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# A NEW INHOMOGENEOUS LOGNORMAL DIFFUSION PROCESS WITH EXOGENOUS FACTORS IN THE DIFFUSION COEFFICIENT

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**Abstract.** We propose a new non-homogeneous one-dimensional stochastic lognormal diffusion process, in which a time function (exogenous factor) is introduced into the diffusion coefficient of the process. This new approach can be considered an extension of the homogeneous lognormal process (see [1] and [12]). From the corresponding Ito's stochastic differential equation, we obtain the probabilistic characteristics of the model, i.e. the transition probability density function and the moments of the process. Finally, we develop the statistical inference of this model, via maximum likelihood with discrete sampling.

*Keywords:* Lognormal diffusion process, Exogenous factors, trends function, Likelihood estimation in diffusion process.

*AMS classification:* 60J60, 62M05.

## §1. Introduction

Stochastic diffusion processes are useful instruments for modelling phenomena, and many scientific disciplines make use of these models to reflect behaviour in areas such as the economy and the financial world, medicine, biology and physics. Accordingly, both in general and in particular cases, these processes are of great importance in the management of probabilistic and statistical problems. For example, problems related to statistical inference (confidence bands, the proposal and testing of hypotheses, etc.) and first-passage time problems for certain types of barriers may be addressed using this approach. Important papers in this field include Tintner et al. [12] and Al Eideh et al. [1] for the lognormal process, Skiadas et al. [11] for the Bass process, Giovanis et al. [5] for the logistic process, Ferrante et al. [4] for the Gompertz process and Gutiérrez et al. [7] for the Gamma process.

However, in most of these studies, the processes considered are homogeneous; in other words, their infinitesimal moments depend only on the state space, which means that the possible influences on the subject variable are functions of this same variable. This fact restricts the scope of application, as well as the possibility of introducing information other than the variable of interest. These limitations are apparent in many applications, in which deviations of the study data from the trend of the homogeneous process can be observed. Accordingly, we must distinguish between external influences (exogenous factors) and the variable actually modelled by the process (i.e., the endogenous variable), whose evolution in time is known. The inclusion of these temporal functions in the drift of the process provides a better fit and, at the same time, enables external control over the behaviour of the variable governed by the process.

The use of exogenous factors with respect to stochastic diffusion processes has been proposed by Tintner and Singupta [12] and by Gutiérrez et al [8] for the lognormal process, by Albano et al. [2], Ferrante et al. [3], Gutiérrez et al. [6] for the Gompertz process, Gutiérrez et al [10] for the Vasicek process and by Nafidi et al. [9] for the Gamma process.

However, all of these prior studies took as their starting point the fact that exogenous factors, including temporal functions, affect drift (the first infinitesimal moment). The relation between the diffusion coefficient (the second infinitesimal moment) and exogenous factors has not been addressed in previous research. The present study considers this question in a particular case of the stochastic lognormal diffusion process.

This paper, thus, discusses the theoretical aspects of a non-homogenous version of the lognormal diffusion process, based on the fundamental fact that the diffusion coefficient is a function of time. The paper is structured as follows: in the second section, from the corresponding Ito's stochastic differential equation, the following probabilistic characteristics of the model are obtained: the explicit expression of the process, its transition probability density function (pdf), its statistical distribution and the moments of the process. In the final section, we develop the statistical inference of this model, using maximum likelihood with discrete sampling.

## §2. The proposed model and its characteristics

### 2.1. The model and the pdf

The proposed model is a lognormal diffusion process with a time-dependent diffusion coefficient. It is defined by the one-dimensional stochastic process $\{x(t); t \in [t_0, T]; t_0 \geq 0\}$ that satisfies Ito's stochastic differential equation (SDE):

$$dx(t) = ax(t)dt + \sigma g(t)x(t)dw(t) \quad ; \quad x(t_0) = x_{t_0},$$

where $\{w(t); t \in [t_0, T]\}$ is a one-dimensional standard Wiener process, $x_{t_0}$ is a fixed real value within $(0, \infty)$. The parameters $a$ and $\sigma$ are real and will be estimated, and the function $g$ is continuous and depends solely on the time.

The analytical expression of the process can be obtained by applying Ito's formula to the following type transform $y(t) = \log(x(t))$. The following SDE is then obtained:

$$dy(t) = \left(a - \frac{\sigma^2 g^2(t)}{2}\right)dt + \sigma g(t)dw(t) \quad , \quad y(t_0) = \log(x_{t_0}),$$

and by integrating, we have

$$y(t) = y(t_0) + \int_{t_0}^{t} \left(a - \frac{\sigma^2 g^2(\theta)}{2}\right)d\theta + \sigma \int_{t_0}^{t} g(\theta)dw(\theta) \quad , \quad y(t_0) = \log(x_{t_0}),$$

from which we can deduce the explicit expression of the process

$$x(t) = \exp\left\{\log(x_{t_0}) + a(t - t_0) - \frac{\sigma^2}{2}\int_{t_0}^{t} g^2(\theta)d\theta) + \sigma \int_{t_0}^{t} g(\theta)dw(\theta)\right\}.$$

As the random variable $\int_s^t g(\theta)dw(\theta)$ has a one-dimensional normal distribution $\mathcal{N}_1\left(0, \int_s^t g^2(\theta)d\theta\right)$, we can deduce that the random variable $x(t)/x(s) = x_s \sim \Lambda_1\left(\mu(s,t,x_s), \sigma^2\nu^2(s,t)\right)$, a one-dimensional log-normal distribution with

$$\mu(s,t,x) = \log(x) + a(t-s) - \frac{\sigma^2}{2}\int_s^t g^2(\theta)d\theta,$$
$$\nu^2(s,t) = \int_s^t g^2(\theta)d\theta.$$

The transition density function of this process $f(y,t \mid x,s)$ takes the form

$$f(y,t \mid x,s) = (2\pi\sigma^2\nu^2(s,t))^{-\frac{1}{2}}x^{-1}\exp\left\{-\frac{[\log(y)-\mu(s,t,x)]^2}{2\sigma^2\nu^2(s,t)}\right\}.$$

## 2.2. The trend functions

Using the following properties of the one-dimensional lognormal distribution:
if $X \sim \Lambda_1\left(\mu, \sigma^2\right)$, then

$$E(X^r) = \exp\left(r\mu + \frac{r^2\sigma^2}{2}\right),$$

the rth conditional moment of the process is given by

$$E\left(x^r(t)/x(s) = x_s\right) = \exp\left(r\mu(s,t,x_s) + \frac{r^2\sigma^2\nu^2(s,t)}{2}\right)$$
$$= x_s^r \exp\left(ra(t-s) + r(r-1)\frac{\sigma^2}{2}\int_s^t g^2(\theta)d\theta\right).$$

For $r = 1$, the conditional trend function is:

$$E\left(x(t)/x(s) = x_s\right) = x_s \exp\left(a(t-s)\right).$$

From this, and considering the initial condition $\mathrm{P}(x(t_1) = x_{t_1}) = 1$, the trend function leads us to

$$E\left(x(t)\right) = x_{t_1}\exp\left(a(t-t_1)\right).$$

# §3. Statistical inference on the model

## 3.1. Maximum likelihood parameter estimation

With discrete sampling, the parameters $a$ and $\sigma^2$ of the model are estimated by means of maximum likelihood. Let us consider a discrete sampling of the process $x(t_1) = x_1, x(t_2) =$

$x_2, \dots, x(t_n) = x_n$ for times $t_1, t_2, \dots, t_n$ and the the initial distribution $\mathrm{P}[x(t_1) = x_1] = 1$. Then, the associated likelihood function can be obtained by the following expression:

$$\mathbb{L}(x_1, \dots, x_n, a, \sigma^2) = \prod_{j=2}^{n} f\left(x_j, t_j \mid x_{j-1}, t_{j-1}\right).$$

An implementation based on the change of variable can be used in order to calculate the maximum likelihood estimators in a simpler way. Consider the following transform: $v_1 = x_1$, $v_i = \nu_i^{-1}(\log(x_i) - \log(x_{i-1}))$, for $i = 2, \dots, n$. Then, given $\xi_i = \nu_i^{-1}(t_i - t_{i-1})$ and $\rho_i = \frac{1}{2}\nu_i$, the log-likelihood function can be expressed as follows:

$$\log(\mathbb{L}) = -\frac{n-1}{2}\log(2\pi) - \frac{n-1}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{j=2}^{n}\left(v_j - \xi_j a + \rho_j \sigma^2\right)^2.$$

By differentiating the log-likelihood function with respect to $a$ and $\sigma^2$ and by equaling this differential to zero, we obtain the following likelihood equations:

$$\sum_{j=2}^{n} \xi_j \left(v_j - \xi_j \hat{a} + \rho_j \hat{\sigma}^2\right) = 0,$$

$$-(n-1)\hat{\sigma}^2 + \sum_{j=2}^{n}\left(v_j - \xi_j \hat{a} + \rho_j \hat{\sigma}^2\right)^2 - 2\hat{\sigma}^2 \sum_{j=2}^{n} \rho_j \left(v_j - \xi_j \hat{a} + \rho_j \hat{\sigma}^2\right) = 0.$$

From the first of these equations, the likelihood estimator $\hat{a}$ can be expressed in terms of the estimator $\hat{\sigma}^2$, as follows:

$$\hat{a} = w + z\hat{\sigma}^2,$$

where

$$w = \left(\sum_{j=2}^{n} \xi_j v_j\right)\left(\sum_{j=2}^{n} \xi_j^2\right)^{-1} \quad \textit{and} \quad z = \left(\sum_{j=2}^{n} \xi_j \rho_j\right)\left(\sum_{j=2}^{n} \xi_j^2\right)^{-1}$$

By substitution in the second likelihood equation, and after various algebraic operations (not shown), we obtain the following second-degree equation in $\hat{\sigma}^2$

$$\left(\sum_{j=2}^{n} \rho_j^2 - z^2 \sum_{j=2}^{n} \xi_j^2\right)\hat{\sigma}^4 + (n-1)\hat{\sigma}^2 - \sum_{j=2}^{n}\left(v_j - w\xi_j\right)^2 = 0. \tag{1}$$

To solve this second-degree equation, we can distinguish two cases.

First case: $\sum_{j=2}^{n} \rho_j^2 - z^2 \sum_{j=2}^{n} \xi_j^2 = 0.$

The likelihood estimators of the parameters in this case are

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{j=2}^{n}\left(v_j - w\xi_j\right)^2,$$

$$\hat{a} = w + z\hat{\sigma}^2.$$

*Remark* 1. An example of this first case occurs when $g(t) = 1$. In this case, we obtain $\xi_j = \rho_j$ and $z = \frac{1}{2}$, and thus:

$$\sum_{j=2}^{n} \rho_j^2 - z^2 \sum_{j=2}^{n} \xi_j^2 = 0,$$

and the resulting estimators are

$$\hat{a} - \frac{\hat{\sigma}^2}{2} = w \quad \text{and} \quad \hat{\sigma}^2 = \frac{1}{n-1} \sum_{j=2}^{n} \left( v_j - w\xi_j \right)^2 .$$

These are precisely the estimators obtained by Al Eideh et al. [1] for the lognormal process with no exogenous factors.

Second case: $\sum_{j=2}^{n} \rho_j^2 - z^2 \sum_{j=2}^{n} \xi_j^2 \neq 0.$

The discriminant of the equation Eq(1) in this case is:

$$\Delta = (n-1)^2 + 4 \left( \sum_{j=2}^{n} \rho_j^2 - z^2 \sum_{j=2}^{n} \xi_j^2 \right) \sum_{j=2}^{n} \left( v_j - w\xi_j \right)^2 .$$

To obtain the sign of this discriminant, we make use of the following result:

**Proposition:**

Let $a_n$ and $b_n$ be two non-negative sequences. Then we have the following inequality:

$$\left( \sum_{k=1}^{n} a_k^2 \right) \left( \sum_{k=1}^{n} b_k^2 \right) \geq \left( \sum_{k=1}^{n} a_k b_k \right)^2 .$$

The proof of this is obtained by induction.

Using the above result in the particular case: $a_k = \rho_k$ and $b_k = \xi_k$ with $1 \leq i \leq n$. Thus, we have:

$$\left( \sum_{k=1}^{n} \rho_j^2 \right) \left( \sum_{k=1}^{n} \xi_j^2 \right) \geq \left( \sum_{k=1}^{n} \rho_j \xi_j \right)^2 .$$

from which we obtain that

$$\sum_{j=2}^{n} \rho_j^2 - z^2 \sum_{j=2}^{n} \xi_j^2 \geq 0.$$

Then we deduce that

$$\Delta > (n-1)^2,$$

and therefore the Eq(1) has two solutions, and the non-negative solution corresponding to $\sigma^2$ is

$$\hat{\sigma}^2 = \frac{\sqrt{\Delta} - (n-1)}{\sum_{j=2}^{n} \rho_j^2 - z^2 \sum_{j=2}^{n} \xi_j^2} .$$

## 3.2. Estimated trend functions

By Zehna's theorem [13], we can obtain the estimated trend function and the estimated conditional trend function of the process, replacing the parameters by their estimators. The estimated conditional trend function is then given by

$$\hat{E}\left(x(t)/x(s)=x_s\right)=x_s\exp\left(\hat{a}(t-s)\right)=\exp\left[\left(w+z\hat{\sigma}^2\right)(t-s)\right],$$

and the estimated trend function is:

$$\hat{E}\left(x(t)\right)=x_{t_1}\exp\left(\hat{a}(t-t_1)\right)=x_{t_1}\exp\left[\left(w+z\hat{\sigma}^2\right)(t-t_1)\right].$$

# References


[1] Al Eideh, B. M., Al Refai, A. S. A., and Sbeiti, W. M. Modelling the cpi using a lognormal diffusion process and implications on forecasting inflation. *IMAA Journal of Management Mathematics 15* (2004), 39–51.

[2] Albano, G., and Giorno, G. A stochastic model in tumor growth. *Journal of Theoretical Biology 242* (2006), 329–336.

[3] Ferrante, L., Bompade, S., Leone., L., and Montanari, M. A stochastic formulation of the gompertzian growth model for in vitro bactericidal kinetics: parameter estimation and extinction probability. *Biometrical Journal 47*, 3 (2005), 309–318.

[4] Ferrante, L., Bompade, S., Possati, L., and Leone., L. Parameter estimation in a gompertzian stochastic model for tumor growth. *Biometrics 56* (2000), 1076–1081.

[5] Giovani, A. N., and Skiadas, C. H. A stochastic logistic innovation diffusion model studying the electricity consumption in greece and the united states. *Technological Forecasting and Social Change 61* (1999), 253–264.

[6] Gutiérrez, R., Gutiérrez-Sánchez, R., and Nafidi, A. Electricity consumption in morocco: stochastic gompertz exogenous factors diffusion analysis. *Appl. Energ.*, 83 (2006), 1139–1151.

[7] Gutiérrez, R., Gutiérrez-Sánchez, R., and Nafidi, A. Statistical inference in the stochastic gamma diffusion process with external information. *Monografías del Seminario Matemático GarcÃŋa de Galdeano*, 35 (2010), 263– 270.

[8] Gutiérrez, R., Rico, N., Román, P., , and Torres, F. Approximate and generalized confidence bands for the mean and mode functions of the lognormal diffusion process. *Computational Statistics and Data Analysis*, 51 (2007), 4038–4053.

[9] Nafidi, A., Gutiérrez, R., Gutiérrez Sánchez, R., Ramos-Abalos, E., and El Hachimi, S. Modelling and predicting electricity consumption in spain using the stochastic gamma diffusion process with exogenous factors. *Energy 15* (2016), 309–318.

[10] R. Gutiérrez, R., Gutiérrez-Sánchez, R., Nafidi, A., and Pascual, A. Detection, modelling and estimation of non-linear trends by using a non-homogeneous vasicek stochastic diffusion. application to co2 emissions in morocco. *Stoch. Environ. Res. Risk Assess. 26* (2012), 533–543.

[11] SKIADAS, C., AND GIOVANI, A. A stochastic bass innovation diffusion model for studying the growth of electricity consumption in greece. *Applied Stochastic Models and Data Analysis 13* (1997), 85–101.

[12] TINTNER, G., AND SENGUPTA, J. K. *Stochastic Economics*. Academic Press, 1972.

[13] ZEHNA, P. W. Invariance of maximum likelihood estimators. *Ann. Math. Stat. 37* (1966), 744.

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