

# PROGRESSIVE ORTHOGONAL WAVELETS: A REVIEW

Florin Avram, Anca Deliu and Baiqiao Deng

**Abstract.** The concept of progressive wavelets, or wavelets supported on a half-line, has been introduced a long time ago by Björn Jawerth, for the purpose of analyzing moving images. Subsequently, it has fallen somewhat in between the fields of wavelet analysis and signal processing. Our paper is an attempt to unify these two strands of literature and review the current state of knowledge on this topic, with accent on splines based examples.

*Keywords:* progressive wavelets, splines, refinement equation, moving average.

*AMS classification:* 42C15, 42C40.

## §1. Introduction

**Progressive wavelets.** In this paper we review and reexamine some results of [9, 3, 4] on a class of wavelets proposed by Björn Jawerth.

**Definition 1.** *Progressive orthogonal wavelets* are real valued functions  $\phi(x) \in L_2(\mathbb{R}_+)$ , with  $\phi(x) = 0$  for  $x < 0$  satisfying:

1. The  $L_2$  orthogonality of integer translates  $\phi(x - k)$ ,  $k \in \mathbb{Z}$ .
2. A functional equation:

$$\sigma(x) = \phi(x) - \sum_{k=1}^{N-1} \alpha_k \phi(x - k) = \left( I - \sum_{k=1}^{N-1} \alpha_k \mathcal{T}^k \right) \phi(x), \quad (1)$$

where  $\alpha_1, \alpha_2, \dots, \alpha_{N-1}$  are real numbers,  $N$  is an integer,  $I$  denotes the identity operator, and  $\mathcal{T}$  denotes the translation operator defined by

$$\mathcal{T}f(x) = f(x - 1).$$

Note that

$$\left\{ \begin{array}{ll} \phi(x) = \sigma(x) & x \text{ in } [0, 1], \\ \phi(x) = \sigma(x) + \alpha_1 \sigma(x - 1) & x \text{ in } [1, 2], \\ \phi(x) = \sigma(x) + \alpha_1 \sigma(x - 1) + (\alpha_2 + \alpha_1^2) \sigma(x - 2) & x \text{ in } [2, 3], \\ \phi(x) = \sigma(x) + \alpha_1 \sigma(x - 1) + (\alpha_2 + \alpha_1^2) \sigma(x - 2) + (\alpha_3 + 2\alpha_2\alpha_1 + \alpha_1^3) \sigma(x - 3) & x \text{ in } [3, 4], \\ \dots & \\ \phi(x) = \sigma(x) + \beta_1 \sigma(x - 1) + \beta_2 \sigma(x - 2) + \dots + \beta_n \sigma(x - n) & x \text{ in } [n, n + 1]. \end{array} \right.$$

Explicit expressions for the coefficients  $\beta_n$  are given in Theorem 4. The fact that  $\phi(x)$  is given by a combination involving more and more of the translates of  $\sigma$  to the right when  $x$  increases suggests the name "progressive function". Since  $\phi(x)$  is completely determined by  $\sigma$ , we call  $\sigma$  an *updating function*. Such a function may be used to profit in designing wavelets via multiresolution analysis [3, 4].

**Solving the transfer functional equation** (1). Looking for solutions of (1) such that  $\phi(x), \phi(x-1), \dots$  are orthogonal on  $L_2(\mathbb{R})$  is formally equivalent to a MA (moving averages) model of time series, from the point of view of "operator calculus" based on the operator  $\mathcal{T}$ . The formal solution of (1) is:

$$\phi(x) = (\alpha(\mathcal{T}))^{-1} \sigma(x), \text{ where } \alpha(z) = 1 - \sum_{k=1}^{N-1} \alpha_k z^k \quad (2)$$

will be called the **transfer function**. There are several ways to make sense out of this, as long as  $\alpha(z) \neq 0$  on the unit circle  $|z| = 1$ , by the famous **Wiener Lemma** (see [17, 12, 14, 7]).

**Definition 2.** A Laurent series  $\alpha(z) = \sum_k \alpha_k z^k$  is said to belong to the **Wiener class**  $W$ , if it has coefficient sequence  $\alpha_k$  in  $\ell_1(\mathbb{Z})$ , and if  $\alpha(z) \neq 0$ , for all  $z$  on the unit circle  $|z| = 1$  (see [1, p. 140]).

Alternatively, if we put  $z = e^{-it}$  and  $\tilde{\alpha}(t) = \alpha(z) = \alpha(e^{-it}) = \sum_k \alpha_k e^{-ikt}$ ,  $t \in [0, 2\pi)$  we say that the periodic function  $\tilde{\alpha}(t)$  is in the Wiener class  $W$  if its Fourier sequence  $\alpha_k$  is in  $\ell_1(\mathbb{Z})$ , and if  $\tilde{\alpha}(t) \neq 0$  for all  $t \in [0, 2\pi)$  (see [7, p.179]). Note that if  $\tilde{\alpha}(t) \in W$ , then  $\tilde{\alpha}(t)$  is continuous on  $[0, 2\pi)$ , since its Fourier series converges absolutely.

**Lemma 1.** a) **Wiener's Lemma.** If  $\alpha(z) \in W$ , then there exists  $\beta(z) = \frac{1}{\alpha(z)} = \sum_{k \in \mathbb{N}} \beta_k z^k \in W$ , for some sequence  $\{\beta_k\} \in \ell_1(\mathbb{Z})$ .

b) **Sylvester's Lemma.** If  $\alpha(z) = (1 - \lambda_1 z)(1 - \lambda_2 z) \dots (1 - \lambda_n z) = 1 + \alpha_1 z + \dots + \alpha_n z^n$ , is a polynomial, then  $\{\beta_k\}$  satisfy the recurrence

$$\alpha(\mathcal{T})\beta_j = 0, j \geq 1, \beta_0 = 1, \beta_i = 0, \forall i < 0, \Leftrightarrow \sum_{k=0}^n \alpha_k \beta_{j-k} = 0, j \geq 1, \beta_0 = 1, \beta_i = 0, \forall i < 0,$$

where  $\mathcal{T}(\beta_j) = \beta_{j-1}$  and  $\alpha_0 = 1$ . Moreover  $\{\beta_j\}$  are given explicitly by the Newton divided differences:

$$\beta_j = (\lambda^{j+n-1})_{[\lambda_1, \lambda_2, \dots, \lambda_n]} := \sum_{i=1}^n \frac{\lambda_i^{j+n-1}}{\prod_{k \neq i} (\lambda_i - \lambda_k)} \quad (3)$$

Furthermore, the solution for the nonhomogeneous recurrence

$$\alpha(\mathcal{T})x_k = \epsilon_k, k \geq 1,$$

is  $x_k = \sum_{j=0}^{k-1} \beta_j \epsilon_{k-j}$ .  $\beta_k$  is called the "fundamental solution" of the operator  $\alpha(\mathcal{T})$  ■

For  $\alpha(z) \in W$ , one may

1. Apply Fourier transform on both sides of (1) and solve for  $\widehat{\phi}(\xi)$ , yielding

$$\widehat{\phi}(\xi) = \frac{\widehat{\sigma}(\xi)}{\alpha(e^{-i\xi})} = \beta(e^{-i\xi}) \widehat{\sigma}(\xi) = \left( \sum_{k=0}^{\infty} \beta_k e^{-ik\xi} \right) \widehat{\sigma}(\xi). \quad (4)$$

2. Or, with  $N$  finite, one may factor  $\alpha(z)$  as in Lemma 1 b), and develop  $\beta(z) = 1/\alpha(z)$  in a power series  $\sum_{k=0}^{\infty} \beta_k z^k$ , using partial fractions. The formal result

$$\phi(x) = \sum_{k=0}^{\infty} \beta_k \sigma(x - k) \tag{5}$$

converges for  $\alpha(z) \in W$  by Wiener's Lemma.

*Remark 1.* A further step, already investigated in [4], is to consider analytic scaling symbols  $s(z)$ . In this paper we restrict ourselves to the easier polynomial case, since here we may exploit the power series expansion of reciprocals of polynomials due to Sylvester (3), and that makes it possible to establish a relation between the scaling symbol  $f(z)$  of  $\phi(x)$  and  $\alpha(z)$  – see (15) below.

**Refinable updating functions.** In this paper we suppose that the updating function satisfies a **two scale refinement equation**

$$\frac{1}{2} \sigma\left(\frac{x}{2}\right) = \sum_{j=0}^N s_j \sigma(x - j) \Leftrightarrow \widehat{\sigma}(w) = s(e^{-iw/2}) \widehat{\sigma}(w/2). \tag{6}$$

Following [5], we assume that the **polynomial symbol**

$$s(z) = \sum_{j=0}^N s_j z^j$$

has real coefficients,  $s_0 > 0, s_N > 0$ , all its roots are in the left half-plane  $z : Re(z) \leq 0$ , at least two of them in  $z : Re(z) < 0$ , and that  $s(-1) = 0, s(1) = 1$ . It follows then [5] that the refinement equation (6) has a unique solution satisfying  $\int_{-\infty}^{\infty} \sigma(x) dx = 1$ , and that  $\sigma(x)$  is continuous, non-negative and has support in  $[0, N]$ .

**Example 1.** In the special case  $s_j \geq 0, \sum_j s_j = 1$ , the solution  $\sigma(x)$  of the refinement equation (6) may be interpreted as the probability density function of an absolutely continuous random variable  $X$  satisfying

$$2X \stackrel{L}{=} X + Y,$$

where  $X, Y$  are independent, and  $P(Y = j) = s_j$  for  $j = 0, \dots, N$ . The probability density of  $X$  satisfies the equation (6).

**Example 2.** In the important special case  $s(z) = (\frac{z+1}{2})^N$ ,  $\sigma(x)$  is the uniform B-spline  $B_N$  of degree  $N - 1$  with knots  $0, \dots, N$ .

**The autocorrelation function.** For any  $\sigma \in L_2(\mathbb{R})$ , let us introduce the functions

$$[\sigma, \sigma](x) = \int_{\mathbb{R}} \sigma(x + y) \overline{\sigma(y)} dy, \quad C_{\sigma}(z) = \sum_{n=-N+1}^{N-1} [\sigma, \sigma](n) z^n \tag{7}$$

called respectively autocorrelation function and autocorrelation symbol. By the Poisson summation formula [10, Thm.6.5.3], the autocorrelation function is related to  $\sigma$  by:

$$\sum_{n \in \mathbb{Z}} |\widehat{\sigma}(\xi + 2\pi n)|^2 = \sum_{n \in \mathbb{Z}} [\sigma, \sigma](n) e^{-in\xi} = C_{\sigma}(e^{-i\xi}). \tag{8}$$

The following orthogonality condition for translates of a scaling function can be found in many references – see for example [11, Thm 5].

**Theorem 2.**

a) For any function  $\phi \in L_2(\mathbb{R})$ ,  $\{\phi(x - n) : n \in \mathbb{Z}\}$  is an orthonormal family if and only if

$$\sum_{n \in \mathbb{Z}} |\hat{\phi}(\xi + 2\pi n)|^2 = 1, \forall \xi \in \mathbb{R}^{\S}. \quad (9)$$

b) **Smith-Barnwell condition.** If  $\phi$  satisfies a two-scale refinement equation  $\phi(x) = 2 \sum_k f_k \phi(2x - k) \iff \widehat{\phi}(2w) = f(e^{-iw})\widehat{\phi}(w)$ , where  $f(z) = \sum_{k \in \mathbb{Z}} f_k z^k$ , then (9) is equivalent to

$$f(z)f(z^{-1}) + f(-z)f(-z^{-1}) = 1 \implies |f(e^{-iw})|^2 + |f(-e^{-iw})|^2 = 1.$$

**Corollary 3.** Let  $\sigma(x)$  be an updating function in  $L_2(\mathbb{R})$  and let  $\phi$  be a progressive function in  $L_2(\mathbb{R})$  satisfying (1) with  $N$  finite. Then,  $\{\phi(x - n) : n \in \mathbb{Z}\}$  is an orthonormal sequence if and only if

$$\sum_{n \in \mathbb{Z}} |\widehat{\sigma}(\xi + 2\pi n)|^2 = C_{\sigma}(e^{-i\xi}) = |\alpha(e^{-i\xi})|^2 \text{ for any } \xi \in \mathbb{R}.$$

**Proof.** By Theorem 2 the translates  $\{\phi(x - n) : n \in \mathbb{Z}\}$  are orthonormal if and only if

$$\sum_{n \in \mathbb{Z}} |\widehat{\phi}(\xi + 2\pi n)|^2 = 1 \quad (10)$$

The corollary follows by equation 4 and the periodicity of  $\alpha(e^{-i\xi})$ :

$$\sum_{n \in \mathbb{Z}} \frac{|\widehat{\sigma}(\xi + 2\pi n)|^2}{|\alpha(e^{-i(\xi + 2\pi n)})|^2} = \frac{\sum_{n \in \mathbb{Z}} |\widehat{\sigma}(\xi + 2\pi n)|^2}{|\alpha(e^{-i\xi})|^2} = 1 \quad (11)$$

## §2. Progressive wavelets with polynomial transfer function

We collect now several results on progressive wavelets scattered in the literature.

**Theorem 4.** Let  $\sigma(x) \in L_2(\mathbb{R})$  with support  $[0, N]$  for a positive integer  $N$ . Assume that  $[\sigma, \sigma](N - 1) \neq 0$ , where  $[\sigma, \sigma]$  denotes the autocorrelation of  $\sigma$  defined in (7), and let  $C(z) = C_{\sigma}(z)$  denote the autocorrelation symbol. Then,

---

<sup>§</sup>As a corollary, the translates of any function  $\phi^{\perp}(x)$  obtained by

$$\widehat{\phi}^{\perp}(w) = \frac{\widehat{\phi}(w)}{\sqrt{\sum_{k \in \mathbb{Z}} |\widehat{\phi}(w + 2\pi k)|^2}}$$

provide an orthonormal base of the space  $V_0$  generated by  $\{\phi(x - k), k \in \mathbb{Z}\}$ . Finally, applying a "mirror filter"  $(-1)^k p_{1-k}$  yields a wavelet function  $\psi(x) = \sum_k (-1)^k p_{1-k} \phi^{\perp}(2x - k)$  whose scaled translates enjoy also orthogonality between the spaces  $V_n, n \in \mathbb{Z}$  at different scales.

1. If  $C(z) = \sum_{n \in \mathbb{Z}} |\widehat{\sigma}(\xi + 2\pi n)|^2 > 0$ , for  $|z| = 1$ , then there exists a unique **canonical spectral factor**  $\alpha(z) \in W$  satisfying

$$\alpha(z)\alpha(z^{-1}) = C(z) \implies |\alpha(z)|^2 = C(z), z = e^{-i\xi},$$

and having no roots in the interior of the unit circle  $|z| = 1$ . If  $z_1, \dots, z_{N-1}$  are the roots of  $C(z)$  outside the unit disc,  $|z_k| > 1, k = 1, \dots, N - 1$ , then

$$\alpha(z) = \sqrt{A} \prod_{k=1}^{N-1} (z - z_k) = \alpha_0 + \alpha_1 z + \dots + \alpha_{N-1} z^{N-1}$$

where  $A = (-1)^{N-1} [\sigma, \sigma] (N - 1) / \prod_{k=1}^{N-1} \bar{z}_k > 0$

Furthermore,  $\alpha_k = 0$  for all  $k \geq N$  in (1) and  $\max\{k : \alpha_k \neq 0\} = N - 1$ .

2. As a consequence, the coefficients  $\beta_k$  of the reciprocal series satisfy the "Yule-Walker recursion" <sup>§</sup>

$$\beta_j = \alpha_1 \beta_{j-1} + \alpha_2 \beta_{j-2} + \dots + \alpha_{N-2} \beta_{j-N+2} + \alpha_{N-1} \beta_{j-N+1}, \beta_0 = 1, \beta_i = 0, \forall i < 0. \quad (12)$$

Moreover the  $\{\beta_j\}$  are given explicitly in terms of the roots of  $\alpha(z)$  by (3), and decay exponentially.

3. The  $\{\beta_n\}$  are given explicitly in terms of the coefficients of  $\alpha(z)$  by

$$\beta_n = \sum_{k_1+2k_2+3k_3+\dots+mk_m=n} \binom{k_1+k_2+\dots+k_m}{k_1, k_2, \dots, k_m} \alpha_1^{k_1} \alpha_2^{k_2} \alpha_3^{k_3} \dots \alpha_m^{k_m}, \quad (13)$$

where  $k_i \geq 0$  for  $0 \leq i \leq N - 1, m = n$  for  $0 \leq n \leq N - 1, m = N - 1$  for  $n > N - 1$ , and

$$\binom{k_1+k_2+\dots+k_m}{k_1, k_2, \dots, k_m} = \frac{(k_1+k_2+\dots+k_m)!}{k_1! k_2! \dots k_m!}$$

are the multinomial coefficients.

4. Let now  $\phi(x)$  be defined by

$$\phi(x) = \sum_{n \geq 0} \beta_n \sigma(x - n), \quad (14)$$

where the convergence is uniform, since  $\beta_n$  decay exponentially. Then,  $\phi$  satisfies the functional equation (1) and  $\{\phi(x - k) : k \in \mathbb{Z}\}$  is an orthonormal sequence.

5. If  $\sigma(x)$  is refinable with symbol  $s(z)$ , then so is the function  $\phi(x)$ , with refinement equation

$$\phi(x) = 2 \sum_k f_k \phi(2x - k),$$

---

<sup>§</sup>This recursion is used a lot in time series, for example when inverting the transfer function, in order to switch from an autoregressive to a moving average model, or viceversa. Note that the famous Fibonacci recursion induced by  $\alpha(z) = 1 - z - z^2$ , is not "acceptable/stationary" due to one root smaller  $z = 0.618034 \notin [-1, 1]$  which induces non-exponential decay. Before using this model, this root must be replaced by its reciprocal.

and scaling symbol  $f(z)$  given by

$$f(z) = s(z) \frac{\alpha(z)}{\alpha(z^2)} \Leftrightarrow F(z) := f(z)f(z^{-1}) = s(z)s(z^{-1}) \frac{C(z)}{C(z^2)} = S(z) \frac{C(z)}{C(z^2)}. \quad (15)$$

**Proof.**

1. This follows by the well-known Fejér-Riesz lemma (see e.g. [8, p. 235], [13, p. 330]), since  $C(z) \in W$  by (8) (i.e.  $C(e^{-i\xi}) > 0, \forall \xi \in \mathbb{R}$ ). Furthermore, The leading term of  $C(z)$  is  $[\sigma, \sigma](N-1)z^{N-1}$ . Comparing coefficients with the (possibly infinite) Laurent series  $\alpha(z)\alpha(z^{-1})$ , we conclude that  $\max\{k : \alpha_k \neq 0\} = N-1$ .
2. This widely used result (for example in signal processing and time series) may be easily checked.
3. For a proof, see for example [4].
4. It is easy to check that (14) satisfies (1). Taking then Fourier transform, we find that

$$\sum_{n \in \mathbb{Z}} |\hat{\phi}(\xi + 2\pi n)|^2 = \frac{\sum_{n \in \mathbb{Z}} |\widehat{\sigma}(\xi + 2\pi n)|^2}{|\alpha(e^{-i\xi})|^2} = 1.$$

and hence the translates  $\phi(x-k)$  are orthonormal by Corollary 3.

5. Following [9, (43)], let us look for a refinement relation  $\widehat{\phi}(2w) = f(e^{-iw})\widehat{\phi}(w)$ . Using  $\widehat{\phi}(w) = \frac{\widehat{\sigma}(w)}{\alpha(e^{-iw})}$  and  $\widehat{\sigma}(2w) = s(e^{-iw})\widehat{\sigma}(w)$  we find

$$\widehat{\phi}(2w) = \frac{\widehat{\sigma}(2w)}{\alpha(e^{-2iw})} = \frac{s(e^{-iw})\widehat{\sigma}(w)}{\alpha(e^{-2iw})} = \frac{s(e^{-iw})\alpha(e^{-iw})}{\alpha(e^{-2iw})}\widehat{\phi}(w).$$

Putting  $z = e^{-iw}$  yields (15)  $\square$

### §3. Linear B-spline progressive wavelets

Recall that the linear B-spline/"roof" function

$$\Lambda(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 2-x & \text{if } 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases},$$

satisfies the two-scale relation

$$\Lambda(x) = \frac{1}{2}\Lambda(2x) + \Lambda(2x-1) + \frac{1}{2}\Lambda(2x-2), \quad (16)$$

with scaling symbol

$$s(z) = \left(\frac{1+z}{2}\right)^2.$$

For  $\nu \geq 0$ , let  $V_R^\nu$  be the closure in  $L_2$  of the subspace spanned by the functions  $\Lambda_{\nu,k}$  for  $k \geq 0$ , and let  $W_R^\nu$  be the orthogonal complement of  $V_R^\nu$  in  $V_R^{\nu+1}$ . We want to construct an orthonormal basis of  $V_R^0$  of the form  $\{\phi(x-k) : k \geq 0\}$ , with a progressive function  $\phi$  defined by  $\phi(x) = C\Lambda(x) + \alpha\phi(x-1)$ , with constants  $\alpha$  and  $C = C_\alpha$  determined so that the set of translates  $\{\phi(x-k)\}$  is orthonormal.

**Theorem 5.** a) *The functional equation*

$$\phi(x) = C\Lambda(x) + \alpha\phi(x-1), \quad (17)$$

admits a unique solution  $\phi = \phi_\alpha \in L_2(\mathbf{R})$  when  $|\alpha| < 1$ . Moreover

$$\phi(x) = C \sum_{k \geq 0} \alpha^k \Lambda(x-k). \quad (18)$$

b) *The function  $\phi$  satisfies the scaling equation*

$$\phi(x) = 2 \sum_{k \geq 0} f_k \phi(2x-k),$$

where the filter coefficients  $f_k$  are given by

$$f_0 = \frac{1}{4}, \quad f_1 = \frac{1}{2} - \frac{\alpha}{4}, \quad f_2 = \frac{1}{4}(1-\alpha),$$

$$f_{2k+1} = f_{2k+2} = \frac{1-\alpha}{4} \alpha^k, \quad \text{for } k \geq 1$$

**Proof** a) The equation (17) can be written as

$$(I - \alpha\mathcal{T})\phi = C\Lambda,$$

and for  $|\alpha| < 1$ , the unique solution  $\phi$  is given by the Neumann series (18).

b) Write

$$\phi(x) = (1 - \alpha\mathcal{T})^{-1} \Lambda(x) = (1 - \alpha\mathcal{T})^{-1} D_2(2s(\mathcal{T}))\Lambda(x) = (1 - \alpha\mathcal{T})^{-1} D_2(2s(\mathcal{T}))(1 - \alpha\mathcal{T})\phi(x), \quad (19)$$

where  $D_2$  is the dilation operator by 2, and  $s(z) = (\frac{1+z}{2})^2$ . Using  $\mathcal{T}^k D_2 = D_2 \mathcal{T}^{2k}$ , we find that

$$\begin{aligned} \phi(x) &= \frac{1}{4} \left( \sum_{k \geq 0} \alpha^k \mathcal{T}^k \right) D_2 (1 + \mathcal{T})^2 (1 - \alpha\mathcal{T}) \phi(x) \\ &= \frac{1}{4} D_2 \left( \sum_{k \geq 0} \alpha^k \mathcal{T}^{2k} \right) (1 + 2\mathcal{T} + \mathcal{T}^2) (1 - \alpha\mathcal{T}) \phi(x) \\ &= \frac{1}{4} D_2 \left( \sum_{k \geq 0} \alpha^k \mathcal{T}^{2k} \right) (1 + \mathcal{T}(2-\alpha) + \mathcal{T}^2(1-2\alpha) - \alpha\mathcal{T}^3) \phi(x) \\ &= \frac{1}{4} D_2 (I + \mathcal{T}(2-\alpha) + \sum_{k \geq 1} \mathcal{T}^{2k} (\alpha^k + \alpha^{k-1}(1-2\alpha)) + \mathcal{T} \sum_{k \geq 1} \mathcal{T}^{2k} (\alpha^k(2-\alpha) - \alpha^{k-1}\alpha)) \phi(x) \\ &= \frac{1}{4} D_2 (I + \mathcal{T}(2-\alpha) + \sum_{k \geq 1} \mathcal{T}^{2k} (\alpha^{k-1}(1-\alpha)) + \mathcal{T} \sum_{k \geq 1} \mathcal{T}^{2k} (\alpha^k(1-\alpha))) \phi(x) \end{aligned}$$

□

**Theorem 6.** The family  $\{\phi_\alpha(x - k) : k \geq 0\}$  is an orthonormal basis for  $V_R^0$  if and only if  $\alpha = -2 + \sqrt{3}$  and  $C^2 = C_\alpha^2 = -6\alpha = \frac{3}{2}(1 + \alpha^2)$ .

**Proof:** It will be enough to determine  $\alpha$  so that

$$\langle \phi_\alpha(x), \phi_\alpha(x - 1) \rangle = 0$$

$$\langle \phi_\alpha(x), \phi_\alpha(x) \rangle = 1.$$

By using that  $\langle \Lambda(x), \Lambda(x) \rangle = \frac{2}{3}$  and  $\langle \Lambda(x), \Lambda(x - 1) \rangle = \frac{1}{6}$ , these equations lead to a system of equations in  $C$  and  $\alpha$

$$C^2 \frac{1}{6} + \alpha = 0$$

$$C^2 \frac{2}{3} + C^2 \alpha \frac{1}{6} = 1$$

with solution  $\alpha = -2 + \sqrt{3} \in (-1, 1)$  and  $C^2 = C_\alpha^2 = -6\alpha$ .

*Remark 2.* We may verify that the symbol  $f(z)$  satisfies the Smith–Barnwell condition  $f(z)f(z^{-1}) + f(-z)f(-z^{-1}) = 1$  when  $\alpha = -2 + \sqrt{3}$ .

#### §4. B-spline progressive wavelets

**B-splines.** The  $m$ th order cardinal B-splines  $N_m(x)$  are the densities of a sum of  $m$  independent  $U[0, 1]$  r.v.'s. They are defined recursively by convolution:

$$N_m(x) = (N_{m-1} * N_1)(x) = \int_{\mathbb{R}} N_{m-1}(x - t)N_1(t)dt, m \geq 2 \quad (20)$$

where  $N_1$  is the characteristic function of the unit interval  $[0, 1)$ . The corresponding Fourier transforms are

$$\int_{-\infty}^{\infty} e^{-iwt} N_m(t) dt = \left( \frac{1 - e^{-iw}}{iw} \right)^m = \left( e^{iw/2} \frac{\sin(w/2)}{iw/2} \right)^m.$$

$N_m$  is supported on  $[0, m]$ , and is symmetric with respect to the center of its support  $\frac{m}{2}$ , i.e.

$$N_m\left(\frac{m}{2} + x\right) = N_m\left(\frac{m}{2} - x\right).$$

The two scale symbol of  $N_m$  is  $s_m(z) = \left(\frac{1+z}{2}\right)^m$  [2, (3.4.6)], and the corresponding two scale relation is [2, (3.4.7)]

$$\frac{1}{2}N_m(x/2) = 2^{-m} \sum_{j=0}^m \binom{m}{j} N_m(x - j) = s_m(T)N_m(x).$$

The autocorrelation of the cardinal B-splines is

$$[N_m, N_m](x) = \int_{\mathbb{R}} N_m(x + y)\overline{N_m(y)}dy = N_{2m}(m + x), m \geq 1,$$

and the autocorrelation symbol (8) is:



$$\sum_{n \in \mathbb{Z}} \left| \widehat{N}_m(\xi + 2\pi n) \right|^2 = \sum_{n \in \mathbb{Z}} [N_m, N_m](n) = \sum_{n \in \mathbb{Z}} N_{2m}(m+n)e^{-in\xi} = \sum_{n=-m+1}^{m-1} N_{2m}(m+n)e^{-in\xi} = E_{2m}(e^{-i\xi}). \tag{21}$$

Here

$$E_m(z) = \sum_{j=-m+1}^{m-1} N_m(m+j)z^j,$$

is the well-known Euler-Frobenius Laurent polynomial [15].

The monic polynomials  $\tilde{E}_m(z) = m!E_m(z)$  satisfy the recursion [15, (2.1.9)]

$$\begin{aligned} \tilde{E}_m(z) &= (1+mz)\tilde{E}_{m-1}(z) + z(1-z)\tilde{E}'_{m-1}(z), \tilde{E}_0(z) = 1 \implies \\ \tilde{E}_1(z) &= z + 1, \\ \tilde{E}_2(z) &= z^2 + 4z + 1, \\ \tilde{E}_3(z) &= z^3 + 11z^2 + 11z + 1, \\ \tilde{E}_4(z) &= z^4 + 26z^3 + 66z^2 + 26z + 1, \\ \tilde{E}_5(z) &= E_5(z) = z^5 + 57z^4 + 302z^3 + 302z^2 + 57z + 1, \\ \tilde{E}_6(z) &= z^6 + 120z^5 + 1191z^4 + 2416z^3 + 1191z^2 + 120z + 1. \end{aligned}$$

The symmetry of  $\tilde{E}_{2m}(z) = z^{-m}E_{2m}(z) = \sum_{k=-m}^m a_k z^k$  is useful for performing the Riesz-Fejer factorization of  $E_{2m}(z)$ , providing a degree reduction from  $2m$  to  $m$  – see [6, Sec. 2.3:Roots method]. Indeed, put  $z + z^{-1} = w$  § . Using

$$z^k + z^{-k} = (z + z^{-1})^k - \sum_{j=1}^{k-1} \binom{k}{j} z^{k-2j}$$

yields

$$(z + z^{-1})^2 = w^2 - 2, \quad (z + z^{-1})^3 = w^3 - 3w$$

and thus

$$\tilde{E}_2(z) = w + 4, \quad \tilde{E}_4(z) = w^2 + 26w + 64, \quad \tilde{E}_6(z) = w^3 + 120w^2 + 1188w + 2176$$

(the first allows to recover the linear spline case by solving  $z + z^{-1} = -4$ ).

For more properties and results about B-splines one can refer to [15, 16]. See also [9, Table II] for a list of "valid scaling symbols"  $F(z)$  associated to B-splines.

**Example 3.** Consider now the quadratic B-spline  $N_3(x)$ . The corresponding progressive function  $\phi$  satisfies the functional equation

$$\phi(x) + \alpha_1 \phi(x-1) + \alpha_2 \phi(x-2) = CN_3(x) \tag{22}$$

---

§ An alternative to express  $\cos(kx), k = 1, \dots, m$  as functions of  $\cos(x)^k, k = 1, \dots, m$ , and ultimately switch to  $y = \cos(x) \in [-1, 1]$ .

where  $C = C_\alpha = \sqrt{\frac{20}{11}(1 + \alpha_1^2 + \alpha_1^2)}$ .

Solving  $w^2 + 26w + 64 = 0$  yields  $w_{1,2} = -13 \pm \sqrt{105}$ , and each of the  $w_i$  roots yields a unique "reciprocal root" in the unit circle

$$z_1 = \frac{2}{-13 - \sqrt{105} - \sqrt{270 + 26\sqrt{105}}}, z_2 = \frac{1}{2} \left( -13 + \sqrt{105} + \sqrt{270 - 26\sqrt{105}} \right).$$

The transfer function is  $(1 - zz_1)(1 - zz_2) = 1 + \alpha_1 z + \alpha_2 z^2$  where  $\alpha_1 = 13 - \sqrt{135 + 4\sqrt{30}}$ ,  $\alpha_2 = 2(8 + \sqrt{30}) - \sqrt{375 + 64\sqrt{30}}$ .

The general results of Theorem 4 are again confirmed in this case.

**Theorem 7.** Let  $\phi$  be the progressive function that satisfies the equation

$$\phi(x) = CN_3(x) + \alpha_1 \phi(x-1) + \alpha_2 \phi(x-2)$$

and let

$$\phi(x) = C \sum_{k \geq 0} \beta_k N_3(x-k), \quad (23)$$

where the coefficients  $\beta_k$  are given inductively by  $\beta_0 = 1$ ,  $\beta_1 = \alpha_1$  and for  $k \geq 2$ ,  $\beta_k = \alpha_1 \beta_{k-1} + \alpha_2 \beta_{k-2}$ . Then the function  $\phi$  satisfies the scaling equation  $\phi(x) = 2 \sum_{k \geq 0} f_k \phi(2x-k)$ , with filter coefficients given by

$$\begin{aligned} 2f_0 &= 1 \\ 2f_{2k-1} &= -\alpha_2 \beta_{k-3} + (1 - 3\alpha_1 - 3\alpha_2) \beta_{k-2} + (3 - \alpha_1) \beta_{k-1} \\ &\vdots \\ 2f_{2k} &= \beta_k + (-\alpha_1 - 3\alpha_2) \beta_{k-2} + (3 - 3\alpha_1 - \alpha_2) \beta_{k-1}. \end{aligned}$$

In particular  $2f_0 = 1$ ,  $2f_1 = 3 - \alpha_1$ ,  $2f_2 = 3 - 2\alpha_1 - \alpha_2 \dots$

*Remark 3.* We may check again using Mathematica for example that (15) holds:

$$f(z) = s(z) \frac{\alpha(z)}{\alpha(z^2)} = \left( \frac{1+z}{2} \right)^3 \frac{(1 - \alpha_1 z - \alpha_2 z^2)}{(1 - \alpha_1 z^2 - \alpha_2 z^4)}.$$

## References

- [1] CHUI, C. K. *An introduction to wavelets*, vol. 1. Academic press, 1992.
- [2] CHUI, C. K. *Wavelets: a mathematical tool for signal analysis*, vol. 1. Siam, 1997.
- [3] DELIU, A., AND DENG, B. Progressive wavelets. In *IPCV: Proceedings of the 2010 International Conference on Image Processing, Computer Vision, and Pattern Recognition* (2010), G. S. Hamid R. Arabnia, Ed., pp. 256–262.
- [4] DELIU, A., AND DENG, B. On progressive functions. *Journal of Mathematical Analysis and Applications* (2014).

- [5] GOODMAN, T., AND MICCHELLI, C. A. On refinement equations determined by pólya frequency sequences. *SIAM journal on mathematical analysis* 23, 3 (1992), 766–784.
- [6] GOODMAN, T. N., MICCHELLI, C. A., RODRIGUEZ, G., AND SEATZU, S. Spectral factorization of laurent polynomials. *Advances in Computational Mathematics* 7, 4 (1997), 429–454.
- [7] GRÖCHENIG, K. Wiener’s lemma: Theme and variations. an introduction to spectral invariance and its applications. In *Four Short Courses on Harmonic Analysis*. Springer, 2010, pp. 175–234.
- [8] HELTON, J. W., AND PUTINAR, M. Positive polynomials in scalar and matrix variables, the spectral theorem and optimization [online]. 2006. Available from: [arXivpreprintmath/0612103](https://arxiv.org/abs/math/0612103).
- [9] HERLEY, C., AND VETTERLI, M. Wavelets and recursive filter banks. *signal Processing, IEEE Transactions on* 41, 8 (1993), 2536–2556.
- [10] HONG, D., WANG, J., AND GARDNER, R. *Real analysis with an introduction to wavelets and applications*. Elsevier, 2004.
- [11] LAI, M.-J. Popular wavelet families and filters and their use. In *Encyclopedia of Complexity and Systems Science*. Springer, 2009, pp. 6855–6913.
- [12] NEWMAN, D. A simple proof of wiener’s 1/f theorem. *Proceedings of the American Mathematical Society* 48, 1 (1975), 264–265.
- [13] PINSKY, M. A. *Introduction to Fourier analysis and wavelets*. Brooks/Cole Pacific Grove, 2002.
- [14] RUDIN, W. *Functional analysis*. international series in pure and applied mathematics, 1991.
- [15] SCHOENBERG, I. J. *Cardinal spline interpolation*, vol. 12. SIAM, 1973.
- [16] STRÖMBERG, J.-O. A modified franklin system and higher order spline systems on  $\mathbb{R}^n$  as unconditional bases for hardy spaces. *Fundamental Paper in Wavelet Theory* (2006), 197–215.
- [17] WIENER, N. Tauberian theorems. *Annals of mathematics* (1932), 1–100.

F. Avram  
Dept de Mathématiques, Univ. de Pau,  
Pau 64000, France 64000  
[Florin.Avram@univ-Pau.fr](mailto:Florin.Avram@univ-Pau.fr)

A. Deliu  
Dept of Mathematics, Georgia Perimeter College  
Clarkston, GA 30021, USA  
[Anca.Deliu@gpc.edu](mailto:Anca.Deliu@gpc.edu)

B. Deng  
Dept of Mathematics, Columbus State University  
Columbus, Georgia 31907, USA  
[bdeng@ColumbusState.edu](mailto:bdeng@ColumbusState.edu)