Monografías Matemáticas *García de Galdeano* **42**, 1–11 (2019)

Generalized resolvent of the Stokes problem with Navier-type boundary conditions

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Abstract. We study in this paper the generalized resolvent of the Stokes problem with Navier-type boundary conditions.

Keywords: Generalized resolvent, Stokes Problem, Navier-type boundary conditions.
AMS classification: 35Q30, 76D05, 76D07, 35K20, 35K22, 76N10, 35A20, 35Q40.

§1. Introduction

This paper is devoted to the existence and uniqueness of weak and strong and very weak solutions to the problem

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = \chi \qquad \text{in} \quad \Omega \times (0, T), \\ \boldsymbol{u} \cdot \boldsymbol{n} = g, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} \qquad \text{on} \quad \Gamma \times (0, T), \end{cases} \tag{1}$$

where we study the generalized resolvent of the Stokes operator with nonstandard Navier-type boundary conditions. Up to now most research concerns the homogeneous boundary conditions, and the case χ=0. Although the case χ $\neq$0 has many important applications, specially in treating more general boundary value problems and using cut-off procedure.

There exists several references on (1) when $\chi = 0$ in Ω. This question was already studied by Solonnikov in [12] for the homogeneous Dirichlet boundary condition (*i.e.* $\boldsymbol{u} = \mathbf{0}$ on Γ). In that work, the author considered the resolvent Problem when $|\arg \lambda| \leq \delta + \pi/2$ where $\delta \geq 0$ is small. Later on, the resolvent of the Stokes operator with Dirichlet boundary condition in bounded domains has been studied by Giga in [6] using the theory of pseudo-differential operators. The results in [6] extends those in [12] in two directions. First, he consider larger set of values of λ. More precisely λ in the sector $|\arg \lambda| \leq \pi - \varepsilon$, for any $\varepsilon > 0$. Second, the resolvent of the Stokes operator is obtained explicitly and this enables him to describe the domains of fractional powers of the Stokes operator with Dirichlet boundary condition.

In exterior domains, Giga and Sohr [7] approximate the resolvent of the Stokes operator with Dirichlet boundary condition with the resolvent of the Stokes operator in the entire space.

Farwig and Sohr [5] investigate the Problem (1) when $\operatorname{div} \boldsymbol{u} \neq 0$ in Ω and $\boldsymbol{u} = \mathbf{0}$ on Γ. Their results include bounded and unbounded domains, for the whole and the half space the proof relies on multiplier technique. The problem is also investigated for bended half spaces and for cones by using perturbation criterion and referring to the half space problem.

The Problem (1) is also studied with Robin boundary conditions by Saal [10], Shibata and Shimada [11]. In [10], Saal proves that the Stokes operator with homogeneous Robin boundary conditions is sectorial and admits an $\boldsymbol{H}^\infty$-calculus on L^p-spaces. Shibata and Shimada proved in [11] a generalized resolvent estimate for the Stokes equations with non-homogeneous Robin boundary conditions and divergence condition in $\boldsymbol{L}^p$-framework in a bounded or exterior domain by extending the argument of Farwig and Shor [5].

Concerning the Navier-type boundary conditions, Miyakawa [9] shows that the Laplacian operator with homogeneous Navier-type boundary conditions generates a holomorphic semi-group on L^p-spaces when the domain Ω is of class C^∞. Mitrea and Monniaux [8] consider the resolvent of the Stokes operator with homogeneous Navier-type boundary conditions in Lipschitz domains using differential forms on Lipschitz sub-domains of a smooth compact Riemannian manifold. In [1] and [2] Al Baba et al. consider the Problem (1) when $\chi = 0$ in Ω and $g = 0$, $\boldsymbol{h} = \boldsymbol{0}$ on Γ and prove the existence of weak, strong and very weak solutions to this problem.

This paper is organized as follows. In Section 2 we give the functional framework and some preliminary results at the basis of our proofs. In Section 3 we prove our main results on the existence of weak, strong and very weak solutions to Problem (1).

§2. Preliminaries

In this subsection we review some basic notations, definitions and functional framework which are essential in our work.

In what follows, if we do not state otherwise, Ω will be considered as an open bounded domain of $\mathbb{R}^3$ of class $C^{2,1}$. Then a unit normal vector to the boundary can be defined almost everywhere it will be denoted by $\boldsymbol{n}$, $\boldsymbol{n}$ is defined everywhere because $\boldsymbol{n}$ is $C^{1,1}$. The generic point in Ω is denoted by $\boldsymbol{x} = (x_1, x_2, x_3)$. The domain Ω is not necessarily simply-connected and the boundary Γ is not necessarily connected.

Let us introduce some functional spaces.
Let $\boldsymbol{L}^p(\Omega)$ denote the usual vector valued $\boldsymbol{L}^p$-space over Ω. Let us define the spaces:

$$\begin{aligned}
\boldsymbol{H}^p(\mathbf{curl}, \Omega) &= \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\},\\
\boldsymbol{H}^p(\mathrm{div}, \Omega) &= \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\ \boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\},\\
\boldsymbol{X}^p(\Omega) &= \boldsymbol{H}^p(\mathbf{curl}, \Omega) \cap \boldsymbol{H}^p(\mathrm{div}, \Omega),
\end{aligned}$$

equipped with their graph norms. Thanks to [4] and [3] we know that $D(\overline{\Omega})$ is dense in $\boldsymbol{H}^p(\mathbf{curl}, \Omega)$, $\boldsymbol{H}^p(\mathrm{div}, \Omega)$ and $\boldsymbol{X}^p(\Omega)$. We also define the subspaces:

$$\begin{aligned}
\boldsymbol{H}^p_0(\mathbf{curl}, \Omega) &= \{\boldsymbol{v} \in \boldsymbol{H}^p(\mathbf{curl}, \Omega);\quad \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\},\\
\boldsymbol{H}^p_0(\mathrm{div}, \Omega) &= \{\boldsymbol{v} \in \boldsymbol{H}^p(\mathrm{div}, \Omega);\quad \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\},\\
\boldsymbol{X}^p_N(\Omega) &= \{\boldsymbol{v} \in \boldsymbol{X}^p(\Omega);\quad \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\},\\
\boldsymbol{X}^p_T(\Omega) &= \{\boldsymbol{v} \in \boldsymbol{X}^p(\Omega);\quad \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}.
\end{aligned}$$

We recall that for all function $\boldsymbol{v} \in \boldsymbol{H}^p(\mathbf{curl}, \Omega)$ (respectively $\boldsymbol{v} \in \boldsymbol{H}^p(\mathrm{div}, \Omega)$), the tangential trace $\boldsymbol{v} \times \boldsymbol{n}$ (respectively the normal trace $\boldsymbol{v} \cdot \boldsymbol{n}$) exists and belongs to $\boldsymbol{W}^{-1/p,p}(\Gamma)$ (respectively to $W^{-1/p,p}(\Gamma)$). Thanks to [4] we know that $D(\Omega)$ is dense in $\boldsymbol{H}^p_0(\mathbf{curl}, \Omega)$ and in $\boldsymbol{H}^p_0(\mathrm{div}, \Omega)$. Finally we denote by $[\boldsymbol{H}^p_0(\mathbf{curl}, \Omega)]'$ and $[\boldsymbol{H}^p_0(\mathrm{div}, \Omega)]'$ the dual spaces of $\boldsymbol{H}^p_0(\mathbf{curl}, \Omega)$ and $\boldsymbol{H}^p_0(\mathrm{div}, \Omega)$ respectively.

Next, we review some known results which are essential in our work. First, we recall that the vector-valued Laplace operator of a vector field $\boldsymbol{v}=(v_1, v_2, v_3)$ is equivalently defined by

$$\Delta\, \boldsymbol{v} = \mathbf{grad}\,(\mathrm{div}\, \boldsymbol{v}) - \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{v}.$$

We have the following lemmas [4]

Lemma 1. *The spaces $\boldsymbol{X}^p_N(\Omega)$ and $\boldsymbol{X}^p_T(\Omega)$ defined above are continuously embedded in $\boldsymbol{W}^{1,p}(\Omega)$.*

In order to consider the case of nonhomogeneous boundary conditions, we introduce the following spaces:

$$\boldsymbol{X}^{1,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\boldsymbol{v} \in L^p(\Omega),\ \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\ \text{and}\ \boldsymbol{v}\cdot\boldsymbol{n} \in W^{1-1/p,p}(\Gamma)\},$$

$$\boldsymbol{Y}^{1,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\boldsymbol{v} \in L^p(\Omega), \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\ \text{and}\ \boldsymbol{v}\times\boldsymbol{n} \in W^{1-1/p,p}(\Gamma)\}.$$

Lemma 2. *The spaces $\boldsymbol{X}^{1,p}(\Omega)$ and $\boldsymbol{Y}^{1,p}(\Omega)$ are continuously embedded in $\boldsymbol{W}^{1,p}(\Omega)$.*

Consider as well the spaces:

$$\boldsymbol{X}^{2,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\boldsymbol{v} \in W^{1,p}(\Omega), \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)\ \text{and}\ \boldsymbol{v}\cdot\boldsymbol{n} \in W^{2-1/p,p}(\Gamma)\},$$

$$\boldsymbol{Y}^{2,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\boldsymbol{v} \in W^{1,p}(\Omega), \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)\ \text{and}\ \boldsymbol{v}\times\boldsymbol{n} \in W^{2-1/p,p}(\Gamma)\}.$$

Theorem 3. *Assume that Ω is of class $C^{2,1}$, then the spaces $\boldsymbol{X}^{2,p}(\Omega)$ and $\boldsymbol{Y}^{2,p}(\Omega)$ are continuously embedded in $\boldsymbol{W}^{2,p}(\Omega)$.*

Consider now the space

$$\boldsymbol{E}^p(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\ \Delta\boldsymbol{v} \in [\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'\},$$

which is a Banach space for the norm $\|\boldsymbol{v}\|_{\boldsymbol{E}^p(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\Delta\boldsymbol{v}\|_{[\boldsymbol{H}^{p'}_o(\mathrm{div},\Omega)]'}$. Thanks to [3, Lemma 4.1] we know that $D(\overline{\Omega})$ is dense in $\boldsymbol{E}^p(\Omega)$. Moreover, (see [3, Corollary 4.2]), the linear mapping $\gamma : \boldsymbol{v} \longmapsto \mathbf{curl}\boldsymbol{v}\times\boldsymbol{n}$ defined on $D(\overline{\Omega})$ can be extended to a linear and continuous mapping $\gamma : \boldsymbol{E}^p(\Omega) \longmapsto \boldsymbol{W}^{-1/p,p}(\Omega)$. Moreover, we have the Green formula: for any $\boldsymbol{v} \in \boldsymbol{E}^p(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{X}^{p'}_\tau(\Omega)$ such that div $\boldsymbol{\varphi}= 0$ in Ω,

$$-\langle\Delta\boldsymbol{v}, \boldsymbol{\varphi}\rangle_{[\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)]'\times\boldsymbol{H}^{p'}_0(\mathrm{div},\Omega)} = \int_\Omega \mathbf{curl}\,\boldsymbol{v}\cdot\mathbf{curl}\,\overline{\boldsymbol{\varphi}}\,\mathrm{d}x - \langle\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n}, \boldsymbol{\varphi}\rangle_\Gamma,$$

where $\langle\cdot,\cdot\rangle_\Gamma = \langle\cdot,\cdot\rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma)\times\boldsymbol{W}^{1/p,p'}(\Gamma)}$.

Next, we introduce the following space

$$\boldsymbol{T}^p(\Omega) = \{\boldsymbol{\phi} \in \boldsymbol{H}^p_0(\mathrm{div},\Omega);\ \mathrm{div}\boldsymbol{\phi} \in W^{1,p}_0(\Omega)\}.$$

The space $\mathcal{D}(\Omega)$ is dense in $\boldsymbol{T}^p(\Omega)$ and for all $\chi \in W^{-1,p}(\Omega)$ and $\boldsymbol{\phi} \in \boldsymbol{T}^{p'}(\Omega)$, we have:

$$\langle\nabla\chi, \boldsymbol{\phi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)} = -\langle\chi, \mathrm{div}\boldsymbol{\phi}\rangle_{W^{-1,p}(\Omega)\times W^{1,p'}_0(\Omega)}. \tag{2}$$

A distribution $\boldsymbol{f}$ belongs to $(\boldsymbol{T}^p(\Omega))'$ if and only if there exist $\boldsymbol{\psi} \in \boldsymbol{L}^{p'}(\Omega)$ and $f_0 \in W^{-1,p'}(\Omega)$, such that $\boldsymbol{f} = \boldsymbol{\psi} + \nabla f_0$. Moreover, we have the estimate

$$\|\boldsymbol{\psi}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|f_0\|_{W^{-1,p'}(\Omega)} \le C\|\boldsymbol{f}\|_{(\boldsymbol{T}^p(\Omega))'}.$$

We will need also the following space

$$\boldsymbol{H}_p(\Delta;\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \Delta\boldsymbol{v} \in (\boldsymbol{T}^{p'}(\Omega))'\},$$

which is a Banach space for the norm $\|\boldsymbol{v}\|_{\boldsymbol{H}_p(\Delta;\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\Delta\boldsymbol{v}\|_{(\boldsymbol{T}^{p'}(\Omega))'}$. The space $\mathcal{D}(\overline{\Omega})$ is dense in $\boldsymbol{H}_p(\Delta;\Omega)$ and The mapping γ: $\boldsymbol{v} \longmapsto \mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n}$ defined on $D(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping $\gamma : \boldsymbol{H}_p(\Delta;\Omega) \longmapsto \boldsymbol{W}^{-1-1/p,p}(\Omega)$. Moreover, we have the Green formula: for any $\boldsymbol{v} \in \boldsymbol{H}_p(\Delta;\Omega)$ and $\boldsymbol{\phi} \in \boldsymbol{Y}_\tau^{p'}(\Omega)$,

$$\langle \Delta\boldsymbol{v}, \boldsymbol{\phi}\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times\boldsymbol{T}^{p'}(\Omega)} = \int_\Omega \boldsymbol{v}\cdot\Delta\overline{\boldsymbol{\phi}}\,\mathrm{d}x + \langle \mathbf{curl}\boldsymbol{v}\times\boldsymbol{n}, \boldsymbol{\phi}\rangle_\Gamma, \tag{3}$$

where $\langle\cdot,\cdot\rangle_\Gamma = \langle\cdot,\cdot\rangle_{\boldsymbol{W}^{-1-1/p,p}(\Gamma)\times\boldsymbol{W}^{1+1/p,p'}(\Gamma)}$ and

$$\boldsymbol{Y}_\tau^p(\Omega) = \{\boldsymbol{\phi} \in \boldsymbol{W}^{2,p}(\Omega);\ \boldsymbol{\phi}\cdot n = 0,\ \mathrm{div}\boldsymbol{\phi} = 0,\ \mathbf{curl}\boldsymbol{\phi}\times n = 0 \text{ on } \Gamma\}.$$

§3. Generalized resolvent problem

In this section we consider the generalized resolvent Problem (1) and we prove the existence and uniqueness of weak, strong and very weak solution to this problem.

3.1. Weak solution

Consider the problem

$$\begin{cases} \lambda\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{f}, & \mathrm{div}\,\boldsymbol{u} = 0 \quad \text{in} \quad \Omega\times(0,T), \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, & \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} \quad \text{on} \quad \Gamma\times(0,T), \end{cases} \tag{4}$$

We start by the existence and uniqueness of weak solution to (4).

Theorem 4. *Let $\varepsilon \in]0,\pi[$ be fixed and $\lambda \in \Sigma_\varepsilon$. Let $p \geq 2$, $\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'$ and $\boldsymbol{h}\times\boldsymbol{n} \in \boldsymbol{W}^{-1/p,p}(\Gamma)$. Then the problem (4) has a unique solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)/\mathbb{R}$ satisfying the following estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(\Omega,p)\Big(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}\Big). \tag{5}$$

Proof. **Step 1 : Existence and uniqueness.** We can easily verify that problem (4) is equivalent to the variational problem: Find $\boldsymbol{u} \in \boldsymbol{V}_\tau^p(\Omega)$ such that for all $\boldsymbol{v} \in \boldsymbol{V}_\tau^{p'}(\Omega)$

$$\lambda\int_\Omega \boldsymbol{u}\cdot\bar{\boldsymbol{v}}\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\bar{\boldsymbol{v}}\,\mathrm{d}x = \langle\boldsymbol{f},\boldsymbol{v}\rangle_\Omega + \langle\boldsymbol{h}\times\boldsymbol{n},\boldsymbol{v}\rangle_\Gamma, \tag{6}$$

where $\langle\cdot,\cdot\rangle_\Omega = \langle\cdot,\cdot\rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]'\times\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}$ and $\langle\cdot,\cdot\rangle_\Gamma = \langle\cdot,\cdot\rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma)\times\boldsymbol{W}^{-1/p,p'}(\Gamma)}$.

The proof is done in two steps:

i) Case $2 \le p \le 6$. The case $p = 2$ can be directly obtained using Lax-Milgram theorem. Suppose that $2 < p \le 6$, then Problem (4) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$. We write (4) in the form:

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} - \lambda \boldsymbol{u} = \boldsymbol{F}, & \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} & \text{on} \quad \Gamma \end{cases}. \tag{7}$$

As $\boldsymbol{H}^1(\Omega) \hookrightarrow \boldsymbol{L}^p(\Omega)$, we have $\boldsymbol{F} \in (\boldsymbol{H}_0^{p'}(\mathrm{div}; \Omega))'$ and

$$\forall\, \boldsymbol{v} \in \boldsymbol{K}_\tau^{p'}(\Omega), \quad \langle \boldsymbol{F}, \boldsymbol{v} \rangle_\Omega + \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{v} \rangle_\Gamma = 0. \tag{8}$$

Theorem 4.4 of [3] implies that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and $\pi \in L^p(\Omega)$.
Let $\boldsymbol{v} \in \boldsymbol{K}_\tau^{p'}(\Omega)$, using the variational formulation we have

$$\langle \boldsymbol{F}, \boldsymbol{v} \rangle_\Omega + \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{v} \rangle_\Gamma = 0.$$

Then our solution $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$.

ii) Case $p \ge 6$. Observe that $(\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega))' \hookrightarrow (\boldsymbol{H}_0^{6/5}(\mathrm{div}, \Omega))'$ and $\boldsymbol{W}^{-1/p,p}(\Gamma) \hookrightarrow \boldsymbol{W}^{-1/6,6}(\Gamma)$. Then Problem (7) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,6}(\Omega) \times L^6(\Omega)/\mathbb{R}$. Thanks to the embedding $\boldsymbol{W}^{1,6}(\Omega) \hookrightarrow \boldsymbol{L}^\infty(\Omega)$ we deduce that $\boldsymbol{F} = \boldsymbol{f} - \lambda \boldsymbol{u} \in (\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega))'$. Moreover, $\boldsymbol{F}$ satisfies the compatibility condition (8), then we conclude that $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$.

Step 2: Estimate. Let $B \in \mathcal{L}(\boldsymbol{V}_\tau^p(\Omega), (\boldsymbol{V}_\tau^{p'}(\Omega))')$ be the operator defined by

$$\forall\, \boldsymbol{u} \in \boldsymbol{V}_\tau^p(\Omega), \forall\, \boldsymbol{v} \in \boldsymbol{V}_\tau^{p'}(\Omega), \quad \langle B\boldsymbol{u}, \boldsymbol{v} \rangle_{(\boldsymbol{V}_\tau^{p'}(\Omega))' \times \boldsymbol{V}_\tau^p(\Omega)} = \lambda \int_\Omega \boldsymbol{u} \cdot \bar{\boldsymbol{v}}\, \mathrm{d}x + \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \bar{\boldsymbol{v}}\, \mathrm{d}x.$$

For all $p \ge 2$, the operator B is an isomorphism from $\boldsymbol{V}_\tau^p(\Omega)$ into $(\boldsymbol{V}_\tau^{p'}(\Omega))'$ and $\|\boldsymbol{u}\|_{\boldsymbol{X}_\tau^p} \approx \|B\boldsymbol{u}\|_{(\boldsymbol{V}_\tau^{p'}(\Omega))'}$ for all $\boldsymbol{u} \in \boldsymbol{V}_\tau^p(\Omega)$. Moreover using the continuous embedding $\boldsymbol{X}_\tau^p(\Omega) \hookrightarrow \boldsymbol{W}^{1,p}(\Omega)$ we have for every $\boldsymbol{u} \in \boldsymbol{V}_\tau^p(\Omega)$ solution of problem (6),

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} \le C(\Omega, p) \|\boldsymbol{u}\|_{\boldsymbol{X}_\tau^p(\Omega)} \le C(\Omega, p) \|B\boldsymbol{u}\|_{(\boldsymbol{V}_\tau^{p'}(\Omega))'}$$

and

$$\begin{aligned} \|B\boldsymbol{u}\|_{(\boldsymbol{V}_\tau^{p'}(\Omega))'} &= \sup_{\substack{\boldsymbol{v} \in \boldsymbol{V}_\tau^{p'}(\Omega) \\ \boldsymbol{v} \neq 0}} \frac{|\langle B\boldsymbol{u}, \boldsymbol{v} \rangle|}{\|\boldsymbol{v}\|_{\boldsymbol{X}_\tau^{p'}(\Omega)}} = \sup_{\substack{\boldsymbol{v} \in \boldsymbol{V}_\tau^{p'}(\Omega) \\ \boldsymbol{v} \neq 0}} \frac{|\langle \boldsymbol{f}, \boldsymbol{v} \rangle_\Omega + \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{v} \rangle_\Gamma|}{\|\boldsymbol{v}\|_{\boldsymbol{X}_\tau^{p'}(\Omega)}} \\ &\le C(\Omega, p) \left(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega))'} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)} \right), \end{aligned}$$

which is estimate (5). □

Theorem 5. *Let* $\lambda \in \Sigma_\varepsilon$. *Let* $p \ge 2$. *Let* $\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega))'$, $\boldsymbol{h} \times \boldsymbol{n} \in \boldsymbol{W}^{-1/p,p}(\Gamma)$, $g \in W^{1-1/p,p}(\Gamma)$ *and* $\chi \in L^p(\Omega)$ *verifying the following compatibility condition*

$$\int_\Omega \chi\, \mathrm{d}x = \int_\Gamma g\, \mathrm{d}\sigma. \tag{9}$$

Then problem (1) *has a unique solution* $(\boldsymbol{u},\pi)\in\boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)/\mathbb{R}$ *satisfying the following estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}+\|\pi\|_{L^p(\Omega)/\mathbb{R}}\leq C(\Omega,p,\lambda)(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'}+\|\chi\|_{L^p(\Omega)}+\|g\|_{W^{1-1/p,p}(\Gamma)}+\|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}).\quad(10)$$

Proof. **i) Existence and uniqueness.** Consider the following Neumann problem

$$\Delta\boldsymbol{\theta}=\chi\quad\text{in}\quad\Omega\qquad\text{and}\qquad\frac{\partial\theta}{\partial n}=g\quad\text{on}\quad\Gamma.\quad(11)$$

Since $g\in W^{1-1/p,p}(\Gamma)$ and $\chi\in L^p(\Omega)$ verifying the compatibility condition (9) this problem has a unique solution $\theta\in W^{2,p}(\Omega)/\mathbb{R}$ such that

$$\|\theta\|_{W^{2,p}(\Omega)/\mathbb{R}}\leq C\left(\|g\|_{W^{1-1/p,p}(\Gamma)}+\|\chi\|_{L^p(\Omega)}\right).\quad(12)$$

Set $\boldsymbol{F}=\boldsymbol{f}-\lambda\nabla\theta+\nabla\chi$ and observe that $\boldsymbol{F}\in(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'$. Then using Theorem 4 we deduce that the problem

$$\begin{cases}\lambda\boldsymbol{z}-\Delta\boldsymbol{z}+\nabla\pi=\boldsymbol{F}, & \mathrm{div}\,\boldsymbol{z}=0\quad\text{in}\quad\Omega\\ \boldsymbol{z}\cdot\boldsymbol{n}=0, & \mathbf{curl}\,\boldsymbol{z}\times\boldsymbol{n}=\boldsymbol{h}\times\boldsymbol{n}\quad\text{on}\quad\Gamma\end{cases}\quad(13)$$

has a unique solution $(\boldsymbol{z},\pi)\in\boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)/\mathbb{R}$ satisfying the following estimate

$$\|\boldsymbol{z}\|_{\boldsymbol{W}^{1,p}(\Omega)}\leq C(\Omega,p)\left(\|\boldsymbol{F}\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'}+\|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}\right).\quad(14)$$

Set $\boldsymbol{u}=\boldsymbol{z}+\nabla\theta$. Then $(\boldsymbol{u},\pi)$ solve (1).

ii) Estimate. Observe that

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}\leq C(\Omega,p)(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'}+|\lambda|\|\nabla\theta\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'}+\|\nabla\chi\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'}+\|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)})+\|\nabla\theta\|_{\boldsymbol{W}^{1,p}(\Omega)}.$$

Then using estimate (12) one gets

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}\leq C(\Omega,p,\lambda)(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'}+\|\chi\|_{L^p(\Omega)}+\|g\|_{W^{1-1/p,p}(\Gamma)}+\|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}).\quad(15)$$

Moreover $\|\pi\|_{L^p(\Omega)/\mathbb{R}}\leq C(\Omega,p)\,\|\nabla\pi\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'}=\|\boldsymbol{f}-\lambda\boldsymbol{u}+\Delta\boldsymbol{u}\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'}$. Thus

$$\|\pi\|_{L^p(\Omega)/\mathbb{R}}\leq C(\Omega,p,\lambda)(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'}+\|\chi\|_{L^p(\Omega)}+\|g\|_{W^{1-1/p,p}(\Gamma)}+\|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}).\quad(16)$$

Combining (15) together with (16) we obtain estimate (10).

$\square$

Theorem 6. *Let* $1 < p < 2$, $\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega))'$ *and* $\boldsymbol{h} \times \boldsymbol{n} \in \boldsymbol{W}^{-1/p,p}(\Gamma)$, $g \in W^{1-1/p,p}(\Gamma)$ *and* $\chi \in L^p(\Omega)$ *verifying the following compatibility condition* (9). *Then Problem* (1) *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$.

Proof. **Step 1:** We suppose that $g = 0$. The problem

$$\begin{cases} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, & \mathrm{div}\, \boldsymbol{u} = \chi, \quad \text{in} \quad \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n}, \quad \text{on} \quad \Gamma, \end{cases} \tag{17}$$

has the following equivalent variational formulation: Find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ satisfying $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ, such that $\forall\ \boldsymbol{w} \in \boldsymbol{W}^{1,p'}$ satisfying $\boldsymbol{w} \cdot \boldsymbol{n} = 0$ and $\mathbf{curl}\ \boldsymbol{w} \times \boldsymbol{n}$=0 on Γ

$$\lambda \int_\Omega \boldsymbol{u} \cdot \overline{\boldsymbol{w}}\, \mathrm{d}x + \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \overline{\boldsymbol{w}}\, \mathrm{d}x - \int_\Omega \pi \cdot \mathrm{div}\, \overline{\boldsymbol{w}}\, \mathrm{d}x = \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]' \times \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)}$$
$$+ \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{w} \rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma) \times \boldsymbol{W}^{-1/p,p'}(\Gamma)} - \int_\Omega \chi \cdot \mathrm{div}\, \overline{\boldsymbol{w}}\, \mathrm{d}x.$$

According to theorem 5, for any $(\boldsymbol{F}, \varphi)$ in $(\boldsymbol{H}_0^p(\mathrm{div}, \Omega))' \times L_0^{p'}(\Omega)$ there exists a unique solution $(\boldsymbol{w}, \eta) \in \boldsymbol{W}^{1,p'}(\Omega) \times L^{p'}(\Omega)/\mathbb{R}$ solution to

$$\begin{cases} \lambda \boldsymbol{w} - \Delta \boldsymbol{w} + \nabla \eta = \boldsymbol{F}, & \mathrm{div}\, \boldsymbol{w} = \varphi, \quad \text{in} \quad \Omega, \\ \boldsymbol{w} \cdot \boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{w} \times \boldsymbol{n} = 0, \quad \text{on} \quad \Gamma, \end{cases} \tag{18}$$

and satisfying

$$\|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p'}(\Omega)} + \|\eta\|_{L^{p'}(\Omega)/\mathbb{R}} \le C(\Omega, p', \lambda)(\|\boldsymbol{F}\|_{(\boldsymbol{H}_0^p(\mathrm{div},\Omega))'} + \|\varphi\|_{\boldsymbol{L}^{p'}(\Omega)}).$$

Let T be a linear form defined from $(\boldsymbol{H}_0^p(\mathrm{div}, \Omega))' \times L_0^{p'}(\Omega)$ onto $\mathbb{C}$ by

$$T : (\boldsymbol{F}, \varphi) \longmapsto \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]' \times \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)} + \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{w} \rangle_\Gamma - \int_\Omega \chi \cdot \overline{\eta}\, \mathrm{d}x.$$

Observe that

$$|T(\boldsymbol{F}, \varphi)| \le \|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'} \|\boldsymbol{w}\|_{\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega))'} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)} \|\boldsymbol{w}\|_{\boldsymbol{W}^{1/p,p'}(\Gamma)} + \|\varphi\|_{L^{p'}(\Omega)}.$$

Then T is continuous on $(\boldsymbol{H}_0^p(\mathrm{div}, \Omega))' \times \boldsymbol{L}^{p'}(\Omega)$ and we deduce that there exists a unique $(\boldsymbol{u}, \pi) \in \boldsymbol{H}_0^p(\mathrm{div}, \Omega) \times L^p(\Omega)/\mathbb{R}$ such that

$$T(\boldsymbol{F}, \varphi) = \langle \boldsymbol{u}, \boldsymbol{F} \rangle_{\boldsymbol{H}_0^p(\mathrm{div},\Omega) \times (\boldsymbol{H}_0^p(\mathrm{div},\Omega))'} - \int_\Omega \pi \cdot \overline{\varphi}\, \mathrm{d}x.$$

As a result

$$\lambda \int_\Omega \boldsymbol{u} \cdot \overline{\boldsymbol{w}}\, \mathrm{d}x + \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \overline{\boldsymbol{w}}\, \mathrm{d}x - \int_\Omega \pi \cdot \mathrm{div}\, \overline{\boldsymbol{w}}\, \mathrm{d}x$$
$$= \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{[\boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)]' \times \boldsymbol{H}_0^{p'}(\mathrm{div},\Omega)} + \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{w} \rangle_{\boldsymbol{W}^{-1/p,p}(\Gamma) \times \boldsymbol{W}^{-1/p,p'}(\Gamma)} - \int_\Omega \chi \cdot \mathrm{div}\, \overline{\boldsymbol{w}}\, \mathrm{d}x.$$

To finish, we shall prove that $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{1,p}(\Omega)$. To this end we write our problem in the form (7) where $\boldsymbol{F} = \boldsymbol{f} - \lambda\boldsymbol{u}$ belongs to $(\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega))'$ and satisfies (8). Then using [3, Remark 4.6] our solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$.

Step 2 : $g \neq 0$. Let $\theta \in W^{2,p}(\Omega)/\mathbb{R}$ be the unique solution of the Neumann problem (11) with $\chi \in L^p(\Omega)$ and $g \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$ satisfying (9). Let $\boldsymbol{F} = \boldsymbol{f} + \nabla\chi - \lambda\nabla\theta \in (\boldsymbol{H}_0^{p'}(\mathrm{div}, \Omega))'$. Then there exists $(z, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$ solution of (13). Set $\boldsymbol{u}$=$\boldsymbol{z} + \nabla\theta$. We can easily verify that $(\boldsymbol{u}, \pi)$ solves (1). □

3.2. Strong solution

Theorem 7. *Let* $1 < p < \infty$. *Let* $\boldsymbol{f} \in L^p(\Omega)$ *and* $\boldsymbol{h} \times \boldsymbol{n} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$. *Then the problem* (4) *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R}$ *satisfying the following estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C(\lambda, p, \Omega)(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}). \tag{19}$$

Proof. We know that problem (4) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)/\mathbb{R}$. Moreover π satisfies

$$\mathrm{div}(\nabla\pi - \boldsymbol{f}) = 0 \text{ in } \Omega, \qquad (\nabla\pi - \boldsymbol{f}) \cdot \boldsymbol{n} = -\mathrm{div}_\Gamma(\boldsymbol{h} \times \boldsymbol{n}) \text{ on } \Gamma.$$

Since $\boldsymbol{h} \times \boldsymbol{n} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$ we deduce that that $\pi \in W^{1,p}(\Omega)$.

Set $z = \mathbf{curl}\,\boldsymbol{u}$. Notice that z verify the following problem:

$$\begin{cases} \lambda z - \Delta z = \mathbf{curl} \boldsymbol{f}, \quad \mathrm{div}\, z = 0, & \text{in} \quad \Omega, \\ \quad z \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n}, & \text{on} \quad \Gamma, \end{cases} \tag{20}$$

where $\mathbf{curl}\,\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega))'$ and $\boldsymbol{h} \times \boldsymbol{n} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$. Then $z \in \boldsymbol{W}^{1,p}(\Omega)$ and satisfies

$$\|z\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(\Omega)(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}).$$

Thus $\boldsymbol{u} \in L^p(\Omega)$, div $\boldsymbol{u}$ =0 $\in W^{1,p}(\Omega)$, $\mathbf{curl}\,\boldsymbol{u} = z \in \boldsymbol{W}^{1,p}(\Omega)$ and $\boldsymbol{u} \cdot \boldsymbol{n} = 0 \in W^{1-1/p,p}(\Gamma)$. Then $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ and

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} \leq C(\lambda, p, \Omega)(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}).$$

Finally proceeding as in step 2 of the proof of theorem 5 , we obtain that the solution $(\boldsymbol{u}, \pi)$ satisfies the estimation (19) which ends the proof. □

Corollary 8. *Let* $1 < p < \infty$. *Let* $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $\boldsymbol{h} \times \boldsymbol{n} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$, $g \in W^{2-1/p,p}(\Gamma)$ *and* $\chi \in W^{1,p}(\Omega)$ *verifying the following compatibility condition* (9). *Then problem* (1) *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R}$ *satisfying*

$$\begin{aligned} \|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C(\Omega, p, \lambda)(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\chi\|_{W^{1,p}(\Omega)} + \|g\|_{W^{2-1/p,p}(\Gamma)} \\ + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}). \end{aligned} \tag{21}$$

Proof. Let $\theta \in W^{2,p}(\Omega)$ be the unique solution of the Neumann problem (11).
Set $\boldsymbol{F} = \boldsymbol{f} - \lambda\nabla\theta + \nabla\chi$ and observe that $\boldsymbol{F} \in \boldsymbol{L}^p(\Omega)$. Thanks to Theorem 7, the Problem (13) has a unique solution $(z, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)/\mathbb{R}$ satisfying

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C(\Omega, p, \lambda)(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}).$$

By setting $\boldsymbol{u} = \boldsymbol{z} + \nabla\theta$, we can easily verify that $(\boldsymbol{u}, \pi)$ solves (1) and verifies (21). □

3.3. Very weak solution

In this subsection we prove the existence of very week solution to Problem (1).

Theorem 9. *Let* $\boldsymbol{f} \in (\boldsymbol{T}^{p'}(\Omega))'$, $\chi \in L^p(\Omega)$, $g \in W^{-1/p,p}(\Gamma)$ *and* $\boldsymbol{h} \times \boldsymbol{n} \in \boldsymbol{W}^{-1-1/p,p}(\Gamma)$ *verifying the compatibility condition* (9)*. Then problem* (1) *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$*. Moreover the following estimate holds*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\pi\|_{W^{-1,p}(\Omega)/\mathbb{R}} \le C(\Omega, p, \lambda)(\|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'} + \|\chi\|_{L^p(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma)} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{-1-1/p,p}(\Gamma)}). \quad (22)$$

Proof. **Step 1.** Problem (1) is equivalent to the variational formulation: find $(\boldsymbol{u}, \pi) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega)/\mathbb{R}$ such that for any $\boldsymbol{\phi} \in \boldsymbol{Y}^{p'}_\tau(\Omega)$, and for any $q \in W^{1,p'}(\Omega)$,

$$\lambda \int_\Omega \boldsymbol{u} \cdot \overline{\boldsymbol{\phi}}\, \mathrm{d}x - \int_\Omega \boldsymbol{u} \cdot \Delta\overline{\boldsymbol{\phi}}\, \mathrm{d}x - \langle \pi, \mathrm{div}\boldsymbol{\phi}\rangle_{W^{-1,p}(\Omega)\times W^{1,p'}_0(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{\phi}\rangle_\Omega + \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{\phi}\rangle_\Gamma \quad (23)$$

$$\int_\Omega \boldsymbol{u} \cdot \nabla\overline{q}\, \mathrm{d}x = -\int_\Omega \chi\overline{q}\, \mathrm{d}x + \langle g, q\rangle_{W^{-1/p,p}(\Gamma)\times W^{1/p,p'}(\Gamma)}, \quad (24)$$

where $\langle\cdot,\cdot\rangle_\Omega = \langle\cdot,\cdot\rangle_{(\boldsymbol{T}^{p'}(\Omega))'\times \boldsymbol{T}^{p'}(\Omega)}$ and $\langle\cdot,\cdot\rangle_\Gamma = \langle\cdot,\cdot\rangle_{\boldsymbol{W}^{-1-1/p,p}(\Gamma)\times \boldsymbol{W}^{1+1/p,p'}(\Gamma)}$.
Indeed, using the Green formula (3), we can verify that every $(\boldsymbol{u}, \pi) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega)$ solution to (1) solves (23)-(24). Conversely, let $(\boldsymbol{u}, \pi) \in \boldsymbol{L}^p(\Omega) \times W^{-1,p}(\Omega)$ be a solution to (23)-(24). Clearly, $-\Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{f}$ and $\mathrm{div}\,\boldsymbol{u} = \chi$ in Ω.
Consequently, $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ and since $\nabla\pi \in (\boldsymbol{T}^{p'}(\Omega))'$, we have $\Delta\boldsymbol{u} = -\boldsymbol{f} + \lambda\boldsymbol{u} + \nabla\pi \in (\boldsymbol{T}^{p'}(\Omega))'$. Then $\boldsymbol{u} \in \boldsymbol{H}_p(\Delta, \Omega)$. Using (2) and (3), we obtain that for any $\boldsymbol{\phi} \in \boldsymbol{Y}^{p'}_\tau(\Omega)$:

$$\lambda \int_\Omega \boldsymbol{u} \cdot \overline{\boldsymbol{\phi}}\, \mathrm{d}x - \int_\Omega \boldsymbol{u} \cdot \Delta\overline{\boldsymbol{\phi}}\, \mathrm{d}x - \langle \mathbf{curl}\boldsymbol{u} \times \boldsymbol{n}, \boldsymbol{\phi}\rangle_\Gamma - \langle \pi, \mathrm{div}\boldsymbol{\phi}\rangle_{W^{-1,p}(\Omega)\times W^{1,p'}_0(\Omega)} = \langle \boldsymbol{f}, \boldsymbol{\phi}\rangle_\Omega.$$

Thus $\langle \mathbf{curl}\boldsymbol{u} \times \boldsymbol{n}, \boldsymbol{\phi}\rangle_\Gamma = \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{\phi}\rangle_\Gamma$. Let $\boldsymbol{\mu} \in \boldsymbol{W}^{1+1/p,p'}(\Gamma)$, there exists a function $\boldsymbol{\phi} \in \boldsymbol{W}^{2,p}(\Omega)$ satisfying

$$\boldsymbol{\phi}_\tau = \boldsymbol{\mu}_\tau \quad \text{and} \quad \frac{\partial\boldsymbol{\phi}}{\partial\boldsymbol{n}} = -\boldsymbol{n}\,\mathrm{div}_\Gamma\boldsymbol{\mu}_\tau + \sum_{j=1}^{2}\Big(\frac{\partial\boldsymbol{\mu}_\tau}{\partial s_j} \times \boldsymbol{T}_j\Big) \times \boldsymbol{n} \quad \text{on } \Gamma.$$

It is clear that $\boldsymbol{\phi} \in \boldsymbol{Y}^{p'}_\tau(\Omega)$ and

$$\langle \mathbf{curl}\boldsymbol{u} \times \boldsymbol{n}, \boldsymbol{\mu}\rangle_\Gamma - \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{\mu}\rangle_\Gamma = \langle \mathbf{curl}\boldsymbol{u} \times \boldsymbol{n}, \boldsymbol{\phi}_\tau\rangle_\Gamma - \langle \boldsymbol{h} \times \boldsymbol{n}, \boldsymbol{\phi}_\tau\rangle_\Gamma = 0.$$

Thus $\mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n}$ on Γ. Next using that $\mathrm{div}\,\boldsymbol{u}{=}\chi$ in Ω, we deduce that for any $q \in W^{1,p'}(\Omega)$, we have
$\langle \boldsymbol{u} \cdot \boldsymbol{n}, q\rangle_{W^{-1/p,p}(\Gamma)\times W^{1/p,p'}(\Gamma)} = \langle g, q\rangle_{W^{-1/p,p}(\Gamma)\times W^{1/p,p'}(\Gamma)}$. Consequently, $\boldsymbol{u} \cdot \boldsymbol{n} = g \in W^{-1/p,p}(\Gamma)$.

Step 2. Let us now solve Problem (23)-(24). We suppose that

$$g = 0 \quad \text{on} \quad \Gamma \qquad \text{and} \qquad \int_\Omega \chi\, \mathrm{d}x = 0.$$

Thanks to Theorem 8, for any pair $(\boldsymbol{F},\xi) \in \boldsymbol{L}^{p'}(\Omega)\times(W_0^{1,p'}(\Omega)\cap L_0^{p'}(\Omega))$ there exists a unique $(\boldsymbol{\phi},q) \in \boldsymbol{W}^{2,p'}(\Omega) \times W^{1,p'}(\Omega)/\mathbb{R}$ satisfying:

$$\begin{cases} \lambda\boldsymbol{\phi} - \Delta\boldsymbol{\phi} + \nabla q = \boldsymbol{F}, \quad \operatorname{div}\boldsymbol{\phi} = \xi, & \text{in} \quad \Omega, \\ \boldsymbol{\phi}\cdot\boldsymbol{n} = 0, \quad \mathbf{curl}\,\boldsymbol{\phi}\times\boldsymbol{n} = 0, & \text{on} \quad \Gamma, \end{cases} \tag{25}$$

with the estimate

$$\|\boldsymbol{\phi}\|_{\boldsymbol{W}^{2,p'}(\Omega)} + \|q\|_{\boldsymbol{W}^{1,p'}(\Omega)/\mathbb{R}} \leq C(\lambda,\Omega,p')(\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|\xi\|_{W^{1,p'}(\Omega)}).$$

Let T be a linear form defined from $\boldsymbol{L}^{p'}(\Omega)\times(W_0^{1,p'}(\Omega)\cap L_0^{p'}(\Omega))$ onto $\mathbb{C}$ by

$$T : (\boldsymbol{F},\xi) \longmapsto \langle \boldsymbol{f},\boldsymbol{\phi}\rangle_\Omega + \langle \boldsymbol{h}\times\boldsymbol{n},\boldsymbol{\phi}\rangle_\Gamma - \int_\Omega \chi q \,\mathrm{d}x.$$

An easy computation shows that

$$|T(\boldsymbol{F},\xi)| \leq C(\Omega,p',\lambda)(\|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1-1/p,p}(\Gamma)} + \|\chi\|_{L^p(\Omega)})(\|\boldsymbol{F}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|\xi\|_{W^{1,p'}(\Omega)}).$$

This means that T defines an element of the dual space of $\boldsymbol{L}^{p'}(\Omega)\times(W_0^{1,p'}(\Omega)\cap L_0^{p'}(\Omega))$ and according to the Riesz's representation theorem, there exists a unique $(\boldsymbol{u},\pi) \in \boldsymbol{L}^p(\Omega)\times W^{-1,p}(\Omega)/\mathbb{R}$ such that

$$T(\boldsymbol{F},\xi) = \langle \boldsymbol{u},\boldsymbol{F}\rangle_{\boldsymbol{T}^{p'}(\Omega)\times(\boldsymbol{T}^{p'}(\Omega))'} - \int_\Omega \pi\xi \,\mathrm{d}x.$$

Then $(\boldsymbol{u},\pi)$ is a solution to (23)-(24) and satisfies (22).

Step 3. Suppose that $g \neq 0$ and the compatibility condition (9) holds. The Neumann problem (11) has a unique solution $\theta \in W^{1,p}(\Omega)/\mathbb{R}$ satisfying the estimate:

$$\|\theta\|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C(\|\chi\|_{L^p(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma)}).$$

Set $\boldsymbol{F} = \boldsymbol{f} - \lambda\nabla\theta + \nabla\chi$. Then $\boldsymbol{F} \in (\boldsymbol{T}^{p'}(\Omega))'$ and the Problem (13) has a unique solution $(\boldsymbol{z},\pi) \in \boldsymbol{L}^p(\Omega)\times W^{-1,p}(\Omega)/\mathbb{R}$ satisfying the following estimate

$$\|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} + \|\pi\|_{W^{-1,p}(\Omega)/\mathbb{R}} \leq C(\lambda,\Omega,p)\left(\|\boldsymbol{F}\|_{(\boldsymbol{T}^{p'}(\Omega))'} + \|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-1-1/p,p}(\Gamma)}\right). \tag{26}$$

Then $(\boldsymbol{u},\pi)$ with $\boldsymbol{u} = \boldsymbol{z} + \nabla\theta$ solves (1) and satisfies (22). □

Remark 1. **i)** Consider the Problem (1) with $\chi \in W^{1,p}(\Omega)$ such that $\int_\Omega \chi \,\mathrm{d}x = 0$, $g = 0$ and $\boldsymbol{h} = \boldsymbol{0}$ on Γ. As in [7] we can prove that the solution $(\boldsymbol{u},\pi)$ satisfies the following estimate

$$|\lambda|\,\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} + \|\nabla\pi\|_{L^p(\Omega)} \leq C\,(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} + \|\nabla\chi\|_{\boldsymbol{L}^p(\Omega)} + |\lambda|\,\|\chi\|_{W^{-1,p}(\Omega)}). \tag{27}$$

Indeed, let $\theta \in W^{2,p}(\Omega)/\mathbb{R}$ solution to $\Delta\theta = \chi$ in Ω, $\frac{\partial\theta}{\partial\boldsymbol{n}} = 0$ on Γ and satisfying $\|\theta\|_{W^{2,p}(\Omega)} \leq C\,\|\chi\|_{W^{1,p}(\Omega)}$. Set $\boldsymbol{F} = \boldsymbol{f} - \lambda\nabla\theta + \nabla\chi$, then $\boldsymbol{F} \in \boldsymbol{L}^p(\Omega))$ and the problem

$$\begin{cases} \lambda\boldsymbol{z} - \Delta\boldsymbol{z} + \nabla\pi = \boldsymbol{F}, \quad \operatorname{div}\boldsymbol{z} = 0 & \text{in} \quad \Omega \\ \boldsymbol{z}\cdot\boldsymbol{n} = 0, \quad \mathbf{curl}\,\boldsymbol{z}\times\boldsymbol{n} = \boldsymbol{0} & \text{on} \quad \Gamma \end{cases}$$

has a unique solution $(\boldsymbol{z},\pi) \in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)/\mathbb{R}$ satisfying the following estimate

$$|\lambda|\,\|\boldsymbol{z}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\nabla\pi\|_{\boldsymbol{L}^p(\Omega)} \leq C(\Omega,p)\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega))} + \|\nabla\chi\|_{\boldsymbol{L}^p(\Omega))} + |\lambda|\|\nabla\theta\|_{\boldsymbol{L}^p(\Omega))}\right)$$

Set $\boldsymbol{u} = \boldsymbol{z} + \nabla\theta$. Then $(\boldsymbol{u},\pi)$ is a solution to (1) and satisfies (27).

ii) Notice that when $\chi = 0$ we recover the resolvent estimate established in [1] and [2].

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