Monografías Matemáticas *García de Galdeano* **41**, 139–147 (2018)

NON-SPECTRAL FREQUENCY INDICATORS

María Antonia Navascués, María Victoria Sebastián and Carlos Ruiz

Abstract. The spectral methods (in terms of trigonometric polynomials) are suitable to model periodic or near periodic phenomena. However some experimental variables are far from periodicity. We describe a numerical procedure to obtain trend curves for historic data or short sampled signals, using orthogonal expansions. The approximants chosen are of Legendre type, and perform a low-pass filtering to the data. Additionally, we propose a numerical quantifyer of the variation of the trend curve whose character is non-spectral. In the second part of the text we perform numerical simulations to test the method: the first one is associated with standard models as Gaussian and sinusoidal functions, the second involves a historic record of the Spanish stock reference index IBEX 35.

Keywords: Orthogonal expansions, Legendre Polynomials, time series.

AMS classification: 41A10, 58C05, 42A58, 62M10.

§1. Introduction

The traditional approach for the processing of sampled signals or historic data has been performed by means of Fourier (spectral) techniques. This procedure has the disadvantage of requiring a hypothesis of periodicity of the records, since the functions involved are trigonometric. However some time series are far from periodicity. Further we believe that the polynomial orthogonal expansions are quite unexplored so far in the treatment of this kind of problems. In this article we present a method to obtain trend curves for historic data or short sampled signals using Legendre poynomials and series expansions. The approximants obtained perform a low-pass filtering to the data. At the same time we propose a quantifyer of the variation of the smoothed curve. Pointwise, uniform and mean-square convergences of the sums are analyzed when the time step tends to zero with the single hypothesis of continuity. At the same time we present numerical simulations in order to compare the method described with traditional numerical integration techniques for the computation of the parameter.

§2. Spectral parameters

We consider the general framework of the space of complex square integrable functions $\mathcal{L}^2(2\pi)$, endowed with the inner product

$$< x, y >= \frac{1}{2\pi} \int_I x(t) y^*(t)\, dt,$$

where $I = [-\pi, \pi]$ and y^* is the complex conjugate of y. $\mathcal{L}^2(2\pi)$ is a Hilbert space ([5]). The norm associated (in quadratic mean or 2-norm) is defined as

$$||x||^2 =< x, x >= \frac{1}{2\pi} \int_I |x(t)|^2 dt.$$

Let us consider $\varphi_n(t) = e^{int}$ for $n \in Z$. The system $\{\varphi_n(t)\}_{n \in Z}$ is an orthonormal basis of $\mathcal{L}^2(2\pi)$. The complex Fourier coefficients of $x(t)$ are defined as

$$k_n =< x, \varphi_n >= \frac{1}{2\pi} \int_I x(t) e^{-int} dt.$$

Then

$$x(t) = \sum_{n=-\infty}^{+\infty} k_n e^{int},$$

in $\mathcal{L}^2$-sense. The inner product can be obtained by means of the coefficients as well:

$$< x, y >= \sum_{n=-\infty}^{+\infty} k_n^x (k_n^y)^*,$$

where k_n^x and k_n^y are the Fourier coefficients of $x(t)$ and $y(t)$, respectively.
The real Fourier coefficients (a_n, b_n) of real functions f are related to the complex according to the equalities

$$k_0 = a_0/2,$$

$$k_n = (a_n - ib_n)/2.$$

From the Fourier coefficients we obtain the spectral moments, and other useful parameters of the signal x.

The spectral moment of order k is defined as:

$$m_k = \int_{-\infty}^{+\infty} \omega^k S(\omega)\, d\omega,$$

where $S(\omega)$ is the power spectrum,

$$S(\omega) = \widehat{x}(\omega) \widehat{x}^*(\omega) = |\widehat{x}(\omega)|^2,$$

and $\widehat{x}(\omega)$ is the Fourier Transform of the real signal $x(t)$, defined as

$$\widehat{x}(\omega) = \frac{1}{2\pi} \int_I x(t) e^{-i\omega t} dt. \tag{1}$$

Since $S(-\omega) = S(\omega)$, the odd moments are null. For the computation of the even moments we consider the discretized frequencies $\omega = \omega_n = n$. In this case

$$S(n) = |\widehat{x}(n)|^2.$$

According to (1)

$$\widehat{x}(n) = \frac{1}{2\pi}\left(\int_I x(t)\cos(nt)\,dt - i\int_I x(t)\sin(nt)\,dt\right),$$

$$\widehat{x}(n) = \frac{1}{2}(a_n - ib_n),$$

where a_n, b_n are the real Fourier coefficients of the real signal. Hence

$$S(n) = |\widehat{x}(n)|^2 = \frac{1}{4}(a_n^2 + b_n^2) = |k_n|^2,$$

where k_n are the complex coefficients, $n = 1, 2, \ldots$
The discrete spectral moment of order k (even) is defined as

$$m_k = \sum_{n=-\infty}^{+\infty} n^k S(n),$$

and thus

$$m_k = \sum_{n=-\infty}^{+\infty} n^k |k_n|^2.$$

The Activity (or energy) A of $x(t)$ is defined as

$$A = ||x||^2 =< x, x >= \frac{1}{2\pi}\int_I |x(t)|^2 dt = \sum_{n=-\infty}^{+\infty} |k_n|^2,$$

due to Parseval's identity. The mobility M of x is the ratio

$$M = \left(\frac{m_2}{m_0}\right)^{1/2} = \left(\frac{\sum_{n=-\infty}^{+\infty} n^2|k_n|^2}{\sum_{n=-\infty}^{+\infty} |k_n|^2}\right)^{1/2}.$$

M is then the square root of the average quadratic frequency, where the weight of the frequency n is the n-th power of the signal. Consequently this parameter is an index of mean frequency of the signal. Moreover, the Mobility is the square root of the quotient of zero-th moment of the derivative divided by the moment of the signal itself. Thus

$$M = \left(\frac{\int_I |x'(t)|^2}{\int_I |x(t)|^2}\right)^{1/2}. \tag{2}$$

By analogy to the periodic case, we propose the use of the parameter defined in (2) as an indicator of the variation of any smooth curve, even in case of no periodicity. From here on, we will call this Variation Parameter (V). It's a descriptor of the change of the signal, with respect to its amplitude.

§3. Numerical Legendre sum

Let $\{p_n\}_{n=0}^{\infty}$ be the system of normalized polynomials of Legendre in the interval $I = [-1, 1]$. These functions are orthonormal with respect to the inner product (see for instance [4])

$$(f, g) = \int_I f(t)g(t)\, dt,$$

whose associate norm is defined as

$$\|f\|_2 = \left(\int_I |f(t)|^2\right)^{1/2}.$$

If the expansion of f in terms of Legendre polynomials is

$$f \sim \sum_{n=0}^{+\infty} c_n p_n,$$

the series converges to f in the $\|\cdot\|_2$-norm (in quadratic mean). Additionally we consider the uniform (or supremum) norm for a continuous function g defined on the compact interval I ($g \in C(I)$),

$$\|g\|_\infty = \max\{|g(t)| : t \in I\}.$$

Let us consider a signal $x(t)$, known by its samples $\{(t_n, x_n)\}_{n=0}^{N}$, and let $h = \max(t_n - t_{n-1})$ be the diameter of the associated partition. Let $\overline{x} = \overline{x}(t)$ be a broken line interpolant of $x(t)$ corresponding to the sampled data, and let us construct a Legendre sum of order m ($S_m\overline{x}(t)$) of the approximant. This function will provide a trend curve for the function $x(t)$.
The modulus of continuity of a continuous function g is defined as

$$\omega_g(\delta) = \sup\{|g(t) - g(t')|; |t - t'| \le \delta, t, t' \in I\}.$$

The following result can be read in [2].

Lemma 1. *If $x \in C(I)$ the uniform distance between x and $\overline{x}$ is bounded by the modulus of continuity ω_x of x as*

$$\|x - \overline{x}\|_\infty \le \omega_x(h).$$

Consequently, x tends to $\overline{x}$ uniformly when h tends to zero.

By $g \in Lip\,\beta$ (g is Hölder-continuous with exponent β) we mean that there exists $M \ge 0$ such that

$$|g(t) - g(t')| \le M|t - t'|^\beta$$

$\forall\, t, t' \in I$.

Lemma 2. *$g \in Lip\,\beta$ if and only if $\omega_g(\delta) \le K\delta^\beta$.*

Proof. See for instance [1]. □

Theorem 3. *Let $x \in C(I)$ be the original function providing the data. The Legendre expansion defined by means of $\overline{x}$ converges in quadratic mean to x as m tends to infinity and h tends to zero.*

Proof. Let $S_m\overline{x}$ be the m-th partial sum of the Legendre series of $\overline{x}$, and let us consider

$$\|x - S_m\overline{x}\|_2 \le \|x - \overline{x}\|_2 + \|\overline{x} - S_m\overline{x}\|_2. \tag{3}$$

In the first term we argue that

$$\|x - \overline{x}\|_2 = \left(\int_I |x(t) - \overline{x}(t)|^2 dt\right)^{1/2} \le \sqrt{2}\, \|x - \overline{x}\|_\infty$$

and hence,

$$\|x - S_m\overline{x}\|_2 \le \sqrt{2}\, \|x - \overline{x}\|_\infty + \|\overline{x} - S_m\overline{x}\|_2.$$

Using Lemma 1, the distance in $\mathcal{L}^2(I)$ between the signal and the finite sum is then bounded as:

$$\|x - S_m\overline{x}\|_2 \le \sqrt{2}\, \omega_x(h) + \|\overline{x} - S_m\overline{x}\|_2.$$

The uniform continuity of x on I, implies that $\lim \omega_x(h) = 0$ as h tends to zero ([1]). The second term of (3) goes to zero as m tends to infinity due to the convergence in quadratic mean of the Legendre series of $\overline{x}$ since $\overline{x} \in C(I) \subseteq \mathcal{L}^2(I)$. □

Lemma 4. *If $f \in C^p[-1,1]$ is such that $f^{(p)} \in Lip\, \delta$, then the m-th Legendre sum of f satisfies the inequality*

$$\|f - \sum_{n=0}^{m} c_n p_n\|_\infty \le \frac{K \ln m}{m^{p+\delta-1/2}}$$

for $p + \delta \ge 1/2$.

Proof. See for instance [6]. □

Theorem 5. *The Legendre expansion of the broken line interpolant $\overline{x}$ converges pointwise and uniformly to $\overline{x}$ on the interval $I = [-1, 1]$.*

Proof. In the reference [3], the author proves that the Legendre series of any function $f \in \mathcal{L}^p(I)$ such that $p > 4/3$ converges pointwisely to f almost everywhere. The interpolant $\overline{x}$ is continuous on a compact interval and then it belongs to $\mathcal{L}^2(I)$, this fact ensures the pointwise convergence almost everywhere.
It is clear that $\overline{x}(t) \in Lip1$. Now, we apply Lemma 4 for $p = 0$ and $\delta = 1$ obtaining

$$\|\overline{x} - \sum_{n=0}^{m} c_n p_n\|_\infty \le \frac{K \ln m}{m^{1/2}}.$$

As m tends to infinity the Legendre sum tends to $\overline{x}$ and the uniform convergence is achieved on the interval $I = [-1, 1]$. This fact ensures the pointwise convergence on the whole interval. □

Remark 1. This result is true for any step h.

Theorem 6. *Let $x \in C(I)$ be the original function providing the data. The Legendre expansion obtained numerically converges uniformly to x as $m \to \infty$ and $h \to 0$.*

Proof. The uniform continuity of $x(t)$ on I implies that $\lim \omega_x(h) = 0$ as h tends to zero ([1]). Let $S_m\overline{x}$ be the m-th partial sum of the Legendre series of $\overline{x}$. Let us consider

$$\|x - S_m\overline{x}\|_\infty \leq \|x - \overline{x}\|_\infty + \|\overline{x} - S_m\overline{x}\|_\infty.$$

The first term goes to zero if $h \to 0$ due to Lemma 1. The second term tends to zero as well when $m \to \infty$ according to the previous Theorem, obtaining the result.

□

Remark 2. The Legendre expansion obtained numerically converges in $\mathcal{L}^2(I)$ as well, since for a continuous function defined in compact intervals, the uniform convergence implies the convergence in quadratic mean. Same is true for pointwise convergence.

Remark 3. The former theorem ensures the goodness of the procedure to represent and evaluate the signal whenever the step and the expansion order are suitably chosen, with the single hypothesis of continuity.

§4. Numerical simulations

In this Section we present some numerical examples in order to compare the method proposed with another standard technique of numerical integration.

4.1. Gaussian function

We considered the Gaussian function $f(t) = e^{-t^2}$, sampled with step $h = 2^{-7}$ in the interval $[-1, 1]$ and obtained its numerical Legendre sums for different degrees. We computed the Fourier-Legendre coefficients using a broken line interpolant, and by means of the compound trapeze rule. We obtained the Variation Parameter for every sum, comparing it with its exact value using the expression (2). We recorded the relative errors, as the absolute value of the difference between the approximate and exact Variations, divided by the exact one. It can be observed that the errors are lower in the polygonal case. The results for every degree and method are collected in Table 1.

4.2. Sinusoidal functions

In this item we dealt with the sinusoidal functions $\sin(\pi t)$, $\sin(2\pi t)$, $\sin(3\pi t)$,$\sin(4\pi t)$, $\sin(5\pi t)$, sampled by a step $h = 8\times10^{-3}$ in the interval $[-1, 1]$, obtained their numerical Legendre sums with different degrees and computed the error in the Variation Parameter. The rest is similar to the previous Subsection (Tables 2, 3, 4).

	Trapeze	Broken line
$n = 5$	1.0×10^{-2}	1.0×10^{-2}
$n = 6$	2.1×10^{-4}	7.8×10^{-4}
$n = 7$	2.1×10^{-4}	7.8×10^{-4}
$n = 8$	2.8×10^{-3}	6.0×10^{-5}
$n = 9$	2.8×10^{-3}	6.0×10^{-5}
$n = 10$	5.7×10^{-3}	1.4×10^{-5}
$n = 11$	5.7×10^{-3}	1.4×10^{-5}
$n = 12$	9.1×10^{-3}	1.6×10^{-5}
$n = 13$	9.1×10^{-3}	1.6×10^{-5}
$n = 14$	8.6×10^{-3}	1.6×10^{-5}

Table 1: Errors in the Variation Parameter for different Legendre sums (with increasing degree n) of the function $f(t) = e^{-t^2}$ obtained by two numerical procedures.

	Trapeze $f(t) = \sin(\pi t)$	Broken line $f(t) = \sin(\pi t)$	Trapeze $f(t) = \sin(2\pi t)$	Broken line $f(t) = \sin(2\pi t)$
$n = 31$	5.7×10^{-3}	8.4×10^{-8}	5.7×10^{-3}	5.5×10^{-7}
$n = 32$	5.7×10^{-3}	8.4×10^{-8}	5.7×10^{-3}	5.5×10^{-7}
$n = 33$	6.3×10^{-3}	7.1×10^{-7}	6.3×10^{-3}	1.2×10^{-6}
$n = 34$	6.3×10^{-3}	7.1×10^{-7}	6.3×10^{-3}	1.2×10^{-6}
$n = 35$	6.9×10^{-3}	4.6×10^{-6}	6.8×10^{-3}	4.0×10^{-6}
$n = 36$	6.9×10^{-3}	4.6×10^{-6}	6.8×10^{-3}	4.0×10^{-6}

Table 2: Errors in the computation of the Variation Parameter for different Legendre sums (with increasing degree n) of the functions $f(t) = \sin(\pi t)$ and $f(t) = \sin(2\pi t)$ obtained by both numerical procedures.

4.3. Historic stock record

In this case we considered the daily closing prices of the IBEX 35 over the year 2004 (256 data) and obtained its Legendre sums with the same procedures (see Figure 1). Since we cannot compare the Parameters with their exact values, we recorded the square root of the mean square error, divided by the maximum of the time series. The results for different degrees and methods are depicted in Table 5. We can observe lower errors for the broken line interpolant in this case as well.

Acknowledgements

The authors wish to thank the project CUD-Universidad de Zaragoza with code UZCUD2014-CIE-09 and research group E106 of Diputación General de Aragón.

	Trapeze $f(t) = \sin(3\pi t)$	Broken line $f(t) = \sin(3\pi t)$	Trapeze $f(t) = \sin(4\pi t)$	Broken line $f(t) = \sin(4\pi t)$
$n = 31$	5.7×10^{-3}	1.3×10^{-6}	5.7×10^{-3}	2.4×10^{-6}
$n = 32$	5.7×10^{-3}	1.3×10^{-6}	5.7×10^{-3}	2.4×10^{-6}
$n = 33$	6.3×10^{-3}	2.0×10^{-6}	6.3×10^{-3}	3.1×10^{-6}
$n = 34$	6.3×10^{-3}	2.0×10^{-6}	6.3×10^{-3}	3.1×10^{-6}
$n = 35$	6.9×10^{-3}	2.9×10^{-6}	6.9×10^{-3}	1.6×10^{-6}
$n = 36$	6.9×10^{-3}	2.9×10^{-6}	6.9×10^{-3}	1.6×10^{-6}

Table 3: Errors in the computation of the Variation Parameter for different Legendre sums (with increasing degree n) of the functions $f(t) = \sin(3\pi t)$ and $f(t) = \sin(4\pi t)$ obtained by both numerical procedures.

	Trapeze	Broken line
$n = 31$	5.7×10^{-3}	3.9×10^{-6}
$n = 32$	5.7×10^{-3}	3.9×10^{-6}
$n = 33$	6.3×10^{-3}	4.5×10^{-6}
$n = 34$	6.3×10^{-3}	4.5×10^{-6}
$n = 35$	6.9×10^{-3}	2.0×10^{-7}
$n = 36$	6.9×10^{-3}	2.0×10^{-7}

Table 4: Errors in the computation of the Variation Parameter for different Legendre sums (with increasing degree n) of the function $f(t) = \sin(5\pi t)$ obtained by both numerical procedures.

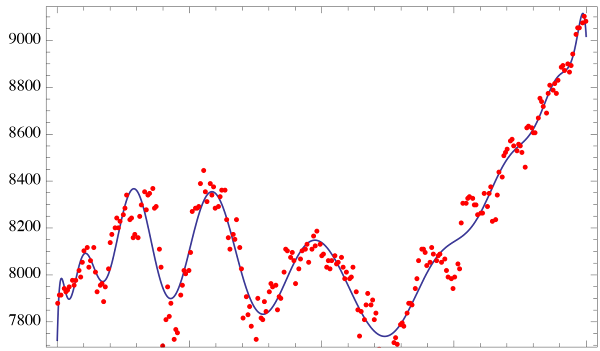

Figure 1: Time series of IBEX 35 over the year 2004 (dotted) along with its trend curve of order 21.

	RMSE trapeze	RMSE broken line
$n = 21$	1.6×10^{-2}	9.0×10^{-3}
$n = 22$	2.1×10^{-2}	8.5×10^{-3}
$n = 23$	2.1×10^{-2}	8.3×10^{-3}
$n = 24$	2.7×10^{-2}	8.1×10^{-3}
$n = 25$	2.7×10^{-2}	7.9×10^{-3}
$n = 26$	3.5×10^{-2}	7.9×10^{-3}
$n = 27$	3.5×10^{-2}	7.8×10^{-3}
$n = 28$	4.6×10^{-2}	7.8×10^{-3}
$n = 29$	4.6×10^{-2}	7.3×10^{-3}
$n = 30$	5.9×10^{-2}	7.3×10^{-3}
$n = 31$	5.9×10^{-2}	7.2×10^{-3}
$n = 32$	7.4×10^{-2}	7.2×10^{-3}
$n = 33$	7.4×10^{-2}	6.6×10^{-3}
$n = 34$	9.1×10^{-2}	6.5×10^{-3}

Table 5: Relative errors for both trapeze and broken line procedures for different Legendre sum calculations (with increasing degree n) for closing daily values of IBEX in year 2004.

References

[1] Cheney, E. W. *Approximation Theory*. AMS Chelsea Publ., 1966.

[2] Navascués, M. A., and Sebastián, M. V. Construction of affine fractal funtions close to classical interpolants. *J. Comp. Anal. Appl. 9*, 3 (2007), 271–285.

[3] Pollard, H. The convergence almost everywhere of legendre series. *Proc. Am. Math. Soc. 35*, 2 (1972), 442–444.

[4] Sansone, G. *Orthogonal Functions*. Krieger Publ., 1977.

[5] Soize, C. *Méthodes mathématiques en analyse du signal*. Masson Paris, 1993.

[6] Suetin, P. K. Representation of continuous and differentiable functions by fourier series of legendre polynomials. *Soviet Math. Dokl. 5* (1964), 1408–1410.

María Antonia Navascués
Departamento de Matemática Aplicada
EINA, Universidad de Zaragoza
C/ M^a de Luna, 50018 Zaragoza
`manavas@unizar.es`

María Victoria Sebastián and Carlos Ruiz
Centro Universitario de la Defensa
Academia General Militar de Zaragoza
Ctra. de Huesca s/n, 50090 Zaragoza
`msebasti@unizar.es` and `cruizl@unizar.es`