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Efficient numerical methods for singularly perturbed systems of reaction-diffusion type

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Abstract. In this work we consider 1D and 2D parabolic singularly perturbed systems of two equations of reaction-diffusion type, when the diffusion parameter is the same at both equations of the system. For these problems, in general, parabolic layers appear at all the boundary of the spatial domain. The solution of the continuous problems is approximated by using a finite difference scheme, which combines a splitting or additive scheme, defined on a uniform mesh, to discretize in time, and the classical central finite difference scheme, defined on a mesh of Shishkin type, to discretize in space. In the case of 1D parabolic systems, this scheme is uniformly convergent with respect to the diffusion parameter, having first order in time and almost second order in space. Some numerical results are given which corroborate the order of convergence of the method. The numerical results given for a 2D parabolic system, indicate the uniform convergence of the proposed splitting scheme.

Keywords: parabolic systems, reaction-diffusion, additive schemes, Shishkin meshes, uniform convergence.

AMS classification: AMS classification codes: 65M06, 65N06, 65N12.

§1. Introduction

In this work we consider 1D parabolic singularly perturbed reaction-diffusion systems of type

$$\begin{cases} L_{1,\varepsilon}\mathbf{u} \equiv \dfrac{\partial \mathbf{u}}{\partial t}(x,t) + \mathcal{L}_{1,x,\varepsilon}\mathbf{u}(x,t) = \mathbf{f}(x,t), \ (x,t) \in Q_1 \equiv (0,1)\times(0,T], \\ \mathbf{u}(x,t) = \mathbf{0}, \ x \in \{0,1\} \ t \in (0,T], \ \mathbf{u}(x,0) = \mathbf{0}, \ x \in [0,1], \end{cases} \tag{1}$$

where the spatial differential operator is given by

$$\mathcal{L}_{1,x,\varepsilon}\mathbf{u} \equiv -\mathcal{D}\frac{\partial^2 \mathbf{u}}{\partial x^2} + \mathcal{A}\mathbf{u}, \tag{2}$$

and 2D parabolic singularly perturbed reaction-diffusion systems of type

$$\begin{cases} L_{2,\varepsilon}\mathbf{u} \equiv \dfrac{\partial \mathbf{u}}{\partial t}(\mathbf{x},t) + \mathcal{L}_{2,\mathbf{x},\varepsilon}\mathbf{u}(\mathbf{x},t) = \mathbf{f}(\mathbf{x},t), \ (\mathbf{x},t) \in Q_2 \equiv \Omega\times(0,T], \\ \mathbf{u}(\mathbf{x},t) = \mathbf{0}, \ \mathbf{x} \in \Gamma \equiv \partial\Omega, \ t \in (0,T], \ \mathbf{u}(\mathbf{x},0) = \mathbf{0}, \ \mathbf{x} \in \overline{\Omega}, \end{cases} \tag{3}$$

where $\Omega = (0,1)^2$ and the spatial differential operator is given by

$$\mathcal{L}_{2,\mathbf{x},\varepsilon}\mathbf{u} \equiv -\mathcal{D}\Delta\mathbf{u} + \mathcal{A}\mathbf{u}. \tag{4}$$

In both problems, $\mathcal{D} = \mathrm{diag}(\varepsilon, \varepsilon)$, $\mathcal{A} = (a_{ij})$, $i, j = 1, 2$, the diffusion parameter ε, $0 < \varepsilon \le 1$, can be sufficiently small, and we suppose that the reaction matrix $\mathcal{A}$ is an M-matrix, i. e., $a_{i1} + a_{i2} \ge \alpha \ge 0,\ a_{ii} > 0,\ i = 1, 2,\ a_{ij} \le 0,\ \text{ if } i \ne j.$

Singularly perturbed 1D elliptic or parabolic systems of reaction-diffusion type and its numerical approximation, has been studied in many works (see for instance [3, 5, 6, 9] and the references therein). From the results in those papers, it follows that the exact solution of problem (1), in general, has a parabolic boundary layer of width $O(\sqrt{\varepsilon})$ at $x = 0, 1$. For example, in [6] the solution of problem (1) was approximated with a finite difference scheme, which uses the Euler method on a uniform mesh for the temporal discretization, and the standard central difference scheme on a piecewise uniform Shishkin mesh for the spatial discretization. The uniform convergence with respect to the diffusion parameter of this scheme was proved in [6]; nevertheless, a high computing time is required to find the solution due to the components of the discrete solution are coupled at each time level. In [1] a splitting (or additive) scheme, to discretize the time variable on a uniform mesh, was used. This scheme decouples the numerical vector solution and simpler problems for each individual unknown are solved at each time level. The main advantage of these methods is that the computational cost is reduced considerably (see [11] for a detailed discussion).

The aim of this paper is to approximate with an additive scheme the solution of 2D parabolic systems given in (3), which up to our knowledge has not been previously considered in the literature. The steady version o f this class of vector problems has been considered in very few papers [7, 8, 10]; in these papers, it is shown that the 2D elliptic singularly perturbed problem has in general a parabolic layer of width $O(\sqrt{\varepsilon})$ near the boundary Γ of the domain. It is expected that the solution of problem (3) exhibits a similar type of layers [2]; then, it is convenient to construct uniformly convergent methods, for which the rate of convergence and the error constant are independent of the diffusion parameter ε. The fully discrete method is obtained by using classical central differences on a mesh of Shishkin type for the spatial discretization. For several test problems, we have observed numerically that this scheme provides reliable solutions using meshes with a reasonable size independently of the value of the diffusion parameter.

The paper is organized as follows. In Section 2.1 we recall the asymptotic behavior of the exact solution $\mathbf{u}$ of problem (1) with respect to the diffusion parameter ε, and we give appropriate bounds of its derivatives. In Section 2.2, we construct the fully discrete scheme, which combines the splitting scheme to discretize in time and the central finite difference scheme to discretize in space. If the spatial discretization is constructed on a nonuniform special mesh of Shishkin type, the resulting method is uniformly convergent of first order in time and almost second order in space. Finally, in Section 3, the additive scheme is applied to the 2D problem in space and we show the results obtained for some test problems of type (1) and (3).

We denote by $\mathbf{v} \le \mathbf{w}$ if $v_i \le w_i$, $i = 1, 2$, $|\mathbf{v}| = (|v_1|, |v_2|)^T$ and $\|\mathbf{f}\|_D = \max\{\|f_1\|_D, \|f_2\|_D\}$ where $\|\cdot\|_D$ is the maximum norm, where D is the corresponding domain. Henceforth, C denotes any positive constant independent of the diffusion parameter ε and the discretization parameters N and M, which can take different values at different places.

§2. Parabolic 1D problems

2.1. Asymptotic behavior of the solution

In this section we recall the asymptotic behavior of the solution of the continuous problem (1) and we give some appropriate bounds for its derivatives. The analysis follows the ideas and techniques developed in [3, 5, 6, 9].

Lemma 1. *(Maximum Principle) ([6]) Let* $\mathbf{v} \in C(\overline{Q}_1) \cap C^2(Q_1)$ *such that* $L_{1,\varepsilon}\mathbf{v} \geq 0$ *on* Q_1 *and* $\mathbf{v} \geq 0$ *on* $\{0, 1\}$*. Then, it holds that* $\mathbf{v} \geq 0$ *on* $\overline{Q}_1$.

To obtain the uniform convergence of the numerical method that we construct later, we need precise bounds of the solution $\mathbf{u}$ of (1) and its derivatives with respect to the diffusion parameter ε. To simplify the notation, we define $B_\varepsilon(\xi) = e^{-\alpha\xi/\sqrt{\varepsilon}} + e^{-\alpha(1-\xi)/\sqrt{\varepsilon}}$, which contains the exponential functions characterizing the behavior of $\mathbf{u}$.

The following result gives appropriate estimates for the partial derivatives of $\mathbf{u}$ which are required for the analysis of the convergence of the finite difference scheme given in the next section.

Lemma 2. *([6]) Assume that problem (1) satisfies enough regularity and compatibility conditions. Then, the following estimates hold*

$$\left|\frac{\partial^k \mathbf{u}}{\partial t^k}(x,t)\right| \leq C,\ 0 \leq k \leq 2,\ \left|\frac{\partial^k \mathbf{u}}{\partial x^k}(x,t)\right| \leq C(1 + \varepsilon^{-k/2} B_\varepsilon(x)), 1 \leq k \leq 4,\ (x,t) \in \overline{Q}_1.$$

Note that these estimates indicate that the solution of problem (1) has boundary layers at $x = 0$ and $x = 1$, both with a width of order $O(\sqrt{\varepsilon})$.

2.2. The fully discrete method: uniform convergence

In this section we construct the numerical method to solve (1). First, we discretize in the time variable; we consider a uniform mesh $\overline{\omega}^M = \{t_n = n\tau,\ 0 \leq n \leq M,\ \tau = T/M\}$, where M is a positive integer. Then, the temporal discretization is given by

$$\begin{array}{l} \mathbf{z}^0 = \mathbf{u}(x,0) = \mathbf{0}, \\ \left\{ \begin{array}{l} \text{For } n = 0, 1, \ldots, M-1, \\ \left(\mathcal{I} + \tau L_{1,x,\varepsilon}^{\mathcal{M}^{n+1}}\right)\mathbf{z}^{n+1}(x) = \tau\mathbf{f}^{n+1}(x) + \left(\mathcal{I} + \tau\mathcal{N}^{n+1}\right)\mathbf{z}^n(x),\ x \in (0,1), \\ \mathbf{z}^{n+1}(x) = \mathbf{0},\ x \in \{0, 1\}, \end{array} \right. \end{array} \tag{5}$$

where $\mathbf{f}^{n+1} = \mathbf{f}(x, t_{n+1}), n = 0, 1, \ldots, M-1$, $\mathcal{I}$ is the identity operator, and

$$L_{1,x,\varepsilon}^{\mathcal{M}^{n+1}}\mathbf{z} \equiv -\mathcal{D}\frac{\partial^2 \mathbf{z}}{\partial x^2} + \mathcal{M}^{n+1}\mathbf{z},$$

with $\mathcal{A}^{n+1} = \mathcal{M}^{n+1} - \mathcal{N}^{n+1}$ and $\mathcal{A}^{n+1} = \mathcal{A}(x, t_{n+1})$.

In this work we consider the additive scheme (see [11]) given by

$$\mathcal{M}^{n+1} = \begin{pmatrix} a_{11}(x, t_{n+1}) & 0 \\ a_{21}(x, t_{n+1}) & a_{22}(x, t_{n+1}) \end{pmatrix}. \tag{6}$$

Lemma 3. *([1]) Let be* $\mathbf{u}$ *and* $\mathbf{z}^{n+1}$ *the solution of problems (1) and (5) respectively. Then, it holds*

$$|\mathbf{u}(x, t_{n+1}) - \mathbf{z}^{n+1}(x)| \leq \mathbf{C}\tau, \ \forall\, x \in [0, 1], \tag{7}$$

and therefore the time discretization is uniformly convergent of first order.

To deduce the fully discrete method, we discretize (5) with the classical central difference scheme defined on a piecewise uniform mesh $I_{x,\varepsilon,N}$, given by $I_{x,\varepsilon,N} = \{0 = x_0 < \ldots < x_N = 1\}$. Taking into account that there are boundary layers at $x = 0$ and $x = 1$, the grid points of the piecewise uniform Shishkin mesh are given by (see [4, 9])

$$x_j = \begin{cases} jh, & j = 0, \ldots, N/4, \\ x_{N/4} + (j - N/4)H, & j = N/4 + 1, \ldots, 3N/4, \\ x_{3N/4} + (j - 3N/4)h, & j = 3N/4 + 1, \ldots, N, \end{cases}$$

with $h = 4\sigma/N$, $H = 2(1 - 2\sigma)/N$, and $\sigma = \min\left\{1/4, 2\sqrt{\varepsilon}\ln N\right\}$, is the transition parameter of the Shishkin mesh.

We denote by $\overline{Q}_1^{N,M} = I_{x,\varepsilon,N} \times \overline{\omega}^M$ the corresponding grid for the (x,t)-variables, by $Q_1^{N,M} = \overline{Q}_1^{N,M} \cap Q_1$, $\Gamma^{N,M} = \overline{Q}_1^{N,M} \backslash Q_1^{N,M}$, by $\mathbf{U} = \{\mathbf{U}^0, \ldots, \mathbf{U}^M\}$ the vector numerical approximation on the grid $\overline{Q}_1^{N,M}$ with $\mathbf{U}^n = \{\mathbf{U}_0^n, \ldots, \mathbf{U}_N^n\}$, $0 \leq n \leq M$. Then, the fully discrete scheme is giving by

$$\begin{aligned} &\mathbf{U}^0 = \mathbf{0}, \\ &\begin{cases} \text{For } n = 0, 1, \ldots, M - 1, \\ [L_{1,\boldsymbol{\varepsilon}}^{N,M}\mathbf{U}]^{n+1} \equiv \dfrac{\mathbf{U}^{n+1} - \mathbf{U}^n}{\tau} - \mathcal{D}\delta_x^2\mathbf{U}^{n+1} + \mathcal{M}^{n+1}\mathbf{U}^{n+1} - \mathcal{N}^{n+1}\mathbf{U}^n = \mathbf{f}^{n+1}, \\ \mathbf{U}_0^{n+1} = \mathbf{U}_N^{n+1} = \mathbf{0}, \end{cases} \end{aligned} \tag{8}$$

where

$$\delta_x^2 Z_i = \frac{2}{h_i + h_{i+1}}\left(\frac{Z_{i+1} - Z_i}{h_{i+1}} - \frac{Z_i - Z_{i-1}}{h_i}\right),$$

is the standard approximation of the second order spatial derivative on a nonuniform mesh, with $h_i = x_i - x_{i-1}$, $i = 1, \ldots, N$.

We write in vector form the solution of problem (5) and it is denoted by $\mathbf{z} = \{\mathbf{z}^0, \ldots, \mathbf{z}^M\}$. Then, the following result holds.

Lemma 4. *([1]) Let* $\mathbf{U}$ *the numerical solution of (8) on the Shishkin mesh and* $\mathbf{z}$ *the solution of (5). Then, it holds*

$$\|\mathbf{U} - \mathbf{z}\|_{\overline{Q}_1^{N,M}} \leq C(N^{-1}\ln N)^2, \tag{9}$$

and therefore the spatial discretization is uniformly convergent of almost second order.

Combining the results of Lemmas 3 and 4, we obtain the following result of convergence, which proves the uniform convergence of the numerical method to the exact solution of problem (1).

Theorem 5. *([1]) Let* $\mathbf{U}$ *be the numerical solution of (8) on the Shishkin mesh and* $\mathbf{u}$ *the solution of (1). Then, it holds*

$$\|\mathbf{U} - \mathbf{u}\|_{\overline{Q}_1^{N,M}} \leq C(\tau + (N^{-1}\ln N)^2), \tag{10}$$

and therefore the fully discrete method is uniformly convergent having first order in time and almost second order in space.

§3. Numerical results

In this section we give the numerical results for two singularly perturbed reaction-diffusion systems which are 1D and 2D in the spatial variable.

3.1. Parabolic 1D problem

We first show the numerical results obtained for a problem of type (1) where the data are defined by

$$\mathcal{A} = \begin{pmatrix} (1+t)(e^x + x) & -(1+x^2)t \\ -x(1+t) & (1+t^2)(1+x^2+\sin(x)) \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} t^2(x^2+\cos(\pi x t)) \\ xt\sin(x+t)(1-e^{-t}) \end{pmatrix}, \tag{11}$$

and $T = 1$. The exact solution of this problem is unknown; in Figure 1 we show the numerical solution using the scheme (8) with $N = M = 32$ for $\varepsilon = 10^{-4}$. We observe that both components of the solution exhibits boundary layer regions at $x = 0$ and $x = 1$.

To approximate the orders of convergence of the scheme, we use a variant of the double-mesh principle [4]. Let $\mathbf{U}_i^n$ denote the numerical solution given by the fully discrete scheme at the grid point (x_i, t_n) with $i = 0, 1, \ldots, N$, $n = 0, 1, \ldots, M$, and $\{\widehat{\mathbf{U}}_i^n\}$ is the numerical solution on a finer mesh $\{(\hat{x}_i, \hat{t}_n)\}$ that consists of the mesh points of the coarse mesh and their midpoints, i.e.,

$$\begin{aligned} &\hat{x}_{2i} = x_i, \;\; i = 0, \ldots, N, \quad \hat{x}_{2i+1} = (x_i + x_{i+1})/2, \;\; i = 0, \ldots, N-1, \\ &\hat{t}_{2n} = t_n, \;\; n = 0, \ldots, M, \quad \hat{t}_{2n+1} = (t_n + t_{n+1})/2, \;\; n = 0, \ldots, M-1. \end{aligned} \tag{12}$$

Then, we compute the two-mesh differences

$$\mathbf{d}_\varepsilon^{N,M} = \max_{0\le n\le M}\max_{0\le i\le N} |\mathbf{U}_i^n - \widehat{\mathbf{U}}_{2i}^{2n}|, \qquad \mathbf{d}^{N,M} = \max_{\varepsilon} \mathbf{d}_\varepsilon^{N,M}, \tag{13}$$

with $\mathbf{d}^{N,M} = (d_1^{N,M}, d_2^{N,M})$. From these values, we obtain the corresponding numerical orders of convergence by

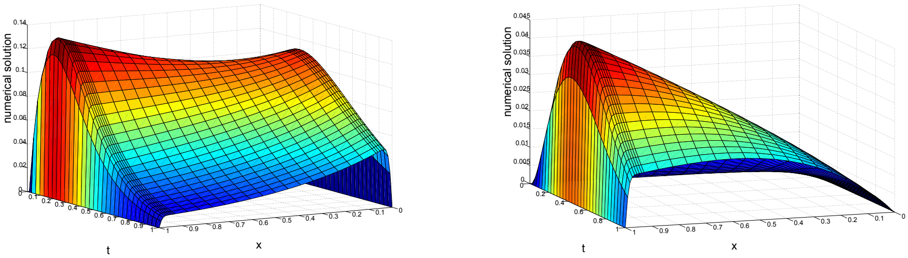
$$\mathbf{p}_\varepsilon^{N,M} = \log\left(\mathbf{d}_\varepsilon^{N,M}/\mathbf{d}_\varepsilon^{2N,2M}\right)/\log 2, \tag{14}$$

and from the the uniform maximum errors $\mathbf{d}^{N,M}$, we obtain the numerical uniform orders of convergence $\mathbf{p}^{N,M} = (p_1^{N,M}, p_2^{N,M})$ given by

$$\mathbf{p}^{N,M} = \log\left(\mathbf{d}^{N,M}/\mathbf{d}^{2N,2M}\right)/\log 2. \tag{15}$$

Tables 1 and 2 display the maximum two-mesh differences and the orders of convergence for each component; from them, we see the uniform convergence of the numerical algorithm. Moreover, we deduce first order of uniform convergence; so, we can conclude that in this example the errors associated to the time discretization dominate into the global error of the numerical method.

To see the influence on the global error of the spatial discretization, which has almost second order in contrast with the time discretization, which has first order, we multiply N by 2 and M by 4. Tables 3 and 4 display the results in this case. From them, we clearly see the almost second order of the numerical algorithm.



(a) Component u_1 (b) Component u_2

Figure 1: Test problem (1) with data (11) and $\varepsilon = 10^{-4}$: numerical solution for $N = M = 32$.

Table 1: Test problem (1) with data (11): maximum and uniform two-mesh differences and their orders of convergence for the component u_1.

	N=32	N=64	N=128	N=256	N=512	N=1024
	M=16	M=32	M=64	M=128	M=256	M=512
$\varepsilon = 1$	4.1890E-4 0.991	2.1072E-4 0.996	1.0567E-4 0.998	5.2913E-5 0.999	2.6476E-5 0.999	1.3243E-5
$\varepsilon = 10^{-1}$	1.9457E-3 0.950	1.0068E-3 0.974	5.1250E-4 0.987	2.5861E-4 0.993	1.2991E-4 0.997	6.5108E-5
$\varepsilon = 10^{-2}$	2.6501E-3 0.938	1.3829E-3 0.968	7.0679E-4 0.984	3.5741E-4 0.992	1.7973E-4 0.996	9.0121E-5
$\varepsilon = 10^{-3}$	3.1427E-3 0.873	1.7154E-3 1.018	8.4682E-4 0.965	4.3373E-4 0.979	2.2004E-4 0.988	1.1096E-4
$\varepsilon = 10^{-4}$	3.9318E-3 0.950	2.0348E-3 0.977	1.0335E-3 0.985	5.2204E-4 0.991	2.6261E-4 0.994	1.3183E-4
$\varepsilon = 10^{-5}$	4.2942E-3 0.968	2.1959E-3 0.983	1.1108E-3 0.991	5.5887E-4 0.995	2.8041E-4 0.997	1.4048E-4
$\varepsilon = 10^{-6}$	4.4130E-3 0.971	2.2510E-3 0.985	1.1372E-3 0.992	5.7158E-4 0.996	2.8658E-4 0.998	1.4349E-4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\varepsilon = 10^{-10}$	4.4745E-3 0.973	2.2796E-3 0.986	1.1507E-3 0.993	5.7816E-4 0.996	2.8978E-4 0.998	1.4507E-4
$d_1^{N,M}$ $p_1^{N,M}$	4.4745E-3 0.973	2.2796E-3 0.986	1.1507E-3 0.993	5.7816E-4 0.996	2.8978E-4 0.998	1.4507E-4

Table 2: Test problem (1) with data (11): maximum and uniform two-mesh differences and their orders of convergence for the component u_2.

	N=32	N=64	N=128	N=256	N=512	N=1024
	M=16	M=32	M=64	M=128	M=256	M=512
$\varepsilon = 1$	1.0876E-4 0.933	5.6962E-5 0.970	2.9078E-5 0.984	1.4700E-5 0.992	7.3910E-6 0.996	3.7057E-6
$\varepsilon = 10^{-1}$	9.6357E-4 0.932	5.0499E-4 0.964	2.5887E-4 0.982	1.3105E-4 0.991	6.5929E-5 0.996	3.3067E-5
$\varepsilon = 10^{-2}$	1.2939E-3 0.936	6.7639E-4 0.968	3.4575E-4 0.983	1.7487E-4 0.992	8.7943E-5 0.996	4.4100E-5
$\varepsilon = 10^{-3}$	3.2608E-3 1.031	1.5960E-3 1.654	5.0709E-4 1.521	1.7663E-4 0.991	8.8840E-5 0.996	4.4553E-5
$\varepsilon = 10^{-4}$	3.2168E-3 0.921	1.6987E-3 1.353	6.6516E-4 1.393	2.5332E-4 1.403	9.5757E-5 1.103	4.4575E-5
$\varepsilon = 10^{-5}$	3.2012E-3 0.919	1.6930E-3 1.354	6.6237E-4 1.393	2.5221E-4 1.406	9.5188E-5 1.094	4.4577E-5
$\varepsilon = 10^{-6}$	3.1962E-3 0.918	1.6911E-3 1.354	6.6146E-4 1.393	2.5185E-4 1.407	9.4999E-5 1.092	4.4577E-5
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\varepsilon = 10^{-10}$	3.1938E-3 0.918	1.6903E-3 1.354	6.6104E-4 1.393	2.5169E-4 1.407	9.4912E-5 1.090	4.4577E-5
$d_2^{N,M}$ $p_2^{N,M}$	3.2608E-3 0.941	1.6987E-3 1.353	6.6516E-4 1.393	2.5332E-4 1.403	9.5757E-5 1.103	4.4577E-5

Table 3: Test problem (1) with data (11): maximum and uniform two-mesh differences and their orders of convergence for the component u_1.

	N=32	N=64	N=128	N=256	N=512	N=1024
	M=16	M=64	M=256	M=1024	M=4096	M =16384
$\varepsilon = 1$	4.1890E-4 1.960	1.0770E-4 1.991	2.7097E-5 1.998	6.7848E-6 1.999	1.6969E-6 2.000	4.2426E-7
$\varepsilon = 10^{-1}$	1.9457E-3 1.909	5.1822E-4 1.976	1.3176E-4 1.994	3.3079E-5 1.998	8.2786E-6 2.000	2.0702E-6
$\varepsilon = 10^{-2}$	2.6501E-3 1.905	7.0771E-4 1.975	1.8001E-4 1.994	4.5199E-5 1.998	1.1312E-5 2.000	2.8289E-6
$\varepsilon = 10^{-3}$	3.1427E-3 0.568	2.1199E-3 1.871	5.7945E-4 1.966	1.4832E-4 1.980	3.7593E-5 1.998	9.4125E-6
$\varepsilon = 10^{-4}$	3.9318E-3 0.749	2.3403E-3 1.422	8.7321E-4 1.472	3.1479E-4 1.612	1.0300E-4 1.673	3.2291E-5
$\varepsilon = 10^{-5}$	4.2942E-3 0.876	2.3394E-3 1.421	8.7364E-4 1.470	3.1529E-4 1.611	1.0322E-4 1.673	3.2364E-5
$\varepsilon = 10^{-6}$	4.4130E-3 0.916	2.3392E-3 1.421	8.7379E-4 1.470	3.1545E-4 1.611	1.0328E-4 1.673	3.2388E-5
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\varepsilon = 10^{-10}$	4.4745E-3 0.936	2.3391E-3 1.420	8.7385E-4 1.470	3.1553E-4 1.611	1.0331E-4 1.673	3.2398E-5
$d_1^{N,M}$ $p_1^{N,M}$	4.4745E-3 0.935	2.3403E-3 1.421	8.7385E-4 1.470	3.1553E-4 1.611	1.0331E-4 1.673	3.2398E-5

Table 4: Test problem (1) with data (11): maximum and uniform two-mesh differences and their orders of convergence for the component u_2.

	N=32	N=64	N=128	N=256	N=512	N=1024
	M=16	M=64	M=256	M=1024	M=4096	M =16384
$\varepsilon = 1$	1.0876E-4 1.917	2.8796E-5 1.980	7.3015E-6 1.995	1.8319E-6 1.999	4.5840E-7 2.000	1.1463E-7
$\varepsilon = 10^{-1}$	9.6357E-4 1.906	2.5714E-4 1.975	6.5387E-5 1.994	1.6417E-5 1.998	4.1088E-6 2.000	1.0275E-6
$\varepsilon = 10^{-2}$	1.2939E-3 1.905	3.4551E-4 1.975	8.7868E-5 1.994	2.2063E-5 1.998	5.5219E-6 2.000	1.3809E-6
$\varepsilon = 10^{-3}$	3.2608E-3 1.148	1.4718E-3 1.856	4.0668E-4 1.955	1.0490E-4 1.978	2.6638E-5 1.997	6.6728E-6
$\varepsilon = 10^{-4}$	3.2168E-3 1.032	1.5730E-3 1.491	5.5947E-4 1.524	1.9453E-4 1.633	6.2726E-5 1.698	1.9329E-5
$\varepsilon = 10^{-5}$	3.2012E-3 1.029	1.5687E-3 1.491	5.5806E-4 1.524	1.9403E-4 1.633	6.2563E-5 1.698	1.9279E-5
$\varepsilon = 10^{-6}$	3.1962E-3 1.028	1.5673E-3 1.491	5.5761E-4 1.524	1.9387E-4 1.633	6.2512E-5 1.698	1.9263E-5
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\varepsilon = 10^{-10}$	3.1938E-3 1.028	1.5666E-3 1.491	5.5740E-4 1.524	1.9380E-4 1.633	6.2488E-5 1.698	1.9256E-5
$d_2^{N,M}$ $p_2^{N,M}$	3.2608E-3 1.052	1.5730E-3 1.491	5.5947E-4 1.524	1.9453E-4 1.633	6.2726E-5 1.698	1.9329E-5

3.2. Parabolic 2D problem

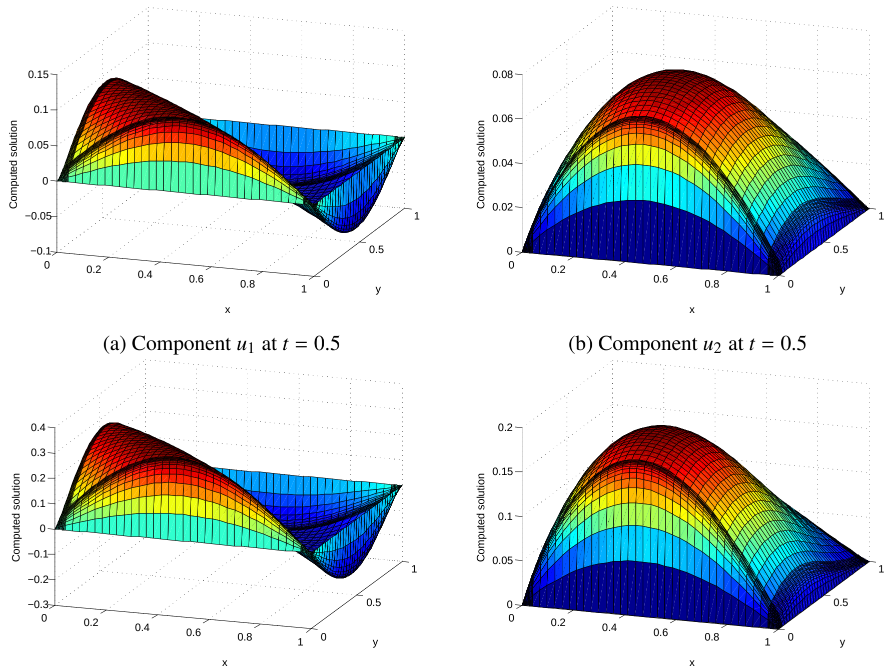
In this section we show that the additive schemes described in (5) for 1D parabolic problems, can be also used to approximate uniformly the exat solution of 2D parabolic problems of type (3). For the numerical approximation, we propose the following finite difference scheme

$$\mathbf{U}^0 = \mathbf{0},\quad \begin{cases} \text{For } n = 0, 1, \dots, M-1, \\ [L_{1,\varepsilon}^{N,M}\mathbf{U}]^{n+1} \equiv \dfrac{\mathbf{U}^{n+1} - \mathbf{U}^n}{\tau} - \mathcal{D}(\delta_x^2 + \delta_y^2)\mathbf{U}^{n+1} + \mathcal{M}^{n+1}\mathbf{U}^{n+1} - \mathcal{N}^{n+1}\mathbf{U}^n = \mathbf{f}^{n+1}, \\ \mathbf{U}_0^{n+1} = \mathbf{U}_N^{n+1} = \mathbf{0}, \end{cases} \tag{16}$$

where the matrices $\mathcal{M}^{n+1}$ and $\mathcal{N}^{n+1}$ were defined for the 1D parabolic problem and δ_x^2 and δ_y^2 are the standard approximation of the second order derivative on a nonuniform mesh in the x and y directions, respectively. We use a uniform mesh in time and the time step is also denoted by $\tau = T/M$. In both space directions we take a piecewise uniform Shishkin mesh, which is the tensor product of 1D piecewise uniform Shishkin meshes, $I_{x,\varepsilon,N} = \{0 = x_0 < \ldots < x_N = 1\}$ and $I_{y,\varepsilon,N} = \{0 = y_0 < \ldots < y_N = 1\}$, where the positive integer N is the spatial discretisation parameter.

We use this scheme to approximate the following test problem

$$\mathcal{A} = \begin{pmatrix} 1 + xy & -x^2y^2 \\ -\cos(0.5(x+y)) & e^{x+y} \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} (1 - e^{-t})\sin(\pi(x+y)) \\ (1 - e^{-t})(3x(1-x) + y(1-y)) \end{pmatrix}, \tag{17}$$



(a) Component u_1 at $t = 0.5$ (b) Component u_2 at $t = 0.5$

(c) Component u_1 at $t = 1$ (d) Component u_2 at $t = 1$

Figure 2: Test problem (3) with data (17) and $\varepsilon = 10^{-4}$: numerical solution for $N = M = 64$.

and the final time is $T = 1$. Figure 2 shows the numerical solution for both components using the scheme (16) with $N = M = 64$, again for $\varepsilon = 10^{-4}$, at $t = 0.5$ and $t = 1$.

The exact solution of problem (17) is unknown and we again use the same variant of the double-mesh principle to approximate the errors: We denote them by

$$\mathbf{D}_\varepsilon^{N,M} = \max_{0\le n\le M} \max_{0\le i,j\le N} |\mathbf{U}_{i,j}^n - \widehat{\mathbf{U}}_{2i,2j}^{2n}|, \qquad \mathbf{D}^{N,M} = \max_{\varepsilon} \mathbf{D}_\varepsilon^{N,M}, \tag{18}$$

where $\{\widehat{\mathbf{U}}_{i,j}^n\}$ is the numerical solution on a finer mesh $\{(\hat{x}_i, \hat{y}_j, \hat{t}_n)\}$ that consists of the mesh points of the coarse mesh and their midpoints, i.e.,

$$\begin{aligned} &\hat{x}_{2i} = x_i, \;\; i = 0,\dots,N, \quad \hat{x}_{2i+1} = (x_i + x_{i+1})/2, \;\; i = 0,\dots,N-1,\\ &\hat{y}_{2j} = y_j, \;\; j = 0,\dots,N, \quad \hat{y}_{2j+1} = (y_j + y_{j+1})/2, \;\; j = 0,\dots,N-1,\\ &\hat{t}_{2n} = t_n, \;\; n = 0,\dots,M, \quad \hat{t}_{2n+1} = (t_n + t_{n+1})/2, \;\; n = 0,\dots,M-1. \end{aligned} \tag{19}$$

From the maximum two-mesh differences $\mathbf{D}_\varepsilon^{N,M}$ and the uniform maximum two-mesh differences $\mathbf{D}^{N,M}$, we obtain the orders of convergence and the uniform orders of convergence by using (14) and (15) respectively. The uniform orders of convergence are denoted by $\mathbf{q}^{N,M} = (q_1^{N,M}, q_2^{N,M})$.

Tables 5 and 6 display the maximum errors and the orders of convergence for both components. These numerical results suggest that the additive scheme is uniformly convergent,

Table 5: Test problem (3) with data (17): maximum and uniform two-mesh differences and their orders of convergence for the component u_1.

	N=M=16	N=M=32	N=M=64	N=M=128
$\varepsilon = 1$	5.622E-5 0.603	3.701E-5 0.648	2.363E-5 0.723	1.431E-5
$\varepsilon = 10^{-1}$	1.326E-3 1.032	6.482E-4 1.014	3.211E-4 1.007	1.598E-4
$\varepsilon = 10^{-2}$	3.258E-3 1.353	1.275E-3 0.942	6.637E-4 0.972	3.383E-4
$\varepsilon = 10^{-3}$	3.780E-3 1.244	1.596E-3 1.035	7.791E-4 0.980	3.949E-4
$\varepsilon = 10^{-4}$	3.730E-3 1.214	1.608E-3 0.970	8.208E-4 0.983	4.154E-4
$\varepsilon = 10^{-5}$	3.716E-3 1.177	1.643E-3 0.973	8.370E-4 0.984	4.233E-4
$\varepsilon = 10^{-6}$	3.712E-3 1.165	1.655E-3 0.972	8.439E-4 0.986	4.261E-4
⋯	⋯	⋯	⋯	⋯
$\varepsilon = 10^{-10}$	3.710E-3 1.160	1.660E-3 0.971	8.469E-4 0.986	4.277E-4
$D_1^{N,M}$ $q_1^{N,M}$	3.780E-3 1.187	1.660E-3 0.971	8.469E-4 0.986	4.277E-4

Table 6: Test problem (3) with data (17): maximum and uniform two-mesh differences and their orders of convergence for the component u_2.

	N=M=16	N=M=32	N=M=64	N=M=128
$\varepsilon = 1$	1.661E-4 0.623	1.079E-4 0.733	6.491E-5 0.853	3.593E-5
$\varepsilon = 10^{-1}$	1.074E-3 0.844	5.982E-4 0.919	3.164E-4 0.960	1.627E-4
$\varepsilon = 10^{-2}$	1.849E-3 1.002	9.236E-4 0.954	4.768E-4 0.977	2.422E-4
$\varepsilon = 10^{-3}$	2.230E-3 1.153	1.003E-3 0.998	5.022E-4 0.981	2.544E-4
$\varepsilon = 10^{-4}$	2.233E-3 1.168	9.936E-4 0.978	5.044E-4 0.982	2.554E-4
$\varepsilon = 10^{-5}$	2.234E-3 1.174	9.905E-4 0.974	5.043E-4 0.981	2.555E-4
$\varepsilon = 10^{-6}$	2.235E-3 1.175	9.894E-4 0.973	5.042E-4 0.981	2.555E-4
⋯	⋯	⋯	⋯	⋯
$\varepsilon = 10^{-10}$	2.235E-3 1.176	9.890E-4 0.972	5.042E-4 0.980	2.555E-4
$D_2^{N,M}$ $q_2^{N,M}$	2.235E-3 1.156	1.003E-3 0.992	5.044E-4 0.981	2.555E-4

when it is used to approximate the solution of 2D parabolic problems. In this case we multiply N and M by 2; from both tables, we see first order of uniform convergence, and therefore we can conclude that the errors associated to the time discretization dominate into the global error of the numerical method.

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