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Stability results for nematic liquid crystals

Haroldo Rodrigues Clark, María Ángeles Rodríguez-Bellido and Marko A. Rojas-Medar

Abstract. In 1994, Ponce et al. analyzed ([11]) the stability of mildly decaying global strong solutions for the Navier-Stokes equations. In this work, we try to apply the same approach for a nematic liquid crystal model, that is a coupled model including a Navier-Stokes type-system for the velocity of the liquid crystal ("liquid part") and a parabolic system for the orientation vector field for the molecules of the liquid crystal ("solid part"). We will focus on the similarities and differences with respect to [11], depending on the boundary data chosen for the solid part.

Keywords: Stability of solutions, nonlinear coupled system, liquid crystal system.

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§1. The model

Liquid crystals are intermediate state of matter (between liquid and solid state), whose applications in physical and technical devices has been fashionable for decades. The modeling of nematic liquid crystals is therefore very interesting for a mathematical point of view in order to reproduce the physical properties of their molecules, specially the refraction of the light. As a consequence, different systems of equations can be found in the literature, most of them including a macroscopic part (equations for the fluid containing the molecules if the liquid crystal) and a microscopic part (equations for the behavior of the molecules of liquid crystals). However, the mathematical analysis is not easy, taking into account that these models contain the Navier-Stokes equations (for the velocity of the fluid) inside, whose external force is now a term depending of the new variable describing the molecules of liquid crystals, and that new variable satisfies its own equation.

Therefore, questions asked for the Navier-Stokes equations can now be asked for these liquid crystals models. Actually, we want to know if the asymptotic stability analyzed in the paper of Ponce et al. (cf. [11]) can be generalized for some liquid crystal model.

In a first attempt, we focus on the model studied by Lin et al. (cf. [8]): If we denote by $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$ the velocity vector, $p(t, \mathbf{x})$ the pressure of the fluid, $\mathbf{e}(t, \mathbf{x})$ the orientation of the liquid crystal molecules, and $\mathbf{x} = (x_1, x_2, x_3) \in \Omega$ a bounded domain (whose boundary is denoted by $\partial\Omega$) then the model for the phenomenon in 3D of liquid crystals of nematic type can be described, for example, by coupled system:

$$\begin{cases} \partial_t \mathbf{v} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla q = -\lambda(\nabla\mathbf{e})^t\Delta\mathbf{e} + \mathbf{g} & \text{in} \quad (0,T)\times\Omega, \\ \nabla\cdot\mathbf{v} = 0 & \text{in} \quad (0,T)\times\Omega, \\ \partial_t\mathbf{e} - \gamma\,(\Delta\mathbf{e} - \mathbf{f}_\delta(\mathbf{e})) + (\mathbf{v}\cdot\nabla)\mathbf{e} = 0 & \text{in} \quad (0,T)\times\Omega, \\ \mathbf{v} = \mathbf{0}, \quad \partial_{\mathbf{n}}\mathbf{e} = \mathbf{0} & \text{on} \quad (0,T)\times\partial\Omega, \\ \mathbf{v}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}), \quad \mathbf{e}(\mathbf{x},0) = \mathbf{e}_0(\mathbf{x}) & \text{in} \quad \Omega\,, \end{cases} \tag{1}$$

where $\nu > 0$ is the fluid viscosity, $\lambda > 0$ is the elasticity constant, $\gamma > 0$ is a relaxation in time constant, the function $\mathbf{f}_\delta$ is defined by

$$\mathbf{f}_\delta(\mathbf{e}) = \frac{1}{\delta^2}(|\mathbf{e}|^2 - 1)\mathbf{e} \quad \text{with} \quad |\mathbf{e}| \le 1, \tag{2}$$

where $|\cdot|$ is the euclidian norm in $\mathbb{R}^3$, $\delta > 0$ is a penalization parameter, and $\mathbf{g}$ is a known function defined in $(0,T)\times\Omega$.

For more details about the penalization function $\mathbf{f}_\delta(\mathbf{e})$ we recommend to readers, the following references (cf. [7], Lin & Liu [8]) and Guillén-González et al (cf. [6]). Here, all those derivatives in problem (1) are in the sense of the distributions of Schwartz (see, L. Schwartz [12]).

Note that system (1) is a simplified model, where the terms modeling the stretching effect (for example) are not contained (see [13] for a more general model). On the other hand, if a tensor variable Q is used to analyze the molecular behavior instead of the director field $\mathbf{e}$, some other more complex Q-tensor models appear (see [10, 9, 5, 3, 4]).

§2. Spaces framework and regularity definitions and results

We will use $\mathbf{V}$ and $\mathbf{H}$ the Lebesgue-Sobolev spaces of type $\mathbf{H}^1(\Omega)$, $\mathbf{H}^1_0(\Omega)$ and $\mathbf{L}^2(\Omega)$ associated with the incompressibility and adherence velocity conditions (as usual in the Navier-Stokes framework), given by

$$\mathbf{V} = \{\mathbf{y}\in\mathbf{H}^1(\Omega);\ \nabla\cdot\mathbf{y} = 0,\ \mathbf{u}_{|\Gamma} = 0\}, \quad \mathbf{H} = \{\mathbf{y}\in\mathbf{L}^2(\Omega);\ \nabla\cdot\mathbf{y} = 0, \mathbf{u}\cdot\mathbf{n}_{|\Gamma} = 0\}.$$

We deal with two different notions of solution for the Navier-Stokes equations and system (1). Namely,

Definition 1. We call

- a weak solution for the Navier-Stokes equations in $(0,T)$ to a function $\mathbf{v}$ satisfying the variational formulation for these equations and with the following regularity:

$$\mathbf{v}\in L^\infty(0,T;\mathbf{H})\cap L^2(0,T;\mathbf{V}), \tag{3}$$

- a weak solution for (1) in $(0,T)$ to a pair $(\mathbf{v},\mathbf{e})$ satisfying the variational formulation for (1), where $\mathbf{v}$ satisfies (3) and $\mathbf{e}$ has the following regularity:

$$\mathbf{e}\in L^\infty(0,T;\mathbf{H}^1(\Omega))\cap L^2(0,T;\mathbf{H}^2(\Omega)). \tag{4}$$

Definition 2. We call

- a strong solution for the Navier-Stokes equations in $(0, T)$ to a function $\mathbf{v}$ satisfying the variational formulation for these equations and with the following regularity:

$$\mathbf{v} \in L^\infty(0, T; \mathbf{H} \cap \mathbf{H}^1(\Omega)) \cap L^2(0, T; \mathbf{H}^2(\Omega) \cap \mathbf{V}), \tag{5}$$

- a strong solution for (1) in $(0, T)$ to a pair $(\mathbf{v}, \mathbf{e})$ satisfying the variational formulation for (1), where $\mathbf{v}$ satisfies (5) and $\mathbf{e}$ has the following regularity:

$$\mathbf{e} \in L^\infty(0, T; \mathbf{H}^2(\Omega)) \cap L^2(0, T; \mathbf{H}^3(\Omega)). \tag{6}$$

The existence of global in time weak solution for (1), that is a weak solution in $(0, T)$ for any given $T \le +\infty$, in the case of Dirichlet boundary conditions for $\mathbf{e}$ ($\mathbf{e}|_\Sigma = \mathbf{h}$ instead of $\partial_\mathbf{n}\mathbf{e}|_\Sigma = \mathbf{0}$) has been treated in the work of Lin et al. (cf. [8]) for the case of time-independent Dirichlet boundary data, and the time-dependent case and the strong regularity of time-periodic solutions were studied by Climent et al. (cf. [2]). In all cases, the following compatibility hypothesis were considered:

$$|\mathbf{e}_0| \le 1 \quad \text{a. e. in } \Omega, \qquad |\mathbf{h}| \le 1 \quad \text{a. e. in } \Sigma, \tag{7}$$

and (among others) $\mathbf{v}_0 \in \mathbf{H}$ and $\mathbf{e}_0 \in \mathbf{H}^1(\Omega)$. On the other hand, under the hypothesis of strong regularity for the data $\mathbf{v}_0 \in \mathbf{V}$, $\mathbf{e}_0 \in \mathbf{H}^2(\Omega)$ it is known the existence and uniqueness of strong solution under some restrictions:

- global in time for big enough viscosity ($\nu >> 1$),
- local in time (the strong solution is defined in $(0, T^*)$, being $T_* \le T$ small enough) for any data.

For the analysis of stability results it is necessary to start from a solution defined in $(0, +\infty)$. Considering homogeneous Neumann boundary conditions for $\mathbf{e}$ ($\partial_\mathbf{n}\mathbf{e}|_\Sigma = \mathbf{0}$), it is possible to prove the existence of a global weak solution $(\mathbf{v}, \mathbf{e})$ in $(0, +\infty)$ for any data $(\mathbf{v}_0, \mathbf{e}_0, \mathbf{g}) \in \mathbf{H} \times \mathbf{H}^1(\Omega) \times L^2(0, +\infty; \mathbf{L}^2(\Omega))$, that is:

$$\begin{aligned} &\mathbf{v} \in L^\infty(0, +\infty; \mathbf{H}) \cap L^2(0, +\infty; \mathbf{V}), \\ &\mathbf{e} \in L^\infty(0, +\infty; \mathbf{H}^1(\Omega)) \cap L^2(0, +\infty; \mathbf{H}^2(\Omega)). \end{aligned} \tag{8}$$

Remark 1. For either non-homogeneous Dirichlet or homogeneous Neumann boundary conditions for $\mathbf{e}$, it is possible to prove that (see, for instance, Appendix A in [6] for a proof):

$$\mathbf{e} \in L^\infty((0, +\infty) \times \Omega) \tag{9}$$

§3. Stability result for (1)

The result of stability that we present is a generalization of a result stablished by Ponce et al. (cf. [11]), made for the classical Navier-Stokes equations. Our stability results for system (1) can be stated as follows:

Theorem 1. *Suppose that there exists a global strong solution* $(\mathbf{v}, q, \mathbf{e})$ *of system* (1) *with the regularity*

$$\begin{aligned} &\mathbf{v} \in L^\infty_{loc}(0, +\infty; \mathbf{V}) \cap L^2_{loc}(0, +\infty; D(A)), \\ &\mathbf{e} \in L^\infty_{loc}(0, +\infty; \mathbf{H}^2(\Omega)) \cap L^2_{loc}(0, +\infty; \mathbf{H}^3(\Omega)), \end{aligned} \tag{10}$$

and that satisfies the Leray global criterion of regularity

$$\|\nabla \mathbf{v}(t)\|^4 \quad and \quad \|\nabla \mathbf{e}(t)\|^4_{H^1(\Omega)} \quad belong\ to \quad L^1(0, \infty). \tag{11}$$

Assume $\mathbf{u}_0 \in \mathbf{V}$*,* $\mathbf{d}_0 \in \mathbf{H}^2(\Omega)$ *and* $P\mathbf{k} \in L^2(0, \infty; \mathbf{H})$*, being P is the Helmholtz projector*

$$P : \mathbf{L}^2(\Omega) \to \mathbf{H} = \overline{\{\mathbf{v} \in \mathbf{L}^2(\Omega) : \nabla \cdot \mathbf{v} = 0\}}^{\mathbf{L}^2(\Omega)}. \tag{12}$$

There exists a $\delta > 0$ *such that if* $(\mathbf{u}_0, \mathbf{d}_0, \mathbf{k})$ *satisfies:*

$$\|\nabla(\mathbf{u}_0 - \mathbf{v}_0)\| + \|\mathbf{d}_0 - \mathbf{e}_0\|_{H^2(\Omega)} + \int_0^\infty \|P(\mathbf{k} - \mathbf{g})(t)\|^2 dt < \delta \tag{13}$$

then there exists a unique global strong solution $(\mathbf{u}, p, \mathbf{d})$ *of system* (1) *with data* $(\mathbf{u}_0, \mathbf{d}_0, P\mathbf{k})$*. Moreover, there exists* $C = C(\delta)$ *with* $C(\delta) \to 0$ *as* $\delta \to 0$ *such that*

$$\sup_{t \ge 0} \|\nabla(\mathbf{u}(t) - \mathbf{v}(t))\| + \sup_{t \ge 0} \|\mathbf{d}(t) - \mathbf{e}(t)\|_{H^2(\Omega)} \le C(\delta). \tag{14}$$

Note that several results are proven: the global regularity in time for the strong solution $(\mathbf{v}, \mathbf{e})$, that is, the “loc" character in (10) can be removed; the uniqueness of solution for system (1) and that solution $(\mathbf{u}, \mathbf{d})$ is also a global in time strong solution near $(\mathbf{v}, \mathbf{e})$ along $(0, +\infty)$.

3.1. Scketch of the proof of Theorem 1

Here we focus on the obtention of (14). The detailed proof of this asymptotic result together with the uniqueness of the global strong solution $(\mathbf{u}, p, \mathbf{d})$ can be found in [1].

We consider the perturbation $(\mathbf{u}, p, \mathbf{d})$ which satisfies the system

$$\begin{cases} \partial_t \mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = -\lambda(\nabla\mathbf{d})^t\Delta\mathbf{d} + \mathbf{k} & \text{in} \quad (0,T)\times\Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in} \quad (0,T)\times\Omega, \\ \partial_t\mathbf{d} - \gamma(\Delta\mathbf{d} - \mathbf{f}_\delta(\mathbf{d})) + (\mathbf{u}\cdot\nabla)\mathbf{d} = 0 & \text{in} \quad (0,T)\times\Omega, \\ \mathbf{u} = 0, \quad \partial_{\mathbf{n}}\mathbf{d} = \mathbf{0} & \text{on} \quad (0,T)\times\partial\Omega, \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{d}(\mathbf{x},0) = \mathbf{d}_0(\mathbf{x}) & \text{in} \quad \Omega, \end{cases} \tag{15}$$

that it is a global in time weak solution, even in $(0, +\infty)$. Therefore, denoting by $A = -P\Delta$ the Stokes operator with domain

$$D(A) = \mathbf{V} \cap \mathbf{H}^2(\Omega), \quad \mathbf{V} = \mathbf{H}_0^1(\Omega) \cap \mathbf{H},$$

where P is the Helmholtz projector defined in (12), we want to analyze the behavior of $(\mathbf{w}, \mathbf{z}) := (\mathbf{v} - \mathbf{u}, \mathbf{e} - \mathbf{d})$, which satisfies:

$$\begin{cases} \partial_t \mathbf{w} + \nu A \mathbf{w} = -P[(\mathbf{v}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{v} - (\mathbf{w}\cdot\nabla)\mathbf{w}] & \\ \qquad -\lambda P[(\nabla\mathbf{e})^t\Delta\mathbf{z} + (\nabla\mathbf{z})^t\Delta\mathbf{e} - (\nabla\mathbf{z})^t\Delta\mathbf{z}] + P(\mathbf{g} - \mathbf{k}) & \text{in} \quad (0,T)\times\Omega, \\ \nabla\cdot\mathbf{w} = 0 & \text{in} \quad (0,T)\times\Omega, \\ \partial_t\mathbf{z} - \gamma\Delta\mathbf{z} = -\gamma[\mathbf{f}_\delta(\mathbf{e}) - \mathbf{f}_\delta(\mathbf{e}-\mathbf{z})] - [(\mathbf{v}\cdot\nabla)\mathbf{z} + (\mathbf{w}\cdot\nabla)\mathbf{e} - (\mathbf{w}\cdot\nabla)\mathbf{z}] & \text{in} \quad (0,T)\times\Omega, \\ \mathbf{w} = 0, \quad \partial_\mathbf{n}\mathbf{z} = \mathbf{0} & \text{on} \quad (0,T)\times\partial\Omega, \\ \mathbf{w}(\mathbf{x},0) = \mathbf{w}_0(\mathbf{x}), \quad \mathbf{z}(\mathbf{x},0) = \mathbf{z}_0(\mathbf{x}) & \text{in} \quad \Omega\,. \end{cases} \tag{16}$$

First, we take $\mathbf{z}$ as test function in $(16)_3$, we obtain:

$$\frac{d}{dt}\|\mathbf{z}\|^2 + \gamma\,\|\mathbf{z}\|^2_{H^1(\Omega)} + \frac{5\gamma}{4\delta^2}\,\|\mathbf{z}\|^4_{L^4(\Omega)} \le C_{\varepsilon,\delta,\gamma}\,\|\mathbf{z}\|^2 + \varepsilon\,\|\nabla\mathbf{w}\|^2 \tag{17}$$

because $\nabla\mathbf{e} \in L^\infty(0,+\infty;\mathbf{L}^2(\Omega))$ thanks to (8) for $T = +\infty$. Then, we take $(\mathbf{w}, -\lambda\,\Delta\mathbf{z})$ as test function in (16), obtaining:

$$\begin{aligned} \frac{d}{dt}\left(\|\mathbf{w}\|^2 + \lambda\,\|\nabla\mathbf{z}\|^2\right) &+ \nu\,\|\nabla\mathbf{w}\|^2 + \gamma\,\lambda\,\|\Delta\mathbf{z}\|^2 \le C\,\left(\|\nabla\mathbf{v}\|^4 + \|\Delta\mathbf{e}\|^4\right)\left(\|\mathbf{w}\|^2 + \lambda\,\|\nabla\mathbf{z}\|^2\right) \\ &+ C\|\mathbf{g} - \mathbf{h}\|^2_{\mathbf{H}^{-1}(\Omega)} + C\,\|\nabla\mathbf{z}\|^2 + \varepsilon_1\|\nabla\mathbf{z}\|^2_{H^1(\Omega)} \end{aligned} \tag{18}$$

Note that ε and ε_1 are small parameters, which will be defined later in order to control the terms containing ε and ε_1 on the right hand side of (20) with the terms appearing on the left hand side of (20); and $C_{\varepsilon,\varepsilon_1}$ denote different constants, increasing (up to $+\infty$) when ε, $\varepsilon_1 \downarrow 0$.

An finally, we test (16) with $(A\mathbf{w}, \Delta^2\mathbf{z})$, obtaining:

$$\begin{aligned} &\frac{d}{dt}\left(\|\nabla\mathbf{w}\|^2 + \|\Delta\mathbf{z}\|^2\right) + (\nu - 11\varepsilon)\,\|A\mathbf{w}\|^2 + (\gamma - 10\varepsilon_1)\,\|\nabla(\Delta\mathbf{z})\|^2 \\ &\le C_{\varepsilon,\varepsilon_1}\,\left(\|\nabla\mathbf{w}\|^6 + \|\mathbf{z}\|^4_{H^2(\Omega)}\|\nabla\mathbf{w}\|^2 + \|\mathbf{z}\|^6_{H^2(\Omega)}\right) \\ &+ C_{\varepsilon,\varepsilon_1}\,\left(\|\nabla\mathbf{v}\|^4 + \|\nabla\mathbf{e}\|^4_{H^1(\Omega)} + 1\right)\left(\|\nabla\mathbf{w}\|^2 + \|\mathbf{z}\|^2_{H^2(\Omega)}\right) \\ &+ C_{\varepsilon,\varepsilon_1}\,\|\mathbf{z}\|^2_{H^2(\Omega)} + C_{\varepsilon,\varepsilon_1}\,\|P(\mathbf{g} - \mathbf{k})\|^2 + \varepsilon\,\|\mathbf{z}\|^2_{H^3(\Omega)} \end{aligned} \tag{19}$$

Observe that the bound for the term $\nabla([\mathbf{f}_\delta(\mathbf{e}) - \mathbf{f}_\delta(\mathbf{e}-\mathbf{z})]), \nabla(\Delta\mathbf{z}))$ can be made as follows: Since $\mathbf{e}$ has the regularity of $(8)_2$, (9) and $\mathbf{f}_\delta(\mathbf{e}) = \frac{1}{\delta^2}\left(|\mathbf{e}|^2 - 1\right)\mathbf{e}$, the mapping

$$\Psi : L^\infty(0,\infty;\mathbf{H}^1(\Omega)) \cap L^2(0,\infty;\mathbf{H}^2(\Omega)) \cap L^\infty((0,\infty)\times\Omega) \to L^\infty(0,\infty;\mathbf{L}^2(\Omega)) \cap L^2(0,\infty;\mathbf{H}^1(\Omega))$$

given by:

$$\Psi(\mathbf{e}) = \nabla \mathbf{f}_\delta(\mathbf{e}) = \frac{1}{\delta^2}\left[(|\mathbf{e}|^2 - 1)\nabla\mathbf{e} + 2(\mathbf{e}\cdot\nabla\mathbf{e})\,\mathbf{e}\right]$$

is well defined. Observe that

$$\delta\Psi(\mathbf{e},\mathbf{h}) = \frac{1}{\delta^2}\left(2\,(\mathbf{e}\cdot\mathbf{h})\,\nabla\mathbf{e} + \left(|\mathbf{e}|^2 - 1\right)\nabla\mathbf{h} + 2\,(\mathbf{h}\cdot\nabla\mathbf{e})\,\mathbf{e} + 2\,(\mathbf{e}\cdot\nabla\mathbf{h})\,\mathbf{e} + 2\,(\mathbf{e}\cdot\nabla\mathbf{e})\,\mathbf{h}\right)$$

Therefore, as $\Psi(\mathbf{e}) - \Psi(\mathbf{e}-\mathbf{z}) = \delta\Psi(\tilde{\mathbf{e}},\mathbf{z})$, being $\delta\Psi(\tilde{\mathbf{e}},\mathbf{z})$ the Gâteaux derivative of Ψ in the direction $\mathbf{z}$, and at the point $\tilde{\mathbf{e}}$ defined as $\tilde{\mathbf{e}} = \theta\,\mathbf{e} + (1-\theta)\,(\mathbf{e}-\mathbf{z})$, $\theta \in (0,1)$ (i.e., $\tilde{\mathbf{e}}$ is any element of the convex line $[\mathbf{e},\mathbf{e}-\mathbf{z}]$), we have:

$$\|\Psi(\mathbf{e}) - \Psi(\mathbf{e}-\mathbf{z})\| \quad\leq\quad \|\delta\Psi(\tilde{\mathbf{e}},\mathbf{z})\| \leq C_\delta\left(\|\nabla\tilde{\mathbf{e}}\|\|\mathbf{z}\|^{1/2}_{H^1(\Omega)}\|\mathbf{z}\|^{1/2}_{H^2(\Omega)} + \|\nabla\mathbf{z}\|\right)$$

Therefore, we can deduce that:

$$(\nabla[\mathbf{f}_\delta(\mathbf{e}) - \mathbf{f}_\delta(\mathbf{e}-\mathbf{z}))], \nabla(\Delta\mathbf{z})) \quad\leq\quad \varepsilon\,\|\nabla(\Delta\mathbf{z})\|^2 + C_{\varepsilon,\gamma,\delta}\,\|\mathbf{z}\|^2_{H^2(\Omega)}$$

because of $\|\nabla\tilde{\mathbf{e}}\| \leq C$ due to $\mathbf{z}$, $\mathbf{e} \in L^\infty(0,+\infty;\mathbf{H}^1(\Omega))$ and because of $\mathbf{e}-\mathbf{z} = \mathbf{d} \in L^\infty(Q)$ thanks to (9).

Adding (17), (18) and (19), and choosing adequate small parameters ε, ε_1 (and their correspondent constants C, $C_{\varepsilon,\varepsilon_1}$), we obtain:

$$\begin{aligned} h'(t) + H(t) \quad&\leq\quad C_1\,h(t)^3 + C\left(\|\nabla\mathbf{v}(t)\|^4 + \|\nabla\mathbf{e}(t)\|^4_{H^1(\Omega)} + \|\widetilde{\mathbf{e}}(t)\|^2_{H^2(\Omega)} + 1\right)h(t) \\ &+\quad C\,\|P(\mathbf{g}-\mathbf{k})(t)\|^2 \end{aligned} \tag{20}$$

for

$$\begin{cases} h(t) &=\quad \|\nabla\mathbf{w}(t)\|^2 + \lambda\|\mathbf{z}(t)\|^2_{H^1(\Omega)} + \|\Delta\mathbf{z}(t)\|^2 \\ H(t) &=\quad \nu\|\nabla\mathbf{w}(t)\|^2 + \|A\mathbf{w}(t)\|^2 + \gamma\lambda\left(\|\mathbf{z}(t)\|^2_{H^1(\Omega)} + \|\Delta\mathbf{z}(t)\|^2\right) + \gamma\,\|\nabla(\Delta\mathbf{z}(t))\|^2 \end{cases}$$

where C (henceforward) is a generic positive constant depending on initial data and smoothness of set Γ. Observe that $\|\widetilde{\mathbf{e}}(t)\|^2_{H^2(\Omega)} \in L^1(0,+\infty)$ thanks to the global in time weak regularity (8).

Due to the structure of $h(t)$ and $H(t)$, it is evident that (for a constant C):

$$C\,h(t) \leq H(t)$$

Therefore, if $\zeta := C\min\{\gamma,\,\nu\} > 0$ and $\varphi(t) := \|\nabla\mathbf{v}(t)\|^4 + \|\nabla\mathbf{e}(t)\|^4_{H^1(\Omega)} + \|\widetilde{\mathbf{e}}(t)\|^2_{H^2(\Omega)}$ then (20) becomes

$$h'(t) + \zeta h(t) \leq C\varphi(t)h(t) + Ch(t) + Ch^3(t) + C\|P(\mathbf{g}-\mathbf{k})\|^2. \tag{21}$$

Note that $h \geq h^3$ if and only if $h \leq 1$. In this case we would have directly that h would be uniformly bounded on whole interval $[0,\infty)$. Thus, we must have to work the case $h^3 \geq h$ in (21), and thus

$$h'(t) + \zeta h(t) \leq C\varphi(t)h(t) + Ch^3(t) + C\|P(\mathbf{g}-\mathbf{k})(t)\|^2. \tag{22}$$

In this point, we employ some similar arguments as in Ponce at al. (cf. [11]).

Due to (13), $h(0) < \delta$. We claim that $h(t) \le \left(\frac{\zeta}{C}\right)^{1/2}$ for any $t \in [0, \infty)$. Indeed, we argue by contradiction: Assume that:

$$\text{there exists a } t^* \text{ such that } h(t^*) = \left(\frac{\zeta}{2C}\right)^{1/2} \tag{23}$$

and $h(s) \le \left(\frac{\zeta}{2C}\right)^{1/2}$ for $0 \le s \le t^*$. Therefore, for any $t \in [0, t^*]$

$$h'(t) + \frac{\zeta}{2}h(t) \le C\varphi(t)h(t) + C\|P(\mathbf{g} - \mathbf{h})(t)\|^2.$$

We denote $\Phi(t) = C \displaystyle\int_0^t \varphi(s)\, ds$ and it is easy to obtain:

$$h(t) \le e^{\Phi(t)} \left(h(0) + C\, e^{-\frac{\zeta t}{2}} \int_0^t e^{\frac{\zeta s}{2}} \|P(\mathbf{g} - \mathbf{k})(s)\|^2\, ds\right) \tag{24}$$

Using (13), if we define δ as follows:

$$\delta = \frac{1}{2\, \max\{1, e^{\Phi(+\infty)}\}} \left(\frac{\zeta}{2C}\right)^{1/2}, \tag{25}$$

then we obtain that:

$$h(t) < \frac{1}{2}\left(\frac{\zeta}{2C}\right)^{1/2} \quad \text{for any } t \in [0, t^*]. \tag{26}$$

In particular, $h(t^*) < \frac{1}{2}\left(\frac{\zeta}{2C}\right)^{1/2}$, which contradicts (23). Observe that the argument is still true for any $t^* \in [0, \infty]$.

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Haroldo Rodrigues Clark
Universidade Federal do Piauí, Parnaiba, Brazil
haroldoclark1@gmail.com

Marko A. Rojas-Medar
Instituto de Alta Investigación
Universidad de Tarapacá, Arica, Chile
marko.medar@gmail.com

María Ángeles Rodríguez-Bellido
Dpto. Ecuaciones Diferenciales y Análisis Numérico
Universidad de Sevilla, Sevilla, Spain
angeles@us.es