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BEYOND WENTZELL-FREIDLIN: SEMI-DETERMINISTIC APPROXIMATIONS FOR DIFFUSIONS WITH SMALL NOISE AND A REPULSIVE CRITICAL BOUNDARY POINT

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Abstract. We extend below a limit theorem [2] for diffusion models used in population theory.

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§1. Introduction

A diffusion with small noise is defined as the solution of a stochastic differential equation (SDE) driven by standard Brownian motion $B_t(\cdot)$ (defined on a probability space and progressively measurable with respect to an increasing filtration)

$$\begin{cases} dX_t^\varepsilon = \mu(X_t^\varepsilon)dt + \sqrt{\varepsilon}\sigma(X_t^\varepsilon)dB_t, & t \geq 0, \\ X_0^\varepsilon = x_0 = \varepsilon, X_t^\varepsilon \in \mathcal{I} := (0, r) \end{cases} \tag{1}$$

where $0 < r \leq +\infty$, $\varepsilon > 0, \mu : \mathcal{I} \mapsto \mathbb{R}$, $\sigma : \mathcal{I} \mapsto \mathbb{R}_{>0}$ and μ, σ satisfy conditions ensuring that (1) has a strong unique solution (for example, μ is locally Lifshitz and σ satisfies the Yamada-Watanabe conditions [14, (2.13), Ch.5.2.C]).[§]

When $\varepsilon \to 0$, (1) is a small perturbation of the dynamical system/ordinary differential equation (ODE):

$$\frac{dx_t}{dt} = \mu(x_t), \quad t \geq 0, \tag{2}$$

which will also be supposed to admit a unique continuous solution $x_t, t \in \mathbb{R}_+$ subject to any $x_0 \in (0, r)$, and the flow of which will be denoted by $\phi_t(x)$.

A basic result in the field is the "fluid limit", which states that when (1) admits a strong unique solution, the effect of noise is negligible as $\varepsilon \to 0$, on any **fixed time interval** $[0, T]$:

[§]For reviews discussing the existence of strong and weak solutions, see for example [5, 13, 8].

Theorem 1. *[Freidlin and Wentzell] [11, Thm 1.2, Ch. 2.1] Let X_t^ε satisfy* (1), *assume μ, σ satisfy the Lifshitz condition, and that $X_0^\varepsilon \xrightarrow[\varepsilon\to 0]{\P} x_0 \in \mathbb{R}_+$, where $\xrightarrow[\varepsilon\to 0]{\P}$ denotes convergence in probability. Then, for any fixed T*

$$\sup_{t\le T} |X_t^\varepsilon - x_t| \xrightarrow[\varepsilon\to 0]{\P} 0,$$

where x_t is the solution of (2) *subject to the initial condition x_0.*[¶]

Although interesting, this result does not give any understanding of the asymptotic behavior of the diffusion process for times converging to infinity; in particular, it does not tell us how the diffusion travels between equilibrium points (which requires times converging to infinity). Following [4, 2], we go here beyond Theorem 1, by analyzing the way a diffusion process leaves an unstable equilibrium point. Precisely, we make the following assumptions:

Assumption 1. Suppose from now on that $l = 0, \mu(0) = 0, \mu'(0) > 0$, which makes zero an **unstable equilibrium point of** (2) and of (1).

Note that under Assumption 1, the Freidlin-Wentzell theorem 1 implies that the solution of (1) started from a small positive initial condition $X_0^\varepsilon = \varepsilon > 0$ converges to zero on any fixed bounded interval

$$\sup_{t\le T} \left|X_t^\varepsilon\right| \xrightarrow[\varepsilon\to 0]{\P} 0, \qquad \forall T \ge 0.$$

Assumption 2. Put now $a(x) = \sigma^2(x)$, and assume that $a(0) = \sigma(0) = 0, a'(0) > 0$, which makes 0 a singular point of the diffusion (1)– see for example [8].

Remark 1. Note that $a'(0) > 0$ rules out important population theory models like the linear Gilpin Ayala diffusion [17] with

$$\mu(x) = \gamma x\Big(1 - (\frac{x}{x_c})^\alpha\Big), \sigma(x) = \sqrt{\varepsilon} x \Leftrightarrow a(x) = \varepsilon x^2, \gamma > 0, x_c > 0, \alpha > 0, \qquad (3)$$

which includes by setting $\alpha = 1$ another favorite, the logistic-type Verlhurst-Pearl diffusion [12, 9, 1].

Recently, a new type of limit theorem [2] was discovered when $T \to \infty$ under Assumptions 1 and 2, when x_0^ε converges to the unstable equilibrium point of (2). Following [2], let

$$T^\varepsilon := \frac{1}{\mu'(0)} \log \frac{1}{\varepsilon} \qquad (4)$$

denote the solution of the equation $\phi_{t,lin}(x_0) = x_0 e^{\mu'(0)t} = 1$ where $\phi_{t,lin}(x_0)$ is the flow of the linearized system of (2) in 0, and divide the evolution of the process in three time-intervals:

$$[0, t_c := cT^\varepsilon], [t_c, t_1 := T^\varepsilon], [t_1, \infty), c \in (1/2, 1) \qquad (5)$$

(the restriction $c > 1/2$ is used in (15)).

It turns out that this partition allows separating the life-time of diffusions with small noise, exiting an unstable point of the fluid limit, into three periods with distinct behaviors:

[¶]For other deterministic limit theorems for one-dimensional diffusions, see also Gikhman and Skorokhod [19], Freidlin and Wentzell [11], Keller et al. [16], and Buldygin et al. [6].

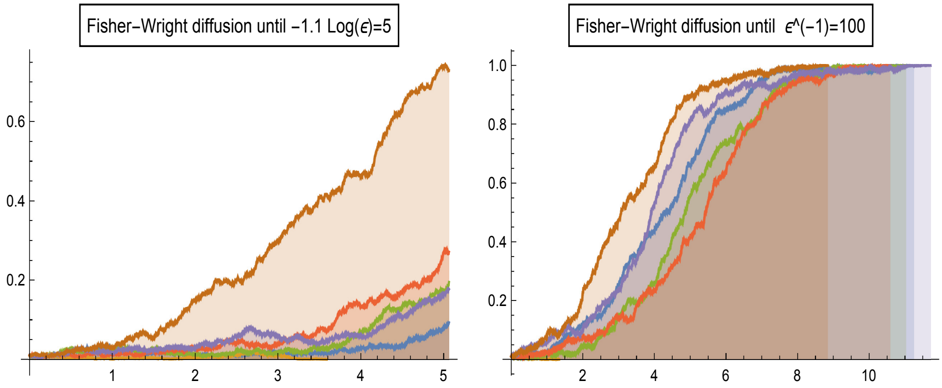

Figure 1: 6 paths of the Kimura-Fisher-Wright diffusion $dX_t = \gamma X_t(1 - X_t)dt + \sqrt{\varepsilon X_t(1 - X_t)}dB_t$, where $x_c = 1$ is an exit boundary, with $\varepsilon = .01$. On the right, three stages of evolution may be discerned

1. In the first stage, the process leaves the neighborhood of the unstable point. The linearization of the SDE implies that here a Feller branching approximation may be used, and this produces a certain exit law W which will be carried over to the next stage as a (random) initial condition.
2. In the second "semi-deterministic stage" (meaning that paths cross very rarely here), the system moves towards its first stable critical point x_c, following the trajectories of its fluid limit (2), again over a time whose length converges to ∞. A further renormalization produces here the main result, the limit exit law (7).
3. In the third stage, after the SDE has approaches the stable critical point of the fluid limit, "randomness is regained" – see crossings of paths in figures 1 and 2); (if the process may reach and overshoot the stable critical point, convergence towards a stationary distribution may occur).

The following result was obtained first in [2], for the "Kimura-Fisher-Wright" diffusion, and extended subsequently to diffusions with bounded volatility.

Theorem 2. Fluid limit with random initial conditions *[2]. Let X_t^ε satisfy Assumption 1,* (1)*, and $X_0^\varepsilon = \varepsilon > 0$. Suppose in addition that the diffusion coefficient $\sigma(\cdot)$ is continuous and* **bounded***, as well as its first derivative, and that $\mu(\cdot)$ satisfies the following drift condition:*

$$\left|\mu(y) - \mu(x)\right| \le \mu'(0)|y - x|, \quad x, y \in \mathbb{R}_+.$$

Let Y_t denote the solution to the **scaled linearized equation**

$$dY_t = \mu'(0)Y_t dt + \sqrt{a'(0)Y_t}dB_t, Y_0 = 1 \Longrightarrow Y_t = 1 + \int_0^t \mu'(0)Y_s ds + \int_0^t \sqrt{a'(0)Y_s}dB_s, \quad (6)$$

known as **Feller branching diffusion***.*

Then, it holds that :

(A)

$$X^{\varepsilon}_{T^{\varepsilon}} \xrightarrow[\varepsilon\to 0]{\P} \widetilde{\phi}(W), \tag{7}$$

where

(i) the random variable W is the a.s. martingale limit

$$W := \lim_{t\to\infty} e^{-\mu'(0)t} Y_t = 1 + \int_0^{\infty} e^{-\frac{\mu'(0)}{2}s} \sqrt{a'(0) Y_s} dB_s \tag{8}$$

(ii) $\widetilde{\phi}(x)$ denotes the **limit of the deterministic flow pushed first backward in time by the linearized deterministic flow** $\phi_{t,lin}(x) = xe^{\mu'(0)t}$ *near the unstable critical point* 0

$$\widetilde{\phi}(x) = \lim_{t\to\infty} \phi_t(\phi_{-t,lin}(x)) = \lim_{t\to\infty} \phi_t(xe^{-\mu'(0)t}), \quad x \geq 0. \tag{9}$$

(B) Also, for any $T > 0$,

$$\sup_{t\in[0,T]} \left| X^{\varepsilon}_{T_{\varepsilon}+t} - x_t \right| \xrightarrow[\varepsilon\to 0]{\P} 0, \tag{10}$$

where x_t is the solution of (2) *subject to the initial condition $X_0 = \widetilde{\phi}(W)$.*

Remark 2. Note that W depends only on the local parameters $\mu'(0), a'(0)$ of the diffusion at the critical point. Assume from now on, without loss of generality that $a'(0) = 1$ (recalling however that this is the only part of the stochastic perturbation that survives in the limiting regime), and put

$$\gamma := \mu'(0) > 0. \tag{11}$$

The Laplace transform of W is well known [18] and easy to compute.

$$Ee^{-\lambda W} = \lim_{t\to\infty} Ee^{-\lambda W_t} = \exp\left(-\frac{2\gamma\lambda}{2\gamma + \lambda}\right) = E\exp\Big(-\lambda \sum_{j=0}^{\Pi} \tau_j\Big), \tag{12}$$

which is the Laplace transform of a Poisson $\Pi \sim$ Poi(2γ) sum of independent random variables $\tau_j \sim$ Exp(2γ).

Remark 3. The main part of Theorem 2 is the equation (7) which identifies the limit after the second stage

$$X^{\varepsilon}_{T^{\varepsilon}} = \Phi^{\varepsilon}_{T^{\varepsilon}}(\varepsilon) \xrightarrow[\varepsilon\to 0]{\P} \lim_{t\to\infty} \phi_t(\phi_{-t,lin}(W)) = \widetilde{\phi}(W), \tag{13}$$

$\Phi^{\varepsilon}_t(x)$ denotes the flow generated by the SDE (1).

Note that $\widetilde{\phi}$ depends only on the dynamical system μ, and that by [2, Prop. 4.1], it is a nontrivial solution of the ODE $\gamma x \widetilde{\phi}'(x) = \mu(\widetilde{\phi}(x)), \quad x > 0, \widetilde{\phi}(0) = 0$.

(13) suggests possible generalizations to multidimensional diffusions and possibly to jump-diffusions (where a CBI might replace the Feller diffusion in the limit).

Remark 4. Part 2. of Theorem 2 follows immediately by a simple change of time: letting $\widetilde{X}^{\varepsilon}_t = X^{\varepsilon}_{T^{\varepsilon}+t}$, and $\widetilde{B}_t = B_{T^{\varepsilon}+t} - B_{T^{\varepsilon}}$ one obtains from (1)

$$\widetilde{X}^{\varepsilon}_t = \widetilde{X}^{\varepsilon}_0 + \int_0^t f(\widetilde{X}^{\varepsilon}_s) ds + \int_0^t \sqrt{\varepsilon\sigma(\widetilde{X}^{\varepsilon}_s)} d\widetilde{B}_s,$$

and the result follows from (7) by the fluid convergence Theorem 1. This part may be viewed as describing "short transitions" (invisible on a long time scale) between the second and third stages.

Remark 5. The limit (7) describing the position after the second stage has been established in [2] for one dimensional distributions with bounded $\sigma(x)$. This assumption seems however restrictive, since for typical diffusions whose fluid limit $\phi_t(x)$ admits a stable critical point x_c, the probability of leaving the neighborhood of the stable point x_c is very small as $\varepsilon \to 0$. This intuition is confirmed by simulations –see Figure 2.

The remark 5 suggests the relation of our problem to that of studying the maximum of X_t. More precisely, we would like to establish and exploit the plausible fact that $\forall \theta > 1$

$$\lim_{\varepsilon\to 0} P[T_{\theta x_c} < T^\varepsilon | X_0 = \varepsilon] = \lim_{\varepsilon \to 0} P[\sup_{0\le t\le T^\varepsilon} X_t^\varepsilon > \theta x_c | X_0 = \varepsilon] = 0, \tag{14}$$

where x_c is the closest critical point towards which the diffusion is attracted, and $T_{\theta x_c}$ is the hitting time of θx_c; clearly, (14) renders unnecessary the assumption that the diffusion coefficient $\sigma(\cdot)$ be bounded.

A weaker statement than (14), but still sufficient for a slight extension, is provided in the elementary Lemma (3) below.

Contents. The paper is organized as follows. In Section 2 we offer, based on Lemma 3, a slight extension of Theorem 2 of [2]. A conjecture (see Problem 1 is presented here as well. We illustrate our new result with the example of the logistic Feller diffusion in Section 3. We include for convenience in Section 4 an outline of the remarkable paper [2].

§2. An extension of Theorem 2 [2]

Recall now from [2] that the restrictive condition $||\sigma||_\infty < \infty$ is used for proving that[‡]

$$||\sigma||_\infty < \infty, c \in (1/2, 1) \Longrightarrow \Phi_{t_c,t_1}(X_{t_c}^\varepsilon) - \phi_{t_c,t_1}(X_{t_c}^\varepsilon) \xrightarrow[\varepsilon\to 0]{L^2} 0, \tag{16}$$

where $t_c = cT^\varepsilon$.

We will show now that it is possible to remove the condition $||\sigma||_\infty < \infty$ in (16), if only convergence in probability is needed, by assuming rather weak and natural conditions on the scale function $s(\cdot)$. Recall that the scale function s is defined (up to two integration constants) as an arbitrary increasing solution of the equation $\mathcal{L}s(x) = 0$, where $\mathcal{L}$ is the generator

[‡]Let us recall the proof of this important piece of the puzzle. Let $\Phi_{s,t}(x)$, $\phi_{s,t}(x)$ denote the stochastic and deterministic flows generated respectively by the SDE (1) and ODE (2), put $\Phi_t^\varepsilon := \Phi_{t_c,t_c+t}(X_{t_c}^\varepsilon)$, $\phi_t := \phi_{t_c,t_c+t}(X_{t_c}^\varepsilon)$ for brevity, and define $\delta_t^\varepsilon = \Phi_t^\varepsilon - \phi_t$. Subtracting equations (1) and (2) and applying the Itô formula:

$$E(\delta_t^\varepsilon)^2 = E\int_0^t 2\delta_s(\mu(\Phi_s^\varepsilon) - \mu(\phi_s))ds + \int_0^t \varepsilon E\sigma(\Phi_s^\varepsilon)ds \le \int_0^t 2\gamma E(\delta_s)^2 ds + \varepsilon t||\sigma||_\infty, t \in \mathbb{R}_+$$

where assumption 2 was used. By Grönwall's inequality

$$E\Big(\Phi_{t_c,t_1}(X_{t_c}^\varepsilon) - \phi_{t_c,t_1}(X_{t_c}^\varepsilon)\Big)^2 = E(\delta_{t_1-t_c}^\varepsilon)^2 \le C_1\varepsilon t_1 e^{2\gamma(t_1-t_c)} \le C_2\varepsilon^{2c-1}\log\frac{1}{\varepsilon} \xrightarrow[\varepsilon\to 0]{} 0 \tag{15}$$

where the convergence holds since $c \in (\frac{1}{2}, 1)$.

operator of the diffusion, and that this function is continuous – see [15, Ch. 15, (3.5), (3.6)] (noting that [15] denote the scale function by $S(\cdot)$).

Lemma 3. *Assume that* 0 *is an attracting boundary and that r is an unattracting boundary, i.e. that* , $s(0_+) > -\infty, s(r-) = \infty$. *Put*

$$\overline{X}^{\varepsilon} = \sup_{0\le t<\infty} X_t^{\varepsilon}, \tag{17}$$

where X^{ε} *is defined in* (1). *Then:*

$$\text{(A)} \quad \forall \varepsilon, \lim_{M\to r} P_{\varepsilon}[\overline{X}^{\varepsilon} > M] = \lim_{M\to r} \frac{s(\varepsilon)-s(0)}{s(M)-s(0)} = (s(\varepsilon)-s(0)) \lim_{M\to r} \frac{1}{s(M)-s(0)} = 0, \tag{18}$$

and

$$\text{(B)} \quad c \in (1/2, 1) \Longrightarrow \Phi_{t_c,t_1}(X_{t_c}^{\varepsilon}) - \phi_{t_c,t_1}(X_{t_c}^{\varepsilon}) \xrightarrow[\varepsilon\to 0]{P} 0. \tag{19}$$

Proof. (18) is straightforward. Indeed, recall that the boundary 0 is attracting. Then,

$$P_{\varepsilon}[\overline{X}^{\varepsilon} > M] = P_{\varepsilon}[T_M < T_0] = \frac{s(\varepsilon)-s(0)}{s(M)-s(0)} \tag{20}$$

where T_0, T_M are the hitting times of X_t^{ϵ} at 0 and M – see [15, Ch. 15, (3.1), (3.10)]. Using now the continuity of the scale function $s(\cdot)$ [15, Ch. 15, (3.5), (3.6)] (note that [15] denote the scale function by $S(\cdot)$) yields $\lim_{M\to r} s(M) = s(r_-) = \infty$ and the result.

(19) follows by a similar argument. Indeed, denote the deterministic and stochastic flows generated by the ODE (2) and SDE (1) (i.e. the solutions of these equations at time t that start at x at time s) by $\phi_{s,t}(x)$ and $\Phi_{s,t}(x)$, respectively, and put $\Phi^{\varepsilon} := \Phi_{t_c,t_1}(X_{t_c}^{\varepsilon})$ and $\phi^{\varepsilon} := \phi_{t_c,t_1}(X_{t_c}^{\varepsilon})$ for brevity and define $\delta^{\varepsilon} = \Phi^{\varepsilon} - \phi^{\varepsilon}$. For fixed ε and M, it holds that

$$\begin{aligned}\forall \delta > 0, P_{\varepsilon}[|\delta^{\varepsilon}| > \delta] &\le P_{\varepsilon}[\overline{X}_{T^{\varepsilon}}^{\varepsilon} \le M] P_{\varepsilon}[|\delta^{\varepsilon}| > \delta | \overline{X}_{T^{\varepsilon}}^{\varepsilon} \le M] + P_{\varepsilon}[\overline{X}_{T^{\varepsilon}}^{\varepsilon} > M] \\ &\le P_{\varepsilon}[\overline{X}_{T^{\varepsilon}}^{\varepsilon} \le M] P_{\varepsilon}[|\delta^{\varepsilon}| > \delta | \overline{X}_{T^{\varepsilon}}^{\varepsilon} \le M] + P_{\varepsilon}[\overline{X}^{\varepsilon} > M].\end{aligned}$$

Letting now ε to 0 makes the first term go to 0 by (16), yielding

$$\forall M < r, \forall \delta > 0, \limsup_{\varepsilon\to 0} P_{\varepsilon}[|\delta^{\varepsilon}| > \delta] \le \lim_{\varepsilon\to 0} \frac{s(\varepsilon)-s(0)}{s(M)-s(0)} = 0$$

where we have used again the continuity of the scale function. □

Theorem 4. *The conclusions of Theorem 2 still hold under the assumptions of Lemma 3.*

Proof. Theorem 2 of [2] only uses the assumption $\|\sigma\|_{\infty} < \infty$ in establishing the unnecessarily strong result (16). Providing weaker conditions for the weaker but still sufficient result (19) establishes therefore our claim. □

Problem 1. Note that essential use of $s(0) > -\infty$ was made in (18). We conjecture however that a finer analysis will reveal that the result of Theorem 4 still holds whenever r is "repelling/unattracting", more precisely when it is natural unattracting or entrance, cf. Feller's classification of boundary points [15, Ch. XV].

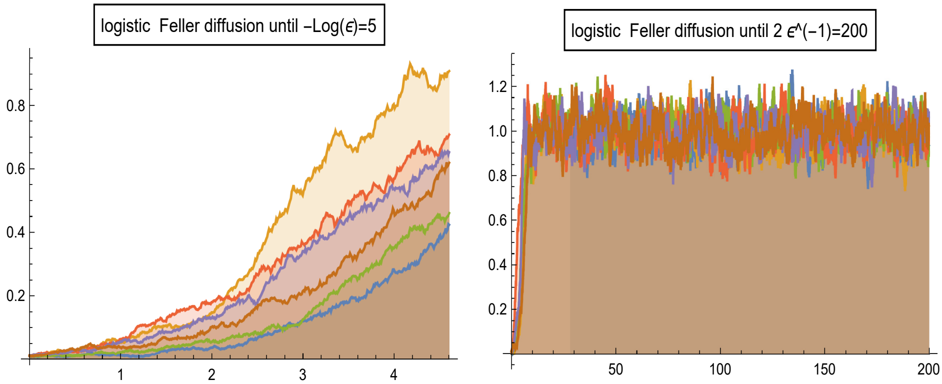

Figure 2: 6 paths of the logistic Feller diffusion ($x_c = 1$ is regular) with $\varepsilon = .01$, until T_ε and after

§3. Examples with $\lim_{t\to\infty} X_t/x_t = 0$: The logistic Feller and Gilpin-Ayala diffusions

We recall now some famous examples for which the conditions of our Lemma 3 hold. The logistic Feller diffusion is defined by

$$dX_t = \gamma X_t\left(1 - \frac{X_t}{x_c}\right)dt + \sqrt{\varepsilon X_t}dB_t,\ X_t \in (0,\infty).$$

The limit point x_c of x_t is a regular point for the diffusion; w.l.o.g. we will take it equal to 1. The scale density $s'(x) = e^{-\frac{2\gamma}{\varepsilon}(x-\frac{x^2}{2})}$ is integrable at 0, but not at ∞, and the speed density [15] $m'(x) = \frac{e^{\frac{2\gamma}{\varepsilon}(x-\frac{x^2}{2})}}{\varepsilon x}$ is integrable at ∞, but not at 0, so that the conditions of Lemma 3 hold.[§]

Therefore, fluid convergence with random initial point before T_ε [2] still holds, with the same deterministic flow and random initial condition as for the Kimura-Fisher Wright diffusion studied in [2]

$$\phi_t(x) = \frac{xe^{\gamma t}}{1 - x + xe^{\gamma t}},\ \widetilde{\phi}(x) = \frac{x}{1+x}, X_0 = \frac{W}{W+1}$$

(since $\mu(.), a'(0)$ did not change)–see Figure 2.

In fact, the paths of the logistic Feller and Kimura-Fisher-Wright diffusions are almost indistinguishable up to T^ε of each other –see Figure 3. After reaching the neighborhood of x_c however, the paths split, reflecting the different natures (regular and exit) of x_c for these two stochastic processes.

[§]Furthermore, conform Feller's boundary classification [15], 0 is an exit boundary since $s'(x)m[x,1]$ is integrable at 0, and absorbtion in 0 occurs with probability 1, and ∞ is an entrance (nonattracting) boundary, since $m'(x)s[1,x]$ is integrable at ∞–see also [7, 3] and [10] for the generalization to continuous-state branching processes with competition.

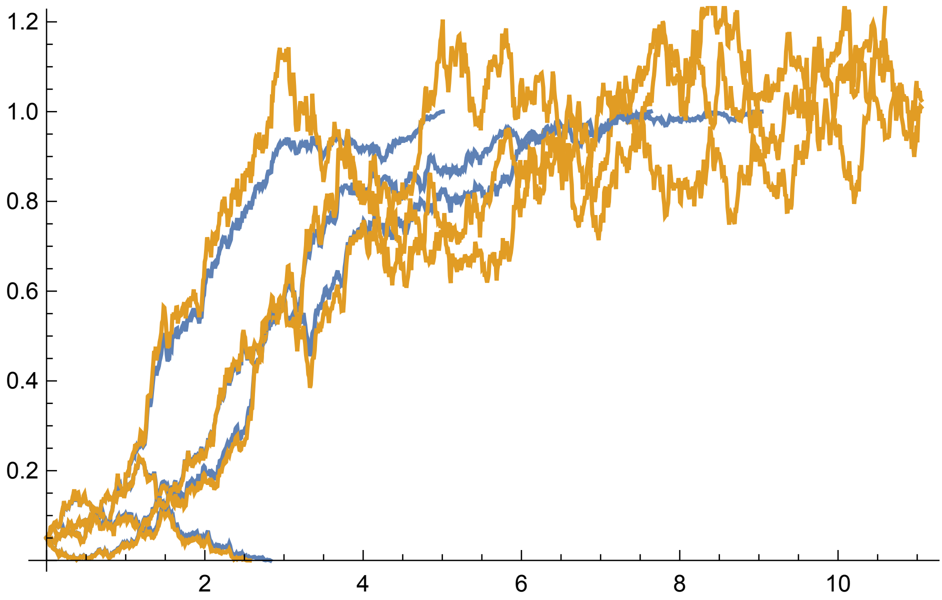

Figure 3: 6 paths of the logistic Feller and Kimura-Fisher-Wright diffusions with $\varepsilon = 1/20$, before and after T_ε

Some other examples of interest in population theory are the diffusion processes defined by the SDEs

$$dX_t = \gamma X_t\Big(1 - (\frac{X_t}{x_c}^{\theta}\Big)dt + \sigma\sqrt{X_t}dB_t, \sigma >, \theta > 0,$$
$$dX_t = \Big[\gamma X_t\Big(1 - \frac{X_t}{x_c} - \beta\frac{X_t^{n-1}}{1+X_t^n}\Big)\Big]dt + \sigma\sqrt{X_t}dB_t, \beta \geq 0, n \geq 1,$$

which are stochastic extensions with square root volatility of deterministic population models introduced by Gilpin and Ayala and Holling respectively.

It is easy to check that adding the exponents θ and n does not affect integrability of the scale and speed densities of these diffusions, so that our extension applies. Furthermore, the rescaled flow $\widetilde{\phi}$ may be computed numerically by [2, Prop. 4.1] (and even symbolically for small integer values of θ, n).

Moving away from the square root volatility case, an interesting, still open question is to investigate whether analogues of the [2] result are available for the processes satisfying $dX_t = \gamma X_t\Big(1 - (\frac{X_t}{x_c})^{\theta}\Big)dt + \sqrt{\varepsilon}(X_t)^{\alpha}dB_t, \quad \alpha > 0.$[§]

[§]The particular case $\alpha = \theta = 1$ is the famous Verlhurst-Pearl diffusion (VP)– see for example [17].

§4. Sketch of the proof of Theorem 2 [2]

Recall that $t_c = ct_1$ with $c \in (1/2, 1)$, arbitrary, and note that $X^\varepsilon_{T^\varepsilon} = \Phi_{t_c,t_1}(X^\varepsilon_{t_c}) = \Phi_{t_c,t_1}(\Phi_{t_c}(\varepsilon))$. The idea of the proof is to approximate this random variable by

$$X^\varepsilon_{T^\varepsilon} \approx \phi_{t_c,t_1}(\Phi_{t_c}(\varepsilon)) \xrightarrow{\varepsilon\to 0} \widetilde{\phi}(W), \tag{21}$$

with the random variable W from (8).

The proof of [2] involves several steps

1. The first idea for establishing the approximation $\widetilde{\phi}(W)$ of $X^\varepsilon_{T^\varepsilon}$ is to **blow-up** the process near the boundary 0
$$\widetilde{X}^\varepsilon_t := \varepsilon^{-1} X^\varepsilon_t,$$
which fixes the initial condition to 1 and changes the SDE to
$$d\widetilde{X}^\varepsilon_t = \varepsilon^{-1}\mu(\varepsilon\widetilde{X}^\varepsilon_t)dt + \sqrt{\frac{a(\varepsilon\widetilde{X}^\varepsilon_t)}{\varepsilon}}dB_t, \quad t \geq 0, \tag{22}$$
it is easy to check that a subsequent **linearization of the SDE** yields
$$\widetilde{X}^\varepsilon_t \approx Y_t$$
where Y_t is a **Feller branching diffusion** started from 1, defined by
$$Y_t = 1 + \int_0^t \mu'(0)Y_s ds + \int_0^t \sqrt{Y_s} dB_s, \quad t \geq 0. \tag{23}$$
One may take advantage then of the well-known nonnegative martingale convergence theorem for the "scaled final position" of the branching process Y_t
$$W := \lim_{t\to\infty} e^{-\mu'(0)t} Y_t. \tag{24}$$
Remark 6. Let us note that the linearization for processes satisfying $a(x) = O(x^2)$ and failing Assumption 2, like the linear Gilpin-Ayala (3), leads to geometric Brownian motion. In this case, (24) holds with $W = 0$, and a different approach seems necessary.
2. After "blowing up" the beginning of the path, the second idea is to **"look from far away"**. We want to break the trajectory at a suitably chosen time point
$$t_c < t_1 = T^\varepsilon = \frac{1}{\gamma}\log\frac{1}{\varepsilon} \tag{25}$$
such that before t_c, the original process is close to Feller's branching diffusion (23), and convergence to the limit W of the Feller diffusion occurs, i.e.
$$X^\varepsilon_{t_c} = \varepsilon\widetilde{X}^\varepsilon_{t_c} = e^{-\gamma t_1}\widetilde{X}^\varepsilon_{t_c} \approx e^{-\gamma t_1} Y_{t_c} = e^{-\gamma(t_1-\tau_c)} e^{-\gamma t_c} Y_{t_c} \approx e^{-\gamma(t_1-\tau_c)} W. \tag{26}$$
The first approximation $e^{-\gamma t_c}\widetilde{X}^\varepsilon_{t_c} \xrightarrow[\varepsilon\to 0]{L^1} Y_{t_c}$ follows from the following lemma [2] showing that the solution of (1) converges, under appropriate scaling, to the Feller branching diffusion (23).

Lemma 5. *Let $\widetilde{X}_t^\varepsilon := \varepsilon^{-1} X_t^\varepsilon$, where X_t^ε is the solution of* (1) *subject to $X_0^\varepsilon = \varepsilon$. Then*

$$\widetilde{X}_t^\varepsilon \xrightarrow[\varepsilon\to 0]{L^1} Y_t, \quad \forall\, t \geq 0,$$

where Y_t is the solution of (23).

Putting these together yields $\phi_{t_c,t_1}(X_{t_c}^\varepsilon) \xrightarrow[\varepsilon\to 0]{\P} \widetilde{\phi}(W)$.

3. The hardest part is proving that in the second portion $[t_c, t_1]$, the influence of the stochasticity is negligible, for example that $\Phi_{t_c,t_1}(X_{t_c}^\varepsilon) - \phi_{t_c,t_1}(X_{t_c}^\varepsilon) \xrightarrow[\varepsilon\to 0]{L^2} 0$, as proved in [2] under the restrictive assumption $||\sigma||_\infty < \infty$.

Putting it all together in one line, one must prove that

$$X_{t_1}^\varepsilon = \Phi_{t_c,t_1}(X_{t_c}^\varepsilon) \approx \Phi_{t_c,t_1}(We^{-\gamma(t_1-\tau_c)}) \approx \phi_{t_c,t_1}(We^{-\gamma(t_1-\tau_c)}) \xrightarrow[\varepsilon\to 0]{\P} \widetilde{\phi}(W). \tag{27}$$

To extend [2], it is sufficient to improve the third approximation step above.

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