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PARAMETER-UNIFORM NUMERICAL METHODS FOR SINGULARLY PERTURBED PROBLEMS

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Abstract. Singularly perturbed differential equations are often characterized by the presence of a small parameter multiplying the highest order derivative term(s) in the differential equation. The solutions typically exhibit steep gradients in narrow regions (often called layers) of the domain [3, 11]. It is important to discuss the choice of norm used to measure the accuracy of any proposed approximate solutions. Standard numerical methods typically fail to accurately capture these layers. This paper outlines the rationale for the definition of parameter-uniform numerical methods and it highlights the central tool used in the numerical analysis associated with singularly perturbed problems [8, 16].

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§1. Introduction

Our interest lies in analysing numerical algorithms, which generate finite dimensional approximations $\bar{U}^N$ to solutions u (defined over a continuum) of a wide class of problems involving partial differential equations. For a large set of problems, we wish to prove that this sequence of computable approximations $\bar{U}^N$ converge (in some norm) to the solution of the continuous problem, as the dimension N of the discrete problem tends to infinity. In addition, it is useful to identify the rate at which this sequence converges. Thus, the main aim here is to establish a theoretical error bound of the form:

$$\|u - \bar{U}^N\|_* \leq CN^{-p}, \quad p > 0.$$

Typically, the exponent p is an integer and C is an undetermined constant, which depends on the problem data and on the value of p; but the error constant C is independent of N; so that as $N \to \infty$, the error tends to zero, in the selected norm $\|\cdot\|_*$.

In practice, limitations need to be placed on the extent of the problem class (e.g. the types of domains admitted, smoothness of the data in the differential operator, the magnitude and sign of certain coefficients in the differential equation, etc.) in order to prove some kind of error estimate. Of course, we hope that we do not place so many restrictions on the problem data that the class corresponds to an empty set or simply to a set of constant coefficient second order boundary values problems. This comment is most pertinent when dealing with classes of nonlinear problems. Moreover, it would be best if all the restrictions on the problem class are explicitly specified. In many cases these restrictions are implicitly stated. In addition, we want to be able to establish theoretical error bounds for some particular

problems (from the problem class) for which an exact solution is **not** explicitly known. If the exact solution is known (in a manageable closed form), then any numerical approximation is rather superfluous. Most importantly, what is an appropriate norm $\|\cdot\|_*$ to measure accuracy ? In the context of theoretical numerical analysis, the choice of norm is open to the individual numerical analyst. Note that it is usually much easier to prove convergence in a weaker norm. In fact, if one is under time pressure to complete a proof, a change to a different norm can greatly facilitate adherence to a publication deadline!

§2. Continuous problem

Throughout this paper we shall *assume* that we are dealing with a bounded domain $\Omega \subset R^n,\ n = 1, 2, 3$; with the boundary denoted by $\partial\Omega$. Consider the continuous problem: Find $u(\mathbf{x}),\ \mathbf{x} \in \overline{\Omega}$, that satisfies the boundary value problem

$$Lu(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \overline{\Omega};$$

where we *assume* that L is a linear second order differential operator[1]. We shall further *assume* that the differential operator is inverse-monotone: That is, for all z within the domain $D(L)$ of the operator L

$$Lz(\mathbf{x}) \geq 0,\ \forall \mathbf{x} \in \overline{\Omega} \quad \text{implies that} \quad z(\mathbf{x}) \geq 0,\ \forall \mathbf{x} \in \overline{\Omega}.$$

This is a pointwise property of the differential operator. If the operator has this property, we can conclude that there exists a positive constant C such that[2] $\|u\|_\infty \leq C\|f\|_\infty$. This stability bound ensures that the problem has a unique solution and is stable with respect to perturbations in the data. The inverse-monotone property allows one bound the continuous solution u, without explicitly identifying the inverse of the operator L. For example, the differential operator L could be of parabolic-type

$$L_1 := -\frac{\partial^2}{\partial x^2} + a(x,t)\frac{\partial}{\partial x} + b(x,t)I + c(x,t)\frac{\partial}{\partial t}, \quad (x,t) \in (0,1)\times(0,T];$$

or of elliptic type $L_2 := -\triangle + \mathbf{a}(\mathbf{x})\cdot\nabla + b(\mathbf{x})I,\ \mathbf{x} \in \Omega := (0,1)\times(0,1)$; and on the boundary we specify, for example, Dirichlet boundary conditions $L := I,\ \mathbf{x} \in \partial\Omega$. In general, restrictions need to be imposed on the sign of coefficients so that the operator L has the inverse-monotone property (e.g., for the parabolic operator L_1, it suffices to require $b \geq 0$ and $c > 0$).

For the numerical analyst, one typically requires the solution to be sufficiently regular for the analysis to be applicable. For the elliptic problem posed on the unit square, $L_2 u = f,\ \mathbf{x} \in \Omega := (0,1)\times(0,1), u = 0,\ \mathbf{x} \in \bar{\Omega} \setminus \Omega$; one often requires $u \in C^{3,\alpha}(\bar{\Omega})$. For this level of regularity[3], one can *assume* that $\mathbf{a}, b, f \in C^{1,\alpha}(\bar{\Omega})$ and the additional local compatibility conditions $f(0,0) = f(1,0) = f(0,1) = f(1,1) = 0$ are satisfied. However, for many elliptic problems posed on a non-rectangular domain $\tilde{G}$, local compatibility conditions do not exist

[1] Normally, the definition of the operator L on the boundary and in the interior of the domain are given separately. For example, $Lu(\mathbf{x}) := -\triangle u(\mathbf{x}) + a(\mathbf{x})u(\mathbf{x}),\ \mathbf{x} \in \Omega := (0,1)\times(0,1);\ Lu(\mathbf{x}) := u(\mathbf{x}),\ \mathbf{x} \in \partial\Omega$.

[2] The stability bound can be established by constructing a bounded function $y(\mathbf{x})$ such that $L(y \pm u)(\mathbf{x}) \geq 0, \forall \mathbf{x} \in \overline{\Omega}$.

[3] See [7] or [9], for example, for a definition of the Hölder spaces $C^{n,\alpha}(\bar{\Omega})$.

[4] to guarantee that the solution $u \in C^{3,\alpha}(G) \cap C^{1,\alpha}(\bar{G})$. Hence, one often comes across implicit assumptions on the data of the form: *Assume* $u \in C^{3,\alpha}(\bar{\Omega})$. These kind of implicit regularity assumptions can significantly limit the scope of the problem class.

§3. Discrete Problem and classical error analysis

Within the numerical analysis literature, there are two standard numerical approaches to generating numerical approximations to the solutions of partial differential equations: the finite difference method and the (far more popular) finite element method. In the finite difference method, the continuous derivatives are approximated by finite differences and the error is typically measured in a pointwise norm,

$$\|u - \bar{U}^N\|_{\overline{\Omega},\infty} := \max_{\mathbf{x}\in\overline{\Omega}} |(u - \bar{U}^N)(\mathbf{x})| \leq CN^{-p}, \quad p > 0;$$

where $\bar{U}^N$ corresponds to some interpolated global approximation generated from the nodal values $U^N(\mathbf{x}_i)$. On the other hand, the finite element method discretizes a weak form of the continuous problem and the accuracy is measured in an L^2-based norm, such as the H^1-norm

$$\|e\|^2_{H^1(\Omega)} := \int_\Omega e_x^2 + e_y^2 + e^2 d\Omega, \quad \text{where} \quad e = u - \bar{U}^N.$$

In general, less regularity is required of the continuous solution u in the case of the finite element method, which is a major attraction of this approach. Let us now discuss these two numerical approaches in some more detail.

For the finite difference method, a finite set of mesh points (or sample points) $\bar{\Omega}^N$ is selected within the continuous domain $\bar{\Omega}$. (Here N is the number of mesh intervals used in any coordinate direction.) Consider a sequence of approximate mesh functions (vectors) $U^N(\mathbf{x_n})$, $\mathbf{x_n} \in \bar{\Omega}^N$, which are the solutions of the discrete problem

$$L^N U^N(\mathbf{x_n}) = f(\mathbf{x_n}), \quad \mathbf{x_n} \in \overline{\Omega}^N;$$

and L^N is a finite difference operator (i.e., a matrix). The finite difference operator could be one of the following [4]

$$L_1^N := -\delta_x^2 + D_t^+, \quad \text{or} \quad L_2^N := -\delta_x^2 - \delta_y^2 + aD_x^- + bD^- y.$$

The discrete first derivatives D_x^+, D_x^- are defined at each mesh point by

$$D_x^+ U^N(x_i, y_j) := \frac{U^N(x_{i+1}, y_j) - U^N(x_i, y_j)}{x_{i+1} - x_i}, \; D_x^- U^N(x_i, y_j) := \frac{U^N(x_i, y_j) - U^N(x_{i-1}, y_j)}{x_i - x_{i-1}}.$$

Both of these discrete first derivatives can be viewed as an approximation to the first order continuous partial derivative $u_x(x_i, y_j)$.

[4] The classical approximation to the second order derivative $u_{xx}(x_i, y_j)$ is

$$\delta_x^2 U^N(x_i, y_j) := \frac{(D^+ U^N - D^- U^N)(x_i, y_j)}{\bar{h}_i}, \qquad \bar{h}_i := (h_i + h_{i+1})/2, \quad h_i := x_i - x_{i-1}.$$

It is desirable to construct the finite difference operator L^N so that the inverse-monotone property is inherited from the differential operator. That is: for any mesh function Z

$$L^N Z(\mathbf{x_i}) \geq 0, \ \forall \mathbf{x_i} \in \overline{\Omega}^N \quad \text{implies that} \quad Z(\mathbf{x_i}) \geq 0, \ \forall \mathbf{x_i} \in \overline{\Omega}^N.$$

In matrix terminology we require the matrix L^N to be monotone [5]. Typically, one must choose appropriate discrete replacements for the first derivatives in the differential operator L so that the associated matrix L^N is an M-matrix[6]. In practice, it is relatively easy to identify an M-matrix, but this set is a distinct subset of the set of monotone matrices. In the case of certain partial differential operators (e.g., containing a mixed second order derivative term), natural discretizations of the various derivatives may lead to a matrix which is not an M-matrix, as not all of the off-diagonal elements may be of one sign. Unfortunately, the characterizations of monotone matrices is not as well developed as it is for M-matrices. For rectangular non-uniform meshes, we can simply *assume* that there are no mixed derivative terms in the differential operator so that the finite difference operator L^N can be assembled according to criteria that guarantee an M-matrix structure.

A classical proof for establishing a nodal error bound, involves combining a truncation error bound $\|L^N(u-U)\|$ with some type of stability argument. Stability can be viewed as implicitly bounding the inverse of the finite difference operator L^N. Truncation error bounds revolve around bounding integrals of the form

$$(D_x^+ u - \frac{\partial u}{\partial x})(x_i, y_j) = \frac{1}{h_i} \int_{t=x_{i-1}}^{x_i} \int_{s=x_i}^{t} u_{ss}(s, y_j)\, ds\, dt, \quad h_i := x_i - x_{i-1}.$$

The argument used to establish a truncation error bound is straightforward. A typical truncation error bound, for second order problems, would be of the form:

$$\|L^N(u-U^N)\|_{\Omega^N} = \|L^N u - f\|_{\Omega^N} = \|(L-L^N)u\|_{\Omega^N} \leq C_1(\|u^{(3)}\|_\Omega + \|u^{(2)}\|_\Omega)N^{-q}, \ q > 0;$$

where $\|u^{(k)}\|_\Omega$ denotes the maximum value of the k^{th} order partial derivatives across the entire domain. So the truncation error depends on the magnitude of the derivatives of the continuous solution across the continuum $\bar{\Omega}$. For our purposes, stability is established using inverse-monotonicity of the finite difference operator. An error bound is established by constructing a barrier function Y, such that $\|Y\|_{\Omega^N} \leq C_2 N^{-p}$ and $(L^N Y)(x_i, y_j) \geq C_1(\|u^{(3)}\|_{\Omega,\infty} + \|u^{(2)}\|_{\Omega,\infty})N^{-q}, \ \forall (x_i, y_j) \in \Omega^N$. Since $L^N(Y \pm (U^N - u))(x_i, y_j) \geq 0$, by inverse-monotonicity, we have the nodal error bound

$$\|U^N - u\|_{\Omega^N,\infty} \leq C_2 N^{-p}.$$

It is important to state that the order of convergence p for the error is not necessarily the same as the order of convergence q of the truncation error.

A nodal error bound is a measure of accuracy at the mesh points Ω^N. What about the other points in the domain Ω ? If there are no mesh points located in the vicinity of some singularity in the solution, then, over a limited range of N, the singularity may never be

[5] A real $n \times n$ matrix A is monotone if for any vector $\mathbf{x}$, $A\mathbf{x} \geq \mathbf{0}$ implies $\mathbf{x} \geq \mathbf{0}$.

[6] A real $n \times n$ matrix $A = (a_{i,j})$ with $a_{i,j} \leq 0, i \neq j$ is an M-matrix if A is nonsingular and $A^{-1} \geq 0$.

observed in the numerical solutions. To compare the performance of two numerical methods, one should compare the global accuracy of both methods.

Let $\phi_i(\mathbf{x})$ be a basis function associated with the mesh point $\mathbf{x_i}$ with $\phi_i(\mathbf{x_i}) = 1,\ \phi_i(\mathbf{x_j}) = 0, \quad i \neq j$; where ϕ_i is a piecewise polynomial. Form the global approximation

$$\bar{U}^N(\mathbf{x}) = \sum_i U^N(\mathbf{x_i})\phi_i(\mathbf{x}).$$

For the purposes of the numerical analysis, one forms the interpolant $\bar{u}(\mathbf{x}) = \sum_i u(\mathbf{x_i})\phi_i(\mathbf{x})$. The global error bound is of the form

$$\|u - \bar{U}^N\|_{\Omega,\infty} \leq \|\bar{u} - \bar{U}^N\|_{\Omega,\infty} + \|u - \bar{u}\|_{\Omega,\infty} \leq C_2 N^{-p} + \|u - \bar{u}\|_{\Omega,\infty}.$$

The interpolation error $\|u - \bar{u}\|_{\Omega,\infty}$ is determined by the choice of basis functions and by the location of the mesh points. Note that the above error bound uses the pointwise L_∞ norm.

For the finite element method, we again select a finite set of nodal points $\bar{\Omega}^N$ within $\bar{\Omega}$. Based on this choice of nodal points, the domain is divided into a finite number of subdomains (or elements). A weak form of the problem is then constructed. For example, for the elliptic problem $-\triangle u + au = f, (x, y) \in \Omega,\ u(x, y) = 0, (x, y) \in \partial\Omega$; an associated weak form would be: Find $u \in H_0^1(\Omega)$ such that

$$B(u, v) := (u_x, v_x) + (u_y, v_y) + (au, v) = (f, v), \quad \forall v \in H_0^1(\Omega); \qquad (u, v) := \int_\Omega uv\, d\Omega.$$

Choose a finite set of basis functions $\{\phi_i(x, y)\}_{i=1}^{N\times N} \in H_0^1(\Omega)$ and let V^N be the linear span of these basis functions. A discrete weak form of the problem is: Find $\bar{U} \in V^N \subset H_0^1(\Omega)$ s.t.

$$B(\bar{U}, \bar{V}) = (f, \bar{V}), \qquad \forall\ \bar{V} \in V^N \subset H_0^1(\Omega).$$

The discretization of the weak form occurs when, rather than testing against all $v \in H_0^1(\Omega)$, we only test against functions in a finite dimensional subset of $H_0^1(\Omega)$. By construction of the subspace V^N, the error is orthogonal to the subspace in the following sense:

$$B(u - \bar{U}, \bar{V}) = B(u, \bar{V}) - B(\bar{U}, \bar{V}) = 0, \forall\ \bar{V} \in V^N.$$

Finite element error analysis crucially relies on the *assumptions* that the bilinear form is both coercive, i.e., there exists a positive constant C_3 such that $B(u, u) \geq C_3\|u\|_V^2,\ \forall\ u \in V$; and bounded, i.e., there exists a positive constant C_4 such that $B(u, v) \leq C_4\|u\|_V\|v\|_V,\ \forall u, v \in V$. If the bilinear form is both coercive and bounded, then

$$C_3\|u - \bar{U}\|_V^2 \leq B(u - \bar{U}, u - \bar{u}) + B(u - \bar{U}, \bar{u} - \bar{U}) = B(u - \bar{U}, u - \bar{u}) \leq C_4\|u - \bar{u}\|_V\|u - \bar{U}\|_V.$$

Hence, the following error bound has been established

$$\|u - \bar{U}\|_V \leq \frac{C_4}{C_3}\|u - \bar{u}\|_V.$$

Assuming all integrals are evaluated exactly, the finite element error analysis reduces to simply estimating the interpolation error $u - \bar{u}$. However, note that a certain norm $\|\cdot\|_V$ is required in this error analysis. How appropriate are these energy norms for a problem containing certain singularities?

§4. Singularly perturbed problems

Consider the following class of singularly perturbed elliptic problems:

Given $\mathbf{a}, b, f \in C^m(\Omega), g \in C^p(\partial\Omega)$ and $\Omega = (0,1)^n, n = 1, 2, 3$; find u such that

$$L_\varepsilon u := -\varepsilon \triangle u + \mathbf{a} \cdot \nabla u + bu = f \text{ in } \Omega, \quad u = g, \text{ on } \quad \partial\Omega; \qquad 0 < \varepsilon \leq 1.$$

Recall that our main aim is to design a numerical method for a large class of problems. Small values for the singular perturbation ε have a significant effect on the character of the solution. Parameter-uniform numerical methods are designed to generate pointwise accurate approximations in the classical case (when $\varepsilon = 1$) of a smooth continuous solution u, in the extreme case of very small values of $\varepsilon \ll CN^{-1}$, when the problem is hyperbolic in character over most of the domain and in the transitional region where $CN^{-1} < \varepsilon < 1$. Thus the accuracy of parameter-uniform numerical methods is guaranteed for all values of the perturbation parameter within the range $0 < \varepsilon \leq 1$.

There are serious issues with classical finite element approaches to constructing numerical methods for singularly perturbed problems. In general, inverse-monotonicity is lost when the singular perturbation parameter ε is small (relative to N^{-1}). The continuous solution may be known to be non-negative, but the finite element approximation will produce negative values. Moreover, large spurious oscillations appear in the computed solution, which have nothing to do with the continuous solution. There is a vast literature on stabilized finite element methods for singularly perturbed problems, which aim to minimize the adverse effects of losing the inverse-monotonicity property of the differential operator. In addition to this defect, we must question the suitability of the energy norm for singularly perturbed problems. A typical corner singularity (appearing in the solution of a singularly perturbed elliptic problem) would be of the form

$$\exp(-(1-x)/\varepsilon)\exp(-(1-y)/\varepsilon).$$

This function is almost zero, except in an ε-neighbourhood of the single point $(1,1)$. For classical finite element analysis, the error is typically measured in the weighted energy norm

$$|||w|||_\varepsilon^2 := \varepsilon(w_x, w_x) + \varepsilon(w_y, w_y) + (w, w);$$

but the magnitude of the corner singularity is negligible in this energy norm. Note that

$$|||e^{-(1-x)/\varepsilon}e^{-(1-y)/\varepsilon}|||_\varepsilon = C\sqrt{\varepsilon}; \quad \text{but} \quad \|e^{-(1-x)/\varepsilon}e^{-(1-y)/\varepsilon}\|_{\Omega,\infty} = 1.$$

This corner singularity is visible in the L_∞ norm, but not in the energy norm.

Under certain restrictions on the data, the derivatives of the continuous solution u satisfy the bounds $\|u^{(k)}\|_{\Omega,\infty} \leq C\varepsilon^{-k}$, $k \leq 3$. A sharp nodal error bound on a classical finite difference method (e.g., a stable scheme on a uniform mesh) will be of the form

$$\|U^N - u\|_{\Omega^N,\infty} \leq CN^{-1}\varepsilon^{-1},$$

where C is independent of both N and ε. If ε is small (say 10^{-6}) then the magnitude of N needs to be at least 10^6 before the above upper bound is of order one. This is observed in practice [3]. Using a uniform mesh, a classical numerical method needs the number of mesh elements in each coordinate direction to be greater than ε^{-1} before convergence begins. This is a severe practical restriction on classical numerical methods.

§5. Parameter-uniform numerical methods

All of the previous discussion on numerical methods for singularly perturbed problems leads us to the following definition: For singularly perturbed problems, a parameter-uniform numerical method is inverse-monotone, satisfies a global error bound (in the pointwise norm) of the form

$$\|\bar{U}^N - u\|_{\Omega,\infty} \leq CN^{-p}, \quad p > 0,$$

and, most importantly, the error constant C is independent of ε and N. How does one construct a parameter-uniform method for a large family of singularly perturbed problems ? The numerical method will require a choice of mesh points (or elements) and a choice of finite difference operator (or basis functions in the finite element context). Let us first consider the fitted operator approach on the simplest of possible meshes, a uniform mesh.

For constant coefficient ordinary differential equations, one can construct a finite difference operator L_*^N such that the numerical method is exact at the nodes ($U^N(x_i) = u(x_i)$). In the case of ordinary differential equations, this fitted operator L_*^N can be easily extended to the corresponding variable coefficient problem. Nodal error bounds of the form

$$\|U^N - u\|_{\Omega^N,\infty} \leq CN^{-1} \quad (\text{or} \quad CN^{-2});$$

(with C independent of ε) exist for convection-diffusion problems of the form $-\varepsilon u'' + au' = f$, $a \geq \alpha > 0$ [5, 11]. This fitted operator approach can also be applied to the following problem classes, posed on the unit interval $x \in (0,1)$ (and $t \in (0,1)$):

$$-\varepsilon u'' + a(x)u' + b(x)u = f(x), \quad a \geq \alpha > 0, b \geq 0; \tag{1}$$

$$-\varepsilon u'' + b(x)u = f(x), \quad b > 0; \tag{2}$$

$$-\varepsilon u_{xx} + a(x,t)u_x + b(x,t)u + c(x,t)u_t = f(x,t), \quad a \geq \alpha > 0, c \geq \delta > 0, b \geq 0. \tag{3}$$

However, for the following class of parabolic problems (which includes the heat equation)

$$-\varepsilon u_{xx} + b(x,t)u + c(x,t)u_t = f(x,t), \quad b > 0, c \geq \delta > 0 \tag{4}$$

then, on a uniform mesh, no fitted operator exists such that the resulting numerical method is parameter-uniform nodally [14]. Moreover, even for the above four problem classes, if we use a uniform mesh and polynomial interpolation, then a nodally exact finite difference scheme will not be globally parameter-uniform. A central message of this paper is to state: For singularly perturbed problems do **not** use a uniform mesh.

Ideally, we should construct a uniform mesh for the dependent variable, which is of course not feasible (in general), without knowing the exact solution u. Bakhvalov (1934-2005) was the first to introduce special non–uniform meshes for solving singularly perturbed boundary value problems [2]. To construct a suitable mesh, consider the use of a mesh generating function $\lambda : \overline{\Omega} \to [-1,1]$, which is continuous and strictly decreasing, with $\lambda(0) = 1$ and $\lambda(1) = -1$. The mesh points $\{x_i\}$ are defined implicitly by the equations

$$\lambda(x_i) = 1 - \frac{2i}{N}.$$

If one takes $\lambda(x) = 1 - 2x$, then the resulting mesh is a uniform mesh in the independent variable. One could also take

$$\lambda_B(x) = e^{-x/\varepsilon};$$

and the resulting mesh is logarithmic in the independent variable. Note that, $\lambda_B(x) \leq \varepsilon, x \geq \tau := \varepsilon \ln \frac{1}{\varepsilon}$. Bakhvalov based his mesh on a combination of a uniform mesh and a logarithmic mesh associated with the mesh generating function λ_B. Twenty years after Bakhvalov's mesh, Shishkin [13, 12] introduced a simplification of the Bakhvalov mesh and subsequently extended his approach to an extensive class of singularly perturbed problems [15].

For the one-dimensional convection-diffusion problem (1) the Shishkin mesh splits the domain in a way that depends on both ε and N. Assign mesh points equally between the two subintervals defined by,

$$[0,1] = [0, 1-\sigma] \cup [1-\sigma, 1], \quad \sigma := \min\{\frac{1}{2}, C\varepsilon \ln N\}, \quad C \geq \frac{1}{\alpha}.$$

Observe that the transition parameter σ is given explicitly, the transition point σ differs from the analytical layer width $\tau = \varepsilon \ln(1/\varepsilon)$ and the piecewise-uniform mesh is only non-uniform at one single point. In the case of the one-dimensional reaction-diffusion problem (2) the standard Shishkin mesh splits the distribution of the mesh points $(N/4 : N/2 : N/4)$ between the three subintervals defined by,

$$[0,1] = [0, \sigma_R] \cup [\sigma_R, 1-\sigma_R] \cup [1-\sigma_R, 1], \quad \sigma_R := \min\{\frac{1}{4}, 2C\varepsilon \ln N\}, \quad C \geq \frac{1}{\sqrt{\beta}}.$$

For both problem classes (1), (2) using an inverse-monotone finite difference operator on such Shishkin meshes we have [3] the global parameter uniform error bound

$$\|u - \bar{U}^N\|_{\Omega,\infty} \leq CN^{-1}(\ln N).$$

For the time-dependent problems (3) and (4), a uniform mesh in time (with M time intervals) coupled with the associated steady-state Shishkin mesh in space will yield the global error bound [8]

$$\|u - \bar{U}^N\|_{\Omega,\infty} \leq CN^{-1}(\ln N) + CM^{-1}.$$

Moreover, this same approach has been successfully extended to a large class of elliptic and parabolic problems in two and three space dimensions [3, 6, 16].

We conclude this paper by briefly discussing a Shishkin-decomposition [6, 16] of the solution, which is the central component used in the analysis of these parameter-uniform numerical methods. The solution is decomposed into a sum of components involving a regular component v and several layer components w. The regular component is defined in such a way that all the first and second order partial derivatives of the regular component are bounded independently of the singular perturbation parameter ε. The layer components are characterized as being the solution of the associated homogenous differential equation and are small (relative to the perturbation parameter) in the domain, except in the neighbourhood of one particular edge or corner of the domain. The Shishkin decomposition is not an asymptotic expansion. There is no remainder term. This solution decomposition allowed Shishkin incorporate classical Schauder *a priori* bounds [7], into the singularly perturbed context, in order to bound the partial derivatives of the components.

We illustrate the Shishkin decomposition in relation to the elliptic problem:

$$L_\varepsilon u := -\varepsilon\triangle u + a_1 u_x + a_2 u_y = f, \quad (x,y) \in \Omega = (0,1)\times(0,1); \quad u = g, \quad (x,y) \in \partial\Omega;$$

with $a_1 > \alpha_1 > 0,\ a_2 > \alpha_2 > 0$ and its associated reduced problem is

$$a_1 r_x + a_2 r_y = f, \quad (x,y) \in \Omega, \quad r = g,\ (x,y) \in \partial\Omega_{in} := \{(x,y) \in \partial\Omega | xy = 0\}.$$

Exponential boundary layers appear near the outflow edges $x = 1,\ y = 1$ and a simple corner layer appears in the vicinity of the corner point $(1,1)$. The regular component $v \in C^{3,\gamma}(\bar{\Omega})$ is the solution of

$$L_\varepsilon v = f,\ (x,y) \in \Omega, \quad v = u,\ (x,y) \in \partial\Omega_{in}$$

and boundary values $v = v^*$ at the boundary, $\partial\Omega_{out} := \{(x,y) \in \partial\Omega | xy \neq 0, (1-x)(1-y) = 0\}$, can be identified so that

$$\|v^{(i)}\|_{\Omega,\infty} \leq C(1+\varepsilon^{2-i}), \quad 0 \leq i \leq 3.$$

The layer components w_R, w_T and w_{TR} are all solutions of the homogeneous problem

$$L_\varepsilon w_R = L_\varepsilon w_T = L_\varepsilon w_{TR} = 0, \quad (x,y) \in \Omega;$$

with the following boundary conditions (where w^* needs to be suitably specified)

$$\begin{aligned}
&w_R(1,y) = (u-v)(1,y), \quad w_R(0,y) = w_R(x,0) = 0, \quad w_R(x,1) = w_R^*(x,1);\\
&w_T(x,1) = (u-v)(x,1), \quad w_T(1,y) = w_T^*(1,y), \quad w_T(0,y) = 0,\ w_T(x,0) = 0;\\
&w_{TR}(x,1) = -w_R(x,1), \quad w_{TR}(1,y) = -w_T(1,y), \quad w_{TR}(0,y) = 0,\ w_{TR}(x,0) = 0.
\end{aligned}$$

By inverse-monotonicity, for all points $(x,y) \in \bar{\Omega}$ the following bounds hold:

$$|w_R(x,y)| \leq Ce^{-\frac{\alpha_1(1-x)}{\varepsilon}}, \quad |w_T(x,y)| \leq Ce^{-\frac{\alpha_2(1-y)}{\varepsilon}}, \quad |w_{TR}(x,y)| \leq Ce^{-\frac{\alpha_1(1-x)}{\varepsilon}}e^{-\frac{\alpha_2(1-y)}{\varepsilon}};$$

and, in addition, (for suitable w^*) one can establish (e.g., [9]) that for all $i, j \leq 3$

$$\begin{aligned}
&\left\|\frac{\partial^j w_R}{\partial x^i}\right\|_{\Omega,\infty} \leq C(1+\varepsilon^{-i}), \qquad \left\|\frac{\partial^j w_R}{\partial y^j}\right\|_{\Omega,\infty} \leq C(1+\varepsilon^{1-j});\\
&\left\|\frac{\partial^j w_T}{\partial x^i}\right\|_{\Omega,\infty} \leq C(1+\varepsilon^{1-i}), \ \left\|\frac{\partial^j w_T}{\partial y^j}\right\|_{\Omega,\infty} \leq C(1+\varepsilon^{-j}); \quad \|w_{TR}^{(i)}\|_{\Omega,\infty} \leq C\varepsilon^{-i}.
\end{aligned}$$

By construction, $u = v + w_R + w_T + w_{TR}$. These bounds on the derivatives of the components are the pivotal ingredients in establishing the error bounds associated with parameter-uniform numerical methods (which incorporate Shishkin meshes). The reader is referred to the papers [1, 6, 9, 10] and the references therein for further details on this decomposition.

§6. Conclusion

This paper indicates that it is important to be aware of the restrictions being imposed on the problem class under investigation. It is also advisable to be somewhat sceptical of exotic new norms introduced to generate a theoret ical result; how does the resulting numerical method perform in practice? For singularly perturbed problems, we advocate aiming to construct a parameter-uniform numerical method for these types of problems.

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