathbb{R}^N} |\nabla w^{(n)}(\eta)|^2\, \mathrm{d}\eta = \sum_n \left\|\nabla w^{(n)}\right\|^2 \text{ as } k \to \infty.
\end{aligned}
$$

Consider now the terms in the sum in third line of (40). Note that the L^2-part of the scalar product converges to zero by Cauchy inequality and by the calculations for the first line of (40). At the light of the orthogonality condition (5) we have to face two cases.

Case 1: The sequence $(j_k^{(n)} - j_k^{(m)})_{k\in\mathbb{N}}$ is unbounded. Assume without loss of generality that $j_k^{(n)} - j_k^{(m)} \to +\infty$ as $k \to \infty$. Then, using changes of variables $\xi = \exp^{-1}_{y_k^{(n)}}(x)$ and $\eta = 2^{j_k^{(n)}}\xi$,

$$
\begin{aligned}
&\left\langle 2^{j_k^{(m)} r} \chi \circ \exp^{-1}_{y_k^{(m)}}(x)\, w^{(m)}\left(2^{j_k^{(m)}} \exp^{-1}_{y_k^{(m)}}(\cdot)\right), 2^{j_k^{(n)} r} \chi \circ \exp^{-1}_{y_k^{(n)}}(x)\, w^{(n)}\left(2^{j_k^{(n)}} \exp^{-1}_{y_k^{(n)}}(\cdot)\right)\right\rangle \\
&\quad = 2^{j_k^{(n)} r} 2^{j_k^{(m)} r} \int_{\mathcal{B}_\rho(y_k^{(m)}) \cap \mathcal{B}_\rho(y_k^{(n)})} \mathrm{d}\left(\chi \circ \exp^{-1}_{y_k^{(m)}}(x)\, w^{(m)}\left(2^{j_k^{(m)}} \exp^{-1}_{y_k^{(m)}}(x)\right)\right) \\
&\qquad\qquad \cdot \mathrm{d}\left(\chi \circ \exp^{-1}_{y_k^{(n)}}(x)\, w^{(n)}\left(2^{j_k^{(n)}} \exp^{-1}_{y_k^{(n)}}(x)\right)\right) \mathrm{d}v_g + o(1)
\end{aligned}
$$

$$= 2^{j_k^{(n)} r} 2^{j_k^{(m)} r} \int_{|\xi|<\rho} \sum_{i,j=1}^{N} g_{ij}^{y_k^{(n)}}(\xi) \partial_i \left(\chi(\xi)\, w^{(n)}(2^{j_k^{(n)}} \xi) \right)$$

$$\cdot\, \partial_j \left(\chi(\exp^{-1}_{y_k^{(m)}}(\exp_{y_k^{(n)}}(\xi)))\, w^{(m)}(2^{j_k^{(m)}} \exp^{-1}_{y_k^{(m)}}(\exp_{y_k^{(n)}}(\xi))) \right) \sqrt{\det g_{ij}^{y_k^{(n)}}(\xi)} \mathrm{d}\xi$$

$$= 2^{-j_k^{(n)} r} 2^{j_k^{(m)} r} \int_{|\eta|<\rho 2^{j_k^{(n)}}} \sum_{i,j=1}^{N} g_{ij}^{y_k^{(n)}}(2^{-j_k^{(n)}} \eta) \partial_i \left((1+o(1))\, w^{(n)}(\eta) \right)$$

$$\cdot\, \partial_j \left((1+o(1))\, w^{(m)}(2^{j_k^{(m)}} \exp^{-1}_{y_k^{(m)}}(\exp_{y_k^{(n)}}(2^{-j_k^{(n)}} \eta))) \right) (1+o(1)) \mathrm{d}\eta + o(1) \to 0,$$

since, by (6),

$$\operatorname*{w-lim}_{k\to\infty} 2^{-j_k^{(n)} r} 2^{j_k^{(m)} r} w^{(m)}(2^{j_k^{(m)}} (\exp^{-1}_{y_k^{(m)}} \circ \exp_{y_k^{(n)}})(2^{-j_k^{(n)}} \cdot)) = \operatorname*{w-lim}_{k\to\infty} 2^{-j_k^{(n)} r} u_k(\cdot) = 0.$$

Case 2: $2^{j_k^{(n)}} d(y_k^{(n)}, y_k^{(m)}) \to \infty$ as $k \to \infty$. Since case 1 has been ruled out, we can assume without restrictions that the sequence $j_k^{(m)} - j_k^{(n)} = j \in \mathbb{R}$ for all large k. Then, by arguing as above (and in particular by taking into account that the L^2-part of the scalar product is negligible), we get that, as $k \to \infty$,

$$\left\langle 2^{j_k^{(m)} r}\, \chi \circ \exp^{-1}_{y_k^{(m)}}\, w^{(m)}(2^{j_k^{(m)}} \exp^{-1}_{y_k^{(m)}}(\cdot)), 2^{j_k^{(n)} r}\, \chi \circ \exp^{-1}_{y_k^{(n)}}\, w^{(n)}(2^{j_k^{(n)}} \exp^{-1}_{y_k^{(n)}}(\cdot)) \right\rangle \to 0,$$

since the values of $w^{(m)}$ and of $w^{(n)}$ are set to concentrate at sufficiently separated points, indeed $d(2^{j_k^{(n)}} y_k^{(n)}, 2^{j_k^{(m)}} y_k^{(m)}) = 2^{j_k^{(n)}} d(y_k^{(n)}, 2^j y_k^{(m)}) \geq 2^{j_k^{(n)}} d(y_k^{(n)}, y_k^{(m)}) \to \infty$.

Then, by applying the estimates obtained for the three lines of inequality (40) we finally deduce (9) concluding the proof of Theorem 1.

Acknowledgements

One of the authors (G.D.) is supported by GNAMPA of the “Istituto Nazionale di Alta Matematica (INdAM)” and by MIUR - FFABR - 2017 research grant.
http://dx.doi.org/10.13039/501100003407

The other author (C.T.) completed this paper as Lady Davis Visiting Professor at Technion.

References

[1] Adimurthi, A., and Tintarev, C. On compactness in the trudinger-moser inequality. *Ann. Sc. Norm. Sup. Pisa Cl. Sci. 5* (2014), 1–18.

[2] Adimurthi, A., and Tintarev, C. Defect of compactness in spaces of bounded variation. *J. Func. Anal.*, 271 (2016), 37–48.

[3] Devillanova, G. Multiscale weak compactness in metric spaces. *J. Elliptic Parabol. Eq. 2*, 131 (2016). `doi:10.1007/BF03377397`.

[4] DEVILLANOVA, G., SOLIMINI, S., AND TINTAREV, C. On weak convergence in metric spaces. *Nonl. Anal. Opt., Contemp. Math. 659* (2016), 43–63.

[5] DEVILLANOVA, G., SOLIMINI, S., AND TINTAREV, C. Profile decomposition in metric spaces. *Pure Appl. Funct. Anal. 2*, 4 (2017), 599–628.

[6] GTEXTBACKSLASH'ERARD, P. Description de compacittextbackslash'e de l'injection de sobolev. *ESAIM Control Optim. Calc. Var.*, 3 (1988), 213–233.

[7] JAFFARD, S. Analysis of the lack of compactness in the critical sobolev embeddings. *J. Funct. Analysis*, 161 (1999), 384–396.

[8] LIEB, E. On the lowest eigenvalue of the laplacian for the intersection of two domains. *Invent. Math.*, 74 (1983), 441–448.

[9] SCHINDLER, I., AND TINTAREV, K. An abstract version of the concentration compactness principle. *Revista Mat. Complutense*, 15 (2002), 1–20.

[10] SOLIMINI, S. A note on compactness-type properties with respect to lorentz norms of bounded subsets of a sobolev space. *Ann. Inst. H. Poincartextbackslash'e Anal. Non Lintextbackslash'eaire*, 12 (1995), 319–337.

[11] SOLIMINI, S., AND TINTAREV, C. Analysis of concentration in the banach space. *Comm. Contemp. Math.*, 18 (2016). doi:10.1142/S0219199715500388.

[12] STRUWE, M. A global compactness result for elliptic boundary value problems involving limiting nonlinearities. *Math. Z.*, 187 (1984), 511–517.

G. Devillanova
Politecnico di Bari,
via Re David, 70125 Bari, Italy
giuseppe.devillanova@poliba.it

C. Tintarev
Technion - Israel Institute of Technology
32000 Haifa, Israel
tammouz@gmail.com