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ON A STOCHASTIC $p(\omega, t, x)$-LAPLACE EQUATION

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Abstract. A stochastic forcing of a non-linear singular/degenerated parabolic problem of $p(\omega, t, x)$-Laplace type is proposed in the framework of Orlicz Lebesgue and Sobolev spaces with variable random exponents. We give a result of existence and uniqueness of the solution, for additive and multiplicative problems.

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§1. Introduction

We are interested in a result of existence, uniqueness and stability of solutions to:

$$(P,h)\begin{cases} du - \Delta_{p(\cdot)}u\ dt = h(\cdot,u)\ dw & \text{in } \Omega\times(0,T)\times D,\\ u=0 & \text{on } \Omega\times(0,T)\times\partial D,\\ u(0,\cdot)=u_0 & \text{in } L^2(D).\end{cases}$$

where $T > 0$, $D \subset \mathbb{R}^d$ is a bounded Lipschitz domain, $Q := (0,T)\times D$, $w = \{w_t, \mathcal{F}_t; 0 \leq t \leq T\}$ is a Wiener process on the classical Wiener space $(\Omega, \mathcal{F}, P)$; $h = h(\omega, t, x, \lambda)$ is a Carathéodory function on $\Omega\times Q\times\mathbb{R}$, uniformly Lipschitz continuous with respect to λ, $\Delta_{p(\cdot)}u = \operatorname{div}(|\nabla u|^{p(\omega,t,x)-2}\nabla u)$ with a variable exponent $p : \Omega\times Q \to (1,\infty)$ satisfying the following conditions:

($p1$) $1 < p^- := \operatorname{ess\,inf}_{(\omega,t,x)} p(\omega,t,x) \leq p^+ := \operatorname{ess\,sup}_{(\omega,t,x)} p(\omega,t,x) < \infty$,

($p2$) ω a.s. in Ω, $(t,x)\mapsto p(\omega,t,x)$, is log-Hölder continuous, *i.e.* there exists $C \geq 0$ (which might depend on ω) such that, for all (t,x), $(s,y)\in Q$,

$$|p(\omega,t,x) - p(\omega,s,y)| \leq \frac{C}{\ln(e + \frac{1}{|(t,x)-(s,y)|})} \tag{1}$$

($p3$) progressive measurability of the variable exponent, *i.e.*

$$\Omega\times[0,t]\times D \ni (\omega,s,x)\mapsto p(\omega,s,x)$$

is $\mathcal{F}_t\times\mathcal{B}(0,t)\times\mathcal{B}(D)$-measurable for all $0\leq t\leq T$.

($p4$) h is a Carathéodory function in the sense that:
for any $\lambda\in\mathbb{R}$, $h(\cdot,\lambda)\in N^2_W(0,T,L^2(D))$, the space of predictable processes with values in $L^2(D)$ (see G. Da Prato *et al.* [3] for example),
and, $P\otimes\mathcal{L}^{d+1}$-a.e., $\lambda\in\mathbb{R}\to h(\omega,t,x,\lambda)\in\mathbb{R}$ is continuous. Moreover, h is a Lipschitz-continuous function of the variable λ, uniformly with respect to the other variables.

Problems with variable exponent (*i.e.* when the exponent p depends on the time-space arguments) have been intensively studied since the years 2000. For the basic definitions and properties of variable exponent Lebesgue and Sobolev spaces we refer to [4]. The main physical motivation was induced by the modelization of electrorheological fluids. For example one can study the case of coupled problems, where the exponent $p = p(v(t, x))$ depends on a solution v of a coupled PDE (see e.g. [1] and the references therein). Since reality is complex, it can be interesting to consider stochastic perturbations acting on both equations, *i.e.*

$$du + A(u, v)\, dt = f\, dw, \quad dv + B(v)\, dt = g\, dw.$$

This motivates our interest to study the toy problem (P, h) with variable exponent p depending on ω, t and x with suitable measurability assumptions with respect to a given filtration. The predictability and the pathwise Hölder continuity of the solution v are formally compatible with the technical assumptions we have to impose on the variable exponent p, since, for technical reasons, we need to consider log-Hölder continuous exponents with respect to (t, x).

§2. Function spaces

Let us define

$$N^2_W(0, T; L^2(D)) := L^2(\Omega \times (0, T); L^2(D))$$

endowed with $dt \otimes dP$ and the predictable σ-algebra $\mathcal{P}_T$ generated by the products $]s, t] \times A$, $0 \leq s < t \leq T, A \in \mathcal{F}_s$, which is the space of predictable and therefore Itô integrable stochastic processes. Let $S^2_W(0, T; H^k_0(D))$ be the subset of simple, predictible processes with values in $H^k_0(D)$ for sufficiently large values of k. Note that $S^2_W(0, T; H^k_0(D))$ is densely imbedded into $N^2_W(0, T; L^2(D))$. The following function space serves as the variable exponent version of the classical Bochner space setting: there exists a full-measure set $\tilde{\Omega} \subset \Omega$ such that we can define

$$X_\omega(Q) := \{u \in L^2(Q) \cap L^1(0, T; W^{1,1}_0(D)) \mid \nabla u \in (L^{p(\omega,\cdot)}(Q))^d\}$$

which is a reflexive Banach space for all $\omega \in \tilde{\Omega}$ with respect to the norm

$$\|u\|_{X_\omega(Q)} = \|u\|_{L^2(Q)} + \|\nabla u\|_{L^{p(\omega,\cdot)}(Q)}.$$

$X_\omega(Q)$ is a parametrization by ω of the space

$$X(Q) := \{u \in L^2(Q) \cap L^1(0, T; W^{1,1}_0(D)) \mid \nabla u \in (L^{p(t,x)}(Q))^d\}$$

which has been introduced in [5] for the case of a variable exponent depending on (t, x). For the basic properties of $X(Q)$, we refer to [5]. For $u \in X_\omega(Q)$, it follows directly from the definition that $u(t) \in L^2(D) \cap W^{1,1}_0(D)$ for almost every $t \in (0, T)$. Moreover, from $\nabla u \in L^{p(\omega,\cdot)}(Q)$ and Fubini's theorem it follows that $\nabla u(t, \cdot)$ is in $L^{p(\omega,t,\cdot)}(D)$ a.e. in $(0, T)$.

Let us introduce the space

$$\mathcal{E} := \{u \in L^2(\Omega \times Q) \cap L^{p^-}(\Omega \times (0, T); W^{1,p^-}_0(D)) \mid \nabla u \in L^{p(\cdot)}(\Omega \times Q)\}$$

which is a reflexive Banach space with respect to the norm

$$u \in \mathcal{E} \mapsto \|u\|_{\mathcal{E}} = \|u\|_{L^2(\Omega\times Q)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega\times Q)}.$$

Thanks to Fubini's theorem, $u \in \mathcal{E}$ implies that $u(\omega) \in X_\omega(Q)$ a.s. in Ω and, since Poincaré's inequality is available with respect to (t,x), independently of ω, $u \in \mathcal{E}$ implies also $u(\omega,t) \in L^2(D) \cap W_0^{1,p(\omega,t,\cdot)}(D)$ for almost all $(\omega,t) \in \Omega \times (0,T)$.

§3. Main result

Definition 1. A solution to (P,h) is a function $u \in L^2(\Omega; C([0,T]; L^2(D))) \cap N^2_W(0,T; L^2(D)) \cap \mathcal{E}$, such that, for almost every $\omega \in \Omega$, $u(0,\cdot) = u_0$, a.e. in D and for all $t \in [0,T]$,

$$u(t) - u_0 - \int_0^t \Delta_{p(\cdot)} u \, ds = \int_0^t h(\cdot,u)\, dw,$$

holds a.s. in D; or, equivalently, in the weak-sense:

$$\partial_t[u(t) - \int_0^t h(\cdot,u)\, dw] - \Delta_{p(\cdot)} u = 0 \text{ in } X'_\omega(Q).$$

Theorem 1. *There exists a unique solution to (P,h). Moreover, if u_1, u_2 are the solutions to (P,h_1), (P,h_2) respectively, then:*

$$E\left[\sup_t \|(u_1-u_2)(t)\|^2_{L^2(D)} + \int_Q \left(|\nabla u_1|^{p(\cdot)-2}\nabla u_1 - |\nabla u_2|^{p(\cdot)-2}\nabla u_2\right)\cdot \nabla(u_1-u_2)\, d(t,x)\right]$$
$$\leq CE \int_Q |h_1(\cdot,u_1) - h_2(\cdot,u_2)|^2 \, d(t,x). \tag{2}$$

§4. Proof of the main result

Our aim is to prove first a result of well-posedness of (P,h) in the additive case, *i.e.* when $h \in N^2_W(0,T;L^2(D))$ is not a function of u:

Proposition 2. *For any $h \in N^2_W(0,T;L^2(D))$, there exists a unique solution to (P,h). Moreover, if u_1, u_2 are the solutions to (P,h_1), (P,h_2) respectively, then:*

$$E\left(\sup_t \|(u_1-u_2)(t)\|^2_{L^2(D)} + \int_Q (|\nabla u_1|^{p(\cdot)-2}\nabla u_1 - |\nabla u_2|^{p(\cdot)-2}\nabla u_2)\cdot \nabla(u_1-u_2)\, d(t,x)\right)$$
$$\leq CE \int_Q |h_1 - h_2|^2 \, d(t,x). \tag{3}$$

Then, with the above Lipschitz principle, one will get the result in the multiplicative case, *i.e.* when h can be a function of u.

4.1. The additive case for $h \in S^2_W(0,T;H^k_0(D))$

Proposition 3. *For* $q \geq \max(2,p^+)$, $0<\varepsilon\leq 1$ *and any* $h \in N^2_W(0,T;L^2(D))$ *there exists*

$$u^\varepsilon \in L^2(\Omega, C([0,T];L^2(D))) \cap N^2_W(0,T;L^2(D)) \cap L^q(\Omega\times(0,T);W^{1,q}_0(D))$$

and a set $\tilde\Omega \subset \Omega$ *of total probability* 1 *on which* $u(0,\cdot)=u_0$ *a.e. in* D *and*

$$u^\varepsilon(t) - u_0 - \int_0^t [\varepsilon\Delta_q u^\varepsilon + \Delta_{p(\cdot)} u^\varepsilon]\, ds = \int_0^t h\, dw. \tag{4}$$

in $W^{-1,q'}(D)$ *for all* $t\in[0,T]$.

Proof: For $q \geq \max(2,p^+)$ and $\varepsilon>0$, the operator

$$A:\Omega\times(0,T)\times W^{1,q}_0(D)\to W^{-1,q'}(D),\quad A(\omega,t,u) = -\varepsilon\Delta_q u - \Delta_{p(\omega,t,x)}u,$$

is monotone with respect to u for a.e. $(\omega,t)\in\Omega\times(0,T)$ and progressively measurable, *i.e.* for every $t\in[0,T]$ the mapping

$$A:\Omega\times(0,t)\times W^{1,q}_0(D)\to W^{-1,q'}(D),\quad (\omega,s,u)\mapsto A(\omega,s,u)$$

is $\mathcal{F}_t\times\mathcal{B}(0,t)\times\mathcal{B}(W^{1,q}_0(D))$-measurable. In particular, $-A$ satisfies the hypotheses of [7, Theorem 2.1, p. 1253], therefore for any $\varepsilon>0$ there exists a continuous process with values in $L^2(D)$ solution to the problem (4). Then, [3, Prop.3.17 p.84] and [7, Theorem 2.3, p. 1254] yield $u^\varepsilon \in L^2(\Omega, C([0,T];L^2(D)))$.

Proposition 4. *For any simple process* $\bar h \in S^2_W(0,T;H^k_0(D))$, *there exist a unique* $u\in\mathcal{E}\cap L^2(\Omega,C([0,T];L^2(D)))$ *and a full-measure set* $\tilde\Omega\in\mathcal{F}$ *such that for all* $\omega\in\tilde\Omega$ *we have* $u(0,\cdot)=u_0$ *a.e. in* D *and*

$$u(t)-u_0-\int_0^t \Delta_{p(\cdot)}u\, ds = \int_0^t \bar h\, dw \tag{5}$$

holds a.e. in D *for all* $t\in[0,T]$. *In particular* u *is a solution to* $(P,\bar h)$ *in the sense of Definition 1.*

Proof: For the first part of the proof, mainly based on deterministic arguments, we can repeat the arguments of [2]: If we set $v^\varepsilon := u^\varepsilon - \int_0^t h\, dw$, such that $v^\varepsilon(0)=u_0$, then u^ε satisfies (4), iff there exists a full-measure set $\tilde\Omega\in\mathcal{F}$ such that

$$\partial_t v^\varepsilon - \varepsilon\Delta_q(v^\varepsilon + \int_0^t \bar h\, dw) - \Delta_{p(\cdot)}(v^\varepsilon + \int_0^t \bar h\, dw) = 0 \tag{6}$$

in $L^{q'}(0,T;W^{-1,q'}(D))$ for all $\omega\in\tilde\Omega$. Testing (6) with v^ε to get *a priori* estimates, we can use classical (monotonicity) arguments to conclude that pointwise for every $\omega\in\tilde\Omega$ we have the following convergence results, passing to a (not relabeled) subsequence if necessary, :

1.) $v^\varepsilon \rightharpoonup v$ in $X_\omega(Q)$ and $L^\infty(0,T;L^2(D))$ weak-*,

2.) for any t, $v^\varepsilon(t)\to v(t)$ in $L^2(D)$, 3.) $\int_Q |\nabla v^\varepsilon - \nabla v|^{p(\omega,t,x)}\, dxdt \to 0$.

Then, passing to the limit in the singular perturbation, v satisfies the problem

$$\partial_t v - \Delta_{p(\cdot)}(v + \int_0^t \bar{h}\, dw) = 0.$$

In particular, $\partial_t v \in X'_\omega(Q)$ (see [5]) and $v \in W_\omega(Q)$ where one denotes by

$$W_\omega(Q) := \{v \in X_\omega(Q) \mid \partial_t v \in X'_\omega(Q)\}.$$

Thanks to [5], $W_\omega(Q) \hookrightarrow C([0,T], L^2(D))$ with a continuity constant depending only on T and the time-integration by parts formula is available. Thus, $v \in C([0,T]; L^2(D))$ and v is a solution of the above problem in $W_\omega(Q)$, for the initial condition u_0. Since this solution is unique, no subsequence is needed in the above limits. Then, denoting by $u = v + \int_0^\cdot \bar{h}\, dw$, the above convergence yields, for all $\omega \in \tilde{\Omega}$:

1.) $u^\varepsilon \to u$ in $L^2(0,T; L^2(D))$ with $\partial_t[u - \int_0^\cdot \bar{h} dw] \in X'_\omega(Q)$,

2.) for any t, $u^\varepsilon(t) \to u(t)$ in $L^2(D)$,

3.) $\Delta_{p(\omega,t,x)} u^\varepsilon \rightharpoonup \Delta_{p(\omega,t,x)} u$ in $X'_\omega(Q)$,

4.) $\int_Q |\nabla u^\varepsilon - \nabla u|^{p(\omega,t,x)}\, dxdt \to 0$.

We continue with the argumentation as in [2]: from the previous convergence results, the *a priori* estimates and since $\nabla \bar{h}$ is bounded, we get uniform estimates that allow us to use Lebesgue Dominated Convergence theorem and therefore it follows that

$$\forall t,\ u^\varepsilon(t) \to u(t) \text{ in } L^2(\Omega, L^2(D)) \quad \text{and} \quad u^\varepsilon \to u \text{ in } \mathcal{E}. \tag{7}$$

Note that the above limits in $L^2(\Omega, L^2(D))$ and $L^2(\Omega, L^2(Q))$ are results in standard Bochner spaces, but the measurability of ∇u with respect to $d(t,x) \otimes dP$ deserves our attention. Since ∇u^ε and $\nabla u^{\epsilon'}$ are globally measurable functions, Lebesgue Dominated Convergence theorem, together with *a priori* estimates yield

$$E \int_Q |\nabla u^\varepsilon - \nabla u^{\varepsilon'}|^{p(\omega,t,x)}\, dxdt \to 0$$

and thus, (∇u^ε) is a Cauchy sequence in $L^{p(\cdot)}(\Omega \times Q)$ and therefore a converging sequence. It is then a direct consequence to see that ∇u is the limit in $L^{p(\cdot)}(\Omega \times Q)$ of ∇u^ε.

Then, passing to a (not relabeled) subsequence if needed, it follows that $u^\varepsilon \to u$ a.e. in $\Omega \times Q$. Hence u satisfies (5), or, in other words, $\partial_t[u - \int_0^t \bar{h} dw] - \Delta_{p(\cdot)} u = 0$.

In particular, since $\bar{h}$ is regular, one gets that $u - \int_0^t \bar{h} dw \in \mathcal{E}$ with $\partial_t[u - \int_0^t \bar{h} dw] \in \mathcal{E}'$.

We need now to prove that $u \in L^2(\Omega, C([0,T], L^2(D)))$. We already know that $u : \Omega \times Q \to L^2(D)$ is a stochastic process. Since $u(\omega, \cdot) \in W_\omega(Q) \hookrightarrow C([0,T], L^2(D))$ for a.e. $\omega \in \Omega$, the measurability follows from [3, Prop.3.17 p.84] with arguments as in [6, Cor. 1.1.2, p.8]. Then, a.s. in Ω, the equation satisfied by u yields $\partial_t v - \Delta_{p(\cdot)} u = 0$, so that, for almoste every $t \in [0,T]$,

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{L^2(D)} + \int_D |\nabla u|^{p(\omega,t,x)-2} \nabla u \cdot \nabla v\, dx = 0.$$

Since, ω a.s.,

$$\sup_{t\in[0,T]} \|v(\omega,t,\cdot)\|^2_{L^2(D)} \leq \|u_0\|^2_{L^2(D)} + 2\int_0^T \int_D \frac{1}{p^-}|\nabla u|^{p(\omega,s,x)} + \frac{1}{(p')^-}\left|\int_0^s \nabla \bar{h}\, dw\right|^{p'(\omega,s,x)} dx\, ds$$

with a right side in $L^1(\Omega)$, one gets that $u, v \in L^2(\Omega; C([0,T], L^2(D))$.

Lemma 5. *Proposition 2 holds for any $h \in S^2_W(0,T; H^k_0(D))$. More precisely, for $h_n, h_m \in S^2_W(0,T; H^k_0(D))$ let u_n be the solution to (P, h_n) and u_m be the solution to (P, h_m). There exist constants K_1, $K_2 \geq 0$ such that for any $m, n \in \mathbb{N}$,*

$$E\left(\|u_n\|^2_{C([0,T];L^2(D))}\right) + E\int_Q |\nabla u_n|^{p(\cdot)} d(t,x) \leq K_1(\|h_n\|^2_{L^2(\Omega\times Q)} + \|u_0\|^2_{L^2(D)}), \tag{8}$$

$$E\left(\|(u_n - u_m)\|^2_{C([0,T];L^2(D))}\right) + E\int_Q (|\nabla u_n|^{p(\cdot)-2}\nabla u_n - |\nabla u_m|^{p(\cdot)-2}\nabla u_m)\cdot\nabla(u_n - u_m)\, d(t,x) \leq K_2\|h_n - h_m\|^2_{L^2(\Omega\times Q)}. \tag{9}$$

Proof: Using the Itô formula in (4) it follows that for all $t \in [0,T]$ a.s. in Ω we have

$$\|u^\varepsilon_n(t)\|^2_{L^2(D)} + 2\int_0^t \int_D |\nabla u^\varepsilon_n|^{p(\cdot)}\, dx\, ds \leq 2\int_0^t \int_D h_n u^\varepsilon_n\, dx\, dw + \int_0^t \int_D h_n^2\, dx\, ds + \|u_0\|^2_{L^2(D)},$$

or, by subtracting (4) with h_m from (4) with h_n,

$$\|[u^\varepsilon_n - u^\varepsilon_m](t)\|^2_{L^2(D)} + 2\int_0^t \int_D (|\nabla u^\varepsilon_n|^{p(\cdot)-2}\nabla u^\varepsilon_n - |\nabla u^\varepsilon_m|^{p(\cdot)-2}\nabla u^\varepsilon_m)\cdot\nabla(u^\varepsilon_n - u^\varepsilon_m)\, dx\, ds \leq 2\int_0^t \int_D [h_n - h_m](u^\varepsilon_n - u^\varepsilon_m)\, dx\, dw + \int_0^t \int_D (h_n - h_m)^2\, dx\, ds.$$

Thus, by passing to the limit with $\varepsilon \to 0$, to the supremum over t and then taking the expectation, it follows that ($c \geq 0$ being a constant)

$$E(\sup_{t\in[0,T]} \|u_n(t)\|^2_{L^2(D)}) + E\int_0^T \int_D |\nabla u_n|^{p(\cdot)}\, dx\, ds \leq cE\left(\sup_{t\in[0,T]} \int_0^t \int_D h_n u_n\, dx\, dw\right) + c\|h_n\|^2_{L^2(\Omega\times Q)} + c\|u_0\|^2_{L^2(D)}, \tag{10}$$

$$E(\sup_{t\in[0,T]} \|[u_n - u_m](t)\|^2_{L^2(D)}) + E\int_0^T \int_D (|\nabla u_n|^{p(\cdot)-2}\nabla u_n - |\nabla u_m|^{p(\cdot)-2}\nabla u_m)\cdot\nabla(u_n - u_m)\, dx\, ds \leq cE\left(\sup_{t\in[0,T]} \int_0^t \int_D [h_n - h_m](u_n - u_m)\, dx\, dw\right) + c\|h_n - h_m\|^2_{L^2(\Omega\times Q)}. \tag{11}$$

Using Burkholder, Hölder and Young inequalities on (10) we get for any $\gamma > 0$

$$E\left(\sup_{t\in[0,T]}\int_0^t\int_D h_n u_n\,dx\,dw\right) \le 3E\left(\int_0^T\left(\int_D h_n u_n\,dx\right)^2 ds\right)^{1/2} \tag{12}$$
$$\le 3E\left(\int_0^T \|h_n\|^2_{L^2(D)}\|u_n\|^2_{L^2(D)}\,dt\right)^{1/2}$$
$$\le 3E\left[\left(\sup_{t\in[0,T]}\|u_n\|^2_{L^2(D)}\right)^{1/2}\left(\int_0^T\|h_n\|^2_{L^2(D)}\right)^{1/2}\right]$$
$$\le 3\gamma E\left(\sup_{t\in[0,T]}\|u_n\|^2_{L^2(D)}\right)+\frac{3}{\gamma}\|h_n\|^2_{L^2(\Omega\times Q)},$$

and similarly on (11),

$$E\left(\sup_{t\in[0,T]}\int_0^t\int_D (h_n-h_m)(u_n-u_m)\,dx\,dw\right) \tag{13}$$
$$\le 3\gamma E\left(\sup_{t\in[0,T]}\|u_n-u_m\|^2_{L^2(D)}\right)+\frac{3}{\gamma}\|h_n-h_m\|^2_{L^2(\Omega\times Q)}.$$

Plugging (12) into (10), (13) into (11) and choosing $\gamma > 0$ small enough yield Lemma 5.

Remark 1. It is an open question if the Itô formula is directly available for a solution of (5) since we are not in Bochner spaces: the stochastic energy has to be defined in different Banach spaces depending on $t \in [0,T]$ and $\omega \in \Omega$. That is why we need to apply the Itô formula to u^ε, and then pass to the limit. But then, only an inequality is obtained.

4.2. Existence for arbitrary $h \in N^2_W(0,T;L^2(D))$

Proposition 6. *For any $h \in N^2_W(0,T;L^2(D))$, there exists a unique $u \in \mathcal{E}\cap L^2(\Omega;C([0,T];L^2(D))\cap N^2_W(0,T;L^2(D))$ such that a.s.*

$$u(t)-u_0-\int_0^t \Delta_{p(\cdot)}u\,ds=\int_0^t h\,dw \tag{14}$$

for all $t \in [0,T]$, a.e. in D.

Proof: For any $h \in N^2_W(0,T;L^2(D))$, there exists a sequence $(h_n)\subset S^2_W(0,T;H^k_0(D))$ converging to h in $N^2_W(0,T;L^2(D))$. Let $(u_n)\in\mathcal{E}\cap L^2(\Omega,C([0,T];L^2(D)))$ be the sequence of corresponding solutions to (P,h_n). From (8) it follows that (u_n) is a bounded sequence in $\mathcal{E}\cap L^2(\Omega,C([0,T];L^2(D)))$ and (9) ensures that (u_n) is a Cauchy sequence in $L^2(\Omega;C([0,T];L^2(D))$. Hence there exists $u\in\mathcal{E}\cap L^2(\Omega;C([0,T];L^2(D)))$ such that $u_n \rightharpoonup u$ in $\mathcal{E}$ and $u_n\to u$ in $L^2(\Omega;C([0,T];L^2(D)))$.

Moreover there exists a full-measure set $\tilde\Omega\in\mathcal{F}$ such that, passing to a (not relabeled) subsequence if necessary, $u_n\to u$ in $C([0,T];L^2(D))$ for all $\omega\in\tilde\Omega$. In particular, $u(0,\cdot)=u_0$ a.e. in D for all $\omega\in\tilde\Omega$.

For $\mu = d(t,x)\otimes dP$ we have

$$\int_{\Omega\times Q}|\nabla u_n-\nabla u_m|^{p(\cdot)}d\mu=\int_{1<p<2}|\nabla u_n-\nabla u_m|^{p(\cdot)}d\mu+\int_{p\ge 2}|\nabla u_n-\nabla u_m|^{p(\cdot)}d\mu$$

Then, from (9) and the fundamental inequality ([8, Section 10]), for any $\xi,\eta\in\mathbb{R}^d$:

$$(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta)\cdot(\xi-\eta)\geq\begin{cases}2^{2-p}|\xi-\eta|^p, \ p\geq 2\\(p-1)|\xi-\eta|^2(1+|\eta|^2+|\xi|^2)^{\frac{p-2}{2}}, \ 1\leq p<2\end{cases}.$$

It follows first that

$$\int_{p\geq 2}|\nabla u_n-\nabla u_m|^{p(\cdot)}d\mu\leq 2^{p^+-2}K_2\|h_n-h_m\|^2_{L^2(\Omega\times Q)}, \tag{15}$$

then, from the generalized Young inequality it follows for any $0<\epsilon<1$,

$$\begin{aligned}&\int_{1<p<2}|\nabla u_n-\nabla u_m|^{p(\cdot)}d\mu\\&=\int_{1<p<2}\frac{|\nabla u_n-\nabla u_m|^{p(\cdot)}}{(1+|\nabla u_n|^2+|\nabla u_m|^2)^{p(\cdot)\frac{2-p(\cdot)}{4}}}(1+|\nabla u_n|^2+|\nabla u_m|^2)^{p(\cdot)\frac{2-p(\cdot)}{4}}d\mu\\&\leq\int_{1<p<2}\epsilon^{\frac{p(\cdot)-2}{p(\cdot)}}\frac{|\nabla u_n-\nabla u_m|^2}{(1+|\nabla u_n|^2+|\nabla u_m|^2)^{\frac{2-p(\cdot)}{2}}}d\mu+\epsilon\int_{1<p<2}(1+|\nabla u_n|^2+|\nabla u_m|^2)^{\frac{p(\cdot)}{2}}d\mu\\&\leq\frac{1}{\epsilon(p^--1)}\int_{1<p<2}(p-1)\frac{|\nabla u_n-\nabla u_m|^2}{(1+|\nabla u_n|^2+|\nabla u_m|^2)^{\frac{2-p(\cdot)}{2}}}d\mu+K_3\epsilon\\&\leq\frac{1}{\epsilon(p^--1)}K_2\|h_n-h_m\|^2_{L^2(\Omega\times Q)}+K_3\epsilon,\end{aligned}\tag{16}$$

since the sequence (u_n) is bounded in $L^{p(\cdot)}(\Omega\times Q)$ and μ is a finite measure.

From (15), (16) and $\lim_{n,m}\|h_n-h_m\|^2_{L^2(\Omega\times Q)}=0$ it follows that ∇u_n is a Cauchy sequence in $L^{p(\cdot)}(\Omega\times Q)$, thus a converging sequence.
In conclusion, u_n converges to u in $\mathcal{E}\cap L^2(\Omega;C([0,T];L^2(D)))\cap N^2_W(0,T;L^2(D))$ and, by a standard argument based on the Nemytskii operator induced by the Carathéodory function $G:(\omega,t,x,\xi)\in\Omega\times Q\times\mathbb{R}^d\mapsto|\xi|^{p(\omega,t,x)-2}\xi\in\mathbb{R}^d$, $|\nabla u_n|^{p(\cdot)-2}\nabla u_n$ converges to $|\nabla u|^{p(\cdot)-2}\nabla u$ in $L^{p'(\cdot)}(\Omega\times Q)$ since $|G(\omega,t,x,\xi)|^{p'(\omega,t,x)}=|\xi|^{p(\omega,t,x)}$.

Let us recall that, for any $n\in\mathbb{N}$, u_n satisfies

$$\partial_t\left(u_n-\int_0^t h_n\,dw\right)-\Delta_{p(\cdot)}u_n=0 \tag{17}$$

in $\mathcal{E}'$. Now we can choose a (not relabeled) subsequence of (u_n) such that all previous convergence results hold true. For any test function $\phi(\omega,t,x)=\rho(\omega)\gamma(t)\nu(x)$ with $\rho\in L^\infty(\Omega)$, $\gamma\in\mathcal{D}([0,T))$ and $\nu\in\mathcal{D}(D)$ we have

$$\begin{aligned}\left\langle\partial_t\left(u_n-\int_0^t h_n\,dw\right),\phi\right\rangle_{\mathcal{E}',\mathcal{E}}&=\int_\Omega\left\langle\partial_t\left(u_n-\int_0^t h_n\,dw\right),\phi\right\rangle_{X'_\omega,X_\omega}dP\\&=-\int_\Omega\left\langle\left(u_n-\int_0^t h_n\,dw\right),\partial_t\phi\right\rangle_{X'_\omega,X_\omega}dP-\int_{\Omega\times D}u_0\varphi(\omega,0,x)\,dx\,dP.\end{aligned}\tag{18}$$

In particular u_n satisfies

$$-\int_{\Omega\times Q}\left(u_n-\int_0^t h_n\,dw\right)\cdot\partial_t\phi+|\nabla u_n|^{p(\cdot)-2}\nabla u_n\cdot\nabla\phi\,d\mu-\int_{\Omega\times D}u_0\varphi(\omega,0,x)\,dx\,dP=0 \quad (19)$$

for all $n\in\mathbb{N}$. Therefore, using our convergence results, we are able to pass to the limit in (19) and obtain

$$\partial_t\left(u-\int_0^t h\,dw\right)-\Delta_{p(\cdot)}u= \qquad 0 \quad (20)$$

in $\mathcal{E}'$. (20), and a classical argument of separability, imply that a.s.

$$\partial_t\left(u-\int_0^t h\,dw\right)=\Delta_{p(\cdot)}u,\ \text{in } X'_\omega(Q)\hookrightarrow L^{\alpha'}(0,T;W^{-1,\alpha'}(D)) \quad (21)$$

with $\alpha\geq p^++2$. Moreover, a.s.

$$u-\int_0^t h\,dw\in C([0,T];L^2(D)).$$

Thus we can integrate (21) to obtain a.s.

$$u(t)-u_0-\int_0^t\Delta_{p(\cdot)}u\,ds=\int_0^t h\,dw \quad (22)$$

in $L^2(D)$ for all $t\in[0,T]$.

If we assume that $u_1,u_2\in\mathcal{E}\cap L^2(\Omega,C([0,T];L^2(D)))\cap N_W^2(0,T;L^2(D))$ are both satisfying (14), it follows that a.s. in Ω

$$\partial_t(u_1-u_2)-(\Delta_{p(\cdot)}u_1-\Delta_{p(\cdot)}u_2)=0\text{ in }(X_\omega(Q))'. \quad (23)$$

Using u_1-u_2 as a test function in (23), and integration by parts in $W_\omega(Q)$ we obtain uniqueness.

4.3. Conclusion

Set $h_1,h_2\in N_W^2(0,T;L^2(D))$ and let u_1,u_2 be solutions to (P,h_1) and (P,h_2). Since

$$E\left(\|(u_1-u_2)\|^2_{C([0,T];L^2(D))}+\int_Q(|\nabla u_1|^{p(\cdot)-2}\nabla u_1-|\nabla u_2|^{p(\cdot)-2}\nabla u_2)\cdot\nabla(u_1-u_2)\,d(t,x)\right)$$
$$\leq C\|h_1-h_2\|^2_{L^2(\Omega\times Q)}, \quad (24)$$

we can repeat the arguments of [2] based on Banach's fixed point theorem applied to

$$\Psi: S\in N_W^2(0,T;L^2(D))\to u_S\in N_W^2(0,T;L^2(D))$$

where u_S is the solution to $(P,h(\cdot,S))$ to deduce the existence of a unique solution u of (P,h) in the sense of Definition 1. From (24) it also follows that (2) holds true and we have finished the proof of Theorem 3.1.

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