Monografías Matemáticas *García de Galdeano* **40**, 1–8 (2015)

L^p-THEORY FOR THE TIME DEPENDENT NAVIER-STOKES PROBLEM WITH NAVIER-TYPE BOUNDARY CONDITIONS

Hind Al Baba, Chérif Amrouche and Miguel Escobedo[1]

Abstract. We study the time dependent Navier-Stokes Problem with Navier-type boundary conditions in L^p- spaces using semi-group theory. Proceeding as in [6] we prove the existence of a unique local in time mild solution to the Navier-Stokes Problem with Navier-type boundary conditions. Then proceeding as in [7] we prove that this solution is a classical one.

Keywords: Navier-Stokes Problem, Navier-type boundary conditions, local existence, mild solution, classical solution.

AMS classification: 35Q30, 76D05, 76D07, 35K20, 35K22, 76N10, 35A20, 35Q40.

§1. Introduction

There is an extensive literature on the solvability of the initial value problem for the Navier-Stokes equation in L^2- spaces. Hopf proved in [8] the existence of a global weak solution using the Faedo-Galerkin approximation and an energy inequality. However the uniqueness and the regularity of Hopf's solution are still open problem for $n \geq 3$. Fujita and Kato [9, 5] use the semi-group theory and the fractional powers of a non-negative operator to study the non-stationary Navier Stokes Problem with Dirichlet boundary condition in the Hilbert space L^2. Furthermore, for $n = 3$ they proved the existence of a unique global strong solution if the initial value has square-summable half derivative. Later on, Giga and Miyakawa [7] prove the existence of a unique local in time strong solution without assuming that the initial velocity is regular. To establish their result, they develop an L^r theory generalising the L^2 theory of Kata and Fujita [9, 5]. In [6], Giga constructs a unique local in time mild solution for a class of semi-linear parabolic equation. He also show that his result includes the semi-linear heat equation and the Navier-Stokes system with Dirichlet boundary condition.

In this paper we consider the time dependent Navier-Stokes problem with Navier-type boundary conditions:

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \boldsymbol{0}, & \operatorname{div}\boldsymbol{u} = 0 \quad \text{in} \quad \Omega\times(0,T), \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, & \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} \quad \text{on} \quad \Gamma\times(0,T), \\ & \boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in} \quad \Omega, \end{cases} \tag{1}$$

where $(\boldsymbol{u}\cdot\nabla) = \sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j}$ and Ω is a bounded domain of $\mathbb{R}^3$ of class $C^{2,1}$. For simplicity the external force is assumed to be zero.

[1]The work of M. E. has been supported by DGES Grant MTM2011-29306-C02-00 and Basque Government Grant IT641-13.

We will show that some informations on the linear problem can be used to obtain local mild and classical solutions to Problem (1). For this reason we will use the semi-group theory developped in [1, 3] for the Stokes operator with Navier-type boundary conditions. First, using the $L^p - L^q$ estimates for the Stokes semi-group with Navier-type boundary conditions (see [3, Theorem 7.4]) and proceeding as Giga in [6] we will prove the existence of a local in time mild solution to the Problem (1). Next, using the fractional powers of the Stokes operator we will estimate the non-linear term $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$. Then, proceeding as Giga and Miyakawa [7] we prove that the solution $\boldsymbol{u} \in \mathbf{D}(A_p)$ for all $t \in (0, T_*]$ for certain $T_* < T$. We will show in this paper some general existence, uniqueness and regularity results for the Navier- Stokes problem with Navier-type boundary conditions with a sketch of their proofs. Detailed results and detailed proofs are given in [2].

§2. Giga's abstract semi-linear problem

Consider the abstract semilinear parabolic equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + \mathcal{A}\boldsymbol{u} = \boldsymbol{F}\boldsymbol{u} \qquad u(0) = \boldsymbol{a}, \qquad \text{in} \quad \Omega \times]0, T[, \tag{2}$$

where Ω is an arbitrary domain of $\mathbb{R}^3$. Giga considers in [6] Problem (2) in L^p-type vector valued function spaces, $1 < p < \infty$, $\boldsymbol{F}\boldsymbol{u}$ represents the non-linear term and $-\mathcal{A}$ is the infinitesimal generator of a strongly continuous semi-group $e^{-t\mathcal{A}}$ in some closed subspace $\boldsymbol{E}^p$ of $\boldsymbol{L}^p(\Omega)$ equipped with norm of $\boldsymbol{L}^p(\Omega)$. He constructs a unique local in time solution in $L^q(0, T_*, \boldsymbol{L}^p(\Omega))$ for the Problem (2). He also proves that the constructed solution is global in time for small initial data. His analysis is based on the regularisation property of the linear part $e^{-t\mathcal{A}}$ and some assumptions on the non-linear term. Giga solves the Problem (2) via the corresponding integral equation

$$\boldsymbol{u}(t) = e^{-t\mathcal{A}}\boldsymbol{a} + \int_0^t e^{-(t-s)\mathcal{A}}\boldsymbol{F}\boldsymbol{u}(s)\,\mathrm{d}\,s. \tag{3}$$

To prove his theorem Giga gives some assumptions on the operator $\mathcal{A}$ and the nonlinear term $\boldsymbol{F}\boldsymbol{u}$.

Having the Navier-Stokes problem (1) in mind, we consider in the sequel a particular case of [6]. We assume that :

(i) There exists a continuous projection P from $L^p(\Omega)$ to $\boldsymbol{E}^p$ for all $1 < p < \infty$ such that the restriction of P on $\mathcal{D}(\Omega)$ is independent of p and $\mathcal{D}(\Omega) \cap \boldsymbol{E}^p$ is dense in $\boldsymbol{E}^p$.

(ii) For a fixed $0 < T < \infty$ the following estimate holds

$$\textbf{(A)} \quad \|e^{-t\mathcal{A}}\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \leq M\, t^{-\frac{3}{2}(\frac{1}{s}-\frac{1}{p})}\|\boldsymbol{f}\|_{\boldsymbol{L}^s(\Omega)}, \qquad \boldsymbol{f} \in \boldsymbol{E}^s, \quad 0 < t < T, \tag{4}$$

with $p \geq s > 1$ and the constant $M = M(p, s, T)$ depends only on p, s and T.

(iii) The nonlinear term $\boldsymbol{F}\boldsymbol{u}$ can be written in the form

$$\boldsymbol{F}\boldsymbol{u} = \boldsymbol{L}\boldsymbol{G}\boldsymbol{u}, \tag{5}$$

where $\boldsymbol{L}$ is the linear part and $\boldsymbol{G}$ is the non-linear part.

(iv) We suppose also that the following estimate holds

$$(\mathbf{N1}) \quad \|e^{-t\mathcal{A}}\boldsymbol{L}\boldsymbol{f}\|_{L^p(\Omega)} \leq N_1 t^{-1/2}\|\boldsymbol{f}\|_{L^p(\Omega)}, \qquad \boldsymbol{f} \in \boldsymbol{E}^p, \quad 0 < t < T, \tag{6}$$

where the constant $N_1 = N(p, T)$ depends only on p and T.
(v) The operator $\boldsymbol{G}$ satisfies the following estimate

$$(\mathbf{N2}) \quad \|\boldsymbol{G}\boldsymbol{v} - \boldsymbol{G}\boldsymbol{w}\|_{L^s(\Omega)} \leq N_2\|\boldsymbol{v} - \boldsymbol{w}\|_{L^p(\Omega)}(\|\boldsymbol{v}\|_{L^p(\Omega)} + \|\boldsymbol{w}\|_{L^p(\Omega)}), \qquad \boldsymbol{G}(\mathbf{0}) = \mathbf{0}, \tag{7}$$

for all $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{E}^p$, for $s = \frac{p}{2} > 1$ and $N_2 = N_2(p)$ depends only on p, $1 < p < \infty$.

Before we state the result of Giga [6, Theorem 1, Theorem 2] who proves the existence and uniqueness of mild solutions of (3) assuming (**A**), (**N1**) and (**N2**) we recall the definition of mild solution (see [4, Chapiter VI, Definition 7.2]).

Definition 1 (Mild Solution)**.** Let $(\mathcal{A}, D(\mathcal{A}))$ be the infinitesimal generator of a strongly continuous semi-group $e^{-t\mathcal{A}}$ on a Banach space X. Let $x \in X$ and $f \in L^1(0, T; X)$. Then the function

$$u(t) = e^{-t\mathcal{A}}x + \int_0^t e^{-(t-s)\mathcal{A}} f\, u(s)\mathrm{d}\, s, \qquad 0 \leq t \leq T,$$

is called mild solution for the abstract Cauchy Problem

$$\frac{\partial u}{\partial t} + \mathcal{A}\, u = f\, u, \qquad u(0) = x, \qquad \text{in} \quad \Omega \times [0, T].$$

In what follows BC denotes the space of bounded and continuous functions and C denotes a positive constant whose value may change from one line to the next.

Theorem 1 (Giga's abstract existence and uniqueness theorem)**.** *Let* $\boldsymbol{u}_0 \in \boldsymbol{E}^p$, $p \geq 3$. *Then there is* $T_0 > 0$ *and a unique mild solution of* (3) *on* $[0, T_0)$ *such that*

$$\boldsymbol{u} \in BC([0, T_0);\, \boldsymbol{E}^p) \cap L^q(0, T_0;\, \boldsymbol{E}^r)$$

with

$$q > p,\ r > p, \qquad \frac{2}{q} + \frac{3}{r} = \frac{3}{p}.$$

Moreover there is a positive constant ε *such that if* $\|\boldsymbol{u}_0\|_{\boldsymbol{E}^p} \leq \varepsilon$, *then* T_0 *can be taken as infinity for* $p = 3$.

In what follows we want to apply Giga's abstract existence theorem to the Stokes operator with Navier-type boundary conditions to get local in-time mild solution to Problem (1). For this reason in the following section, we recall some properties of the Stokes operator with Navier-type boundary conditions.

§3. Stokes operator with Navier-type boundary conditions

In this section we recall some properties of the Stokes operator with Navier-type boundary conditions that are crucial in our work. This operator has been studied in details in [1, 3].

The Stokes operator with Navier-type boundary conditions is a closed linear densely defined operator $A_p\,:\,\mathbf{D}(A_p)\subset \boldsymbol{L}^p_{\sigma,\tau}(\Omega)\longmapsto \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, where

$$\mathbf{D}(A_p)=\Big\{\boldsymbol{u}\in \boldsymbol{W}^{2,p}(\Omega);\ \operatorname{div}\boldsymbol{u}=0 \text{ in } \Omega,\ \boldsymbol{u}\cdot\boldsymbol{n}=0,\ \ \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}=\mathbf{0} \text{ on } \Gamma\Big\}.$$

and For all $\boldsymbol{u}\in\mathbf{D}(A_p)$, $A_p\,\boldsymbol{u}=-\Delta\boldsymbol{u}$ (see [1, Proposition 3.1]). The space $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ is given by

$$\boldsymbol{L}^p_{\sigma,\tau}(\Omega)=\Big\{\boldsymbol{f}\in \boldsymbol{L}^p(\Omega);\quad \operatorname{div}\boldsymbol{f}=0 \text{ in } \Omega,\quad \boldsymbol{f}\cdot\boldsymbol{n}=0 \text{ on } \Gamma\Big\}.$$

Thanks to [1, Theorem 4.12] we know that the Stokes operator A_p is a sectorial operator and generates a bounded analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, for all $1<p<\infty$. We denote by e^{-tA_p} the analytic semi-group associated to the operator A_p.

The authors study in [3, Proposition 6.7, Corollary 6.8] the domains of fractional powers of the Stokes operator A_p and prove that for all $1<p<\infty$ and for all $\alpha\in\mathbb{R}$ such that $0<\alpha<3/2p$ one has

$$\mathbf{D}(A_p^\alpha)\hookrightarrow \boldsymbol{L}^q(\Omega),\qquad \frac{1}{q}=\frac{1}{p}-\frac{2\alpha}{3}. \tag{8}$$

They also prove that for all $\boldsymbol{u}\in\mathbf{D}(A_p^\alpha)$ the following estimate holds

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^q(\Omega)}\ \le\ C(\Omega,p)\,\|A_p^\alpha\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{9}$$

Using estimate (9), the authors prove in [3, Theorem 7.4] an estimate of type L^p-L^q for the Stokes semi-group e^{-tA_p}. More precisely, they prove that for all $1<p\le q<\infty$ and for all $\boldsymbol{f}\in\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, the following estimates holds:

$$\|e^{-tA_p}\boldsymbol{f}\|_{\boldsymbol{L}^q(\Omega)}\ \le\ C\,t^{-3/2(1/p-1/q)}\|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{10}$$

We also know thanks to [3, Theorem 6.5] that $\mathbf{D}(A_p^{1/2})\ =\ \boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega)$, where

$$\boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega)=\{\boldsymbol{u}\in\boldsymbol{W}^{1,p}(\Omega);\ \operatorname{div}\boldsymbol{u}=0 \text{ in } \Omega \text{ and } \boldsymbol{u}\cdot\boldsymbol{n}=0 \text{ on } \Gamma\}$$

with equivalent norms.

§4. Main results

In this section we prove the existence of a unique local in time mild solution for the Problem (1) and we prove that this solution is regular and belongs to $\mathbf{D}(A_p)$.

Applying the Helmholtz projection P to the first equation of system (1) we get

$$\frac{\partial\boldsymbol{u}}{\partial t}+A_p\boldsymbol{u}=-P(\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\qquad \boldsymbol{u}(0)=\boldsymbol{u}_0\in\boldsymbol{L}^p_{\sigma,\tau}(\Omega). \tag{11}$$

The Helmholtz projection $P:\boldsymbol{L}^p(\Omega)\longmapsto\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ is defined by:

$$\forall\,\boldsymbol{f}\in\boldsymbol{L}^p(\Omega),\qquad P\boldsymbol{f}\ =\ \boldsymbol{f}\ -\ \mathbf{grad}\,\pi,$$

where π is the unique solution of Problem

$$\operatorname{div}(\mathbf{grad}\,\pi\,-\,\boldsymbol{f})=0\qquad \text{in } \Omega,\qquad (\mathbf{grad}\,\pi\,-\,\boldsymbol{f})\cdot\boldsymbol{n}=0\qquad \text{on } \Gamma.$$

Now we verify assumptions (**A**), (**N1**) and (**N2**) for the Stokes operator A_p. Observe that in our case $\boldsymbol{E}^p = \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. First, notice that the assumption (**A**) is the $L^p - L^q$ estimate (10). Thus assumption (**A**) holds. We next verify the assumptions for the non-linear term

$$\boldsymbol{F}\boldsymbol{u} = -P(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}. \tag{12}$$

Since $\operatorname{div}\boldsymbol{u} = 0$ in Ω, then for all $1 \le i \le 3$, $(\boldsymbol{u}\cdot\nabla\boldsymbol{u})_i = \sum_{j=1}^3 \frac{\partial(u_j u_i)}{\partial x_j}$.

As in [6] let $(g_{ij})_{1\le i,\, j\le 3}$ be a matrix and for all $1 \le i \le 3$ we set $\boldsymbol{g}_i = (g_{ij})_{1\le j\le 3}$. We define $\boldsymbol{L}$ by

$$\boldsymbol{L}\boldsymbol{g}_i = P\operatorname{div}\boldsymbol{g}_i \tag{13}$$

which is a linear operator. The non-linear term $\boldsymbol{F}\boldsymbol{u}$ is expressed by $\boldsymbol{L}\boldsymbol{G}\boldsymbol{u}$, where $(\boldsymbol{G}\boldsymbol{u})(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{u}(\boldsymbol{x}))$ and $\boldsymbol{g}(\boldsymbol{u}) : \mathbb{R}^3 \longmapsto \mathbb{R}^9$, $(\boldsymbol{g}(\boldsymbol{u}))_{ij} = -u_i u_j$, $1 \le i, j \le 3$. It is easy to see that for all $\boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^3$, $\boldsymbol{g}$ satisfies

$$|\boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{g}(\boldsymbol{z})| \le N_2\, |\boldsymbol{y} - \boldsymbol{z}|\, (|\boldsymbol{y}| + |\boldsymbol{z}|), \qquad \boldsymbol{g}(\boldsymbol{0}) = \boldsymbol{0},$$

with $|\cdot|$ denotes the norm on $\mathbb{R}^k$, $(k = 3, 9)$. Thus $\boldsymbol{G}$ satisfies (**N2**).
It remains to verify the assumption (**N1**). To this end we prove the following proposition :

Proposition 2. *Let* $\boldsymbol{L}$ *be the operator defined in* (13)*, then the following estimate holds*

$$\forall \boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \qquad \|e^{-tA_p}\boldsymbol{L}\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \le \frac{C(p,T)}{t^{1/2}} \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \tag{14}$$

where e^{-tA_p} *is the semi-group generated by the Stokes operator* A_p *on* $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ *and* $C(p,T)$ *is a constant depending on* p *and* T.

Proof. Since the Stokes operator A_p generates a bounded analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ for all $< p < \infty$, then for all $t > 0$, $e^{-tA_p} = e^t e^{-t(I+A_p)}$, where $e^{-t(I+A_p)}$ is the analytic semi-group generated by the operator $-(I + A_p)$ on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. Let $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ one has

$$\begin{aligned} \|e^{-tA_p}\boldsymbol{L}\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} &= e^t\, \|(I + A_p)^{1/2}\, e^{-t(I+A_p)}\, (I + A_p)^{-1/2}\, \boldsymbol{L}\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \\ &\le \frac{Ce^T}{t^{1/2}}\, \|(I + A_p)^{-1/2}\, \boldsymbol{L}\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)} \\ &\le \frac{Ce^T}{t^{1/2}} \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}. \end{aligned}$$

The last inequality comes from the fact that the operator $(I + A_p)^{-1/2}\boldsymbol{L}$ is a bounded operator from $\boldsymbol{L}^p(\Omega)$ into $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. Indeed as in the proof of [7, Lemma 2.1], for all $1 \le j \le 3$, the operator

$$\frac{\partial}{\partial x_j}\, I\, (I + A_p)^{-1/2} \;:\quad \boldsymbol{L}^p_{\sigma,\tau}(\Omega) \longmapsto \mathbf{D}((I + A_p)^{1/2}) \longmapsto \boldsymbol{W}^{1,p}(\Omega) \longmapsto \boldsymbol{L}^p(\Omega)$$

is continuous for each p, $1 < p < \infty$, where I denotes the continuous embedding of $\mathbf{D}((I + A_p)^{1/2}) = \boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega)$ in $\boldsymbol{W}^{1,p}(\Omega)$. As a result, the adjoint operator $(I+A_{p'})^{-1/2}P\frac{\partial}{\partial x_j}$ is continuous from $\boldsymbol{L}^{p'}(\Omega)$ to $\boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. We recall that for all $1 < p < \infty$ $(\boldsymbol{L}^p(\Omega))' \simeq \boldsymbol{L}^{p'}(\Omega)$ and $(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))' \simeq \boldsymbol{L}^{p'}_{\sigma,\tau}(\Omega)$. We recall also that the adjoint of the canonical embeddings I is the Helmholtz projection P. □

Notice that Proposition 2 means the Stokes operator with Navier-type boundary conditions satisfies the assumption (**N1**).

We thus have checked all assumptions that guarantee the existence and uniqueness of local in time mild solution for the Navier-Stokes Problem (11). As a result applying Theorem1 to the Stokes operator A_p with $\boldsymbol{E}^p = \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ we have the following theorem :

Theorem 3 (Existence and uniqueness)**.** *Let* $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega),\ p \geq 3$. *There is a* $T_0 > 0$ *and a unique mild solution of* (11) *on* $[0, T_0)$ *such that*

$$\boldsymbol{u} \in BC([0, T_0);\ \boldsymbol{L}^p_{\sigma,\tau}(\Omega)) \cap L^q(0, T_0;\ \boldsymbol{L}^r_{\sigma,\tau}(\Omega))$$

with

$$q > p,\ \ r > p, \qquad \frac{2}{q} + \frac{3}{r} = \frac{3}{p}.$$

Moreover there is a positive constant ε *such that if* $\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p_{\sigma,\tau}(\Omega)} \leq \varepsilon$ *then* T_0 *can be taken as infinity for* $p = 3$.

The mild solution obtained above is a classical solution. More precisely we have the following theorem

Theorem 4. *Let* $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ *with* $p \geq 3$. *There exists a maximal interval of time* $T_* \in]0, T[$ *such that the unique solution* $\boldsymbol{u}(t)$ *of Problem* (11) *satisfies* $\boldsymbol{u} \in C((0, T_*], \mathbf{D}(A_p)) \cap C^1((0, T_*];\ \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$.

The proof of Theorem 4 follows arguments similar to those in [7]. First we prove that the solution $\boldsymbol{u}(t)$ is in $\mathbf{D}(A_p^\alpha)$ for all $t \in (0, T_*]$ and for all $0 < \alpha < 1$. Then we deduce that $\boldsymbol{u} \in C((0, T_*], \mathbf{D}(A_p)) \cap C^1((0, T_*];\ \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$.

Theorem 5. *Let* $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ *with* $p \geq 3$. *There exists a maximal interval of time* $T_* \in]0, T_0[$ *such that the unique solution* $\boldsymbol{u}(t)$ *of Problem* (11) *is in* $C((0, T_*];\ \mathbf{D}(A_p^\alpha))$ *for all* $0 < \alpha < 1$. *Moreover the solution* $\boldsymbol{u}(t)$ *satisfies*

$$\|(I + A_p)^\alpha \boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq K_\alpha t^{-\alpha}, \tag{15}$$

for some constant $K_\alpha > 0$.

Proof. First we note that thanks to Theorem 3, there exists a $T_0 > 0$ such that the unique solution $\boldsymbol{u}(t)$ of Problem (11) obtained in Theorem 3 is in $BC([0, T_0);\ \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$. Moreover for all $0 \leq t \leq T_0,\ \boldsymbol{u}(t)$ is given by

$$\boldsymbol{u}(t) = \boldsymbol{u}_0(t) + \boldsymbol{S}\boldsymbol{u}(t) \tag{16}$$

with

$$\boldsymbol{u}_0(t) = e^{-tA_p}\boldsymbol{u}_0 \qquad \text{and} \qquad \boldsymbol{S}\boldsymbol{u}(t) = \int_0^t e^{-(t-s)A_p}\boldsymbol{F}\boldsymbol{u}(s)\,\mathrm{d}\,s, \tag{17}$$

with $\boldsymbol{F}\boldsymbol{u} = -P(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$. In addition, thanks to [6, Theorem 1], we know that by construction there exists a sequence $(\boldsymbol{u}_m(t))_{m\geq 0}$ such that $(\boldsymbol{u}_m)_m$ converges to $\boldsymbol{u}$ in $BC([0, T_0];\ \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$ and $\boldsymbol{u}_m(t)$ is defined recursively by

$$\boldsymbol{u}_0(t) = e^{-tA_p}\boldsymbol{u}_0, \qquad \forall\, m \geq 1, \qquad \boldsymbol{u}_{m+1}(t) = \boldsymbol{u}_0(t) + \boldsymbol{S}\boldsymbol{u}_m(t). \tag{18}$$

Since e^{-tA_p} is a bounded analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, then $\boldsymbol{u}_0(t) \in \mathbf{D}(A_p) \hookrightarrow \mathbf{D}(A_p^\alpha)$ and

$$\|(I+A_p)^\alpha \boldsymbol{u}_0(t)\|_{\boldsymbol{L}^p(\Omega)} = \|(I+A_p)^\alpha e^{-tA_p}\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} = e^t\|(I+A_p)^\alpha e^{-t(I+A_p)}\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \leq K_{\alpha 0}t^{-\alpha}, \quad (19)$$

with

$$K_{\alpha 0} = \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \sup_{0<t\leq T_0} e^t t^\alpha \|(I+A_p)^\alpha e^{-t(I+A_p)}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))}. \quad (20)$$

Proceeding by induction, in the same way as in the proof of [7, Theorem 2.3], we prove that for all $m \geq 1$, $\boldsymbol{u}_m(t) \in \mathbf{D}(A_p^\alpha)$ for all $0 < t \leq T_0$ and satisfies

$$\forall\ \alpha,\ \ 0<\alpha<1, \qquad \|(I+A_p)^\alpha \boldsymbol{u}_m(t)\|_{\boldsymbol{L}^p(\Omega)} \leq K_{\alpha m}t^{-\alpha}, \quad (21)$$

for some constant $K_{\alpha m} > 0$. Furthermore as in the proof [7, Theorem 2.3] under some assumptions on $K_{\alpha 0}$ one has for all $m \geq 0$

$$\|(I+A_p)^\alpha \boldsymbol{u}_m(t)\|_{\boldsymbol{L}^p(\Omega)} \leq K_\alpha t^{-\alpha}, \quad (22)$$

for some constant K_α independent of m. As a result, for all $0 < T < T_0$ the sequence $(\boldsymbol{u}_m(t))_{m\geq 0}$ is bounded in $\mathbf{D}(A_p^\alpha)$ and thus it converges weakly in $\mathbf{D}(A_p^\alpha)$ to a function denoted by $\boldsymbol{v}(t)$ and $(I+A_p)^\alpha \boldsymbol{u}_m(t)$ converges weakly to $(I+A_p)^\alpha \boldsymbol{v}(t)$ in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. In the other hand $\boldsymbol{u}_m(t)$ converges to $\boldsymbol{u}(t)$ in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ thus $\boldsymbol{u}(t) = \boldsymbol{v}(t)$ and $\boldsymbol{u}(t) \in \mathbf{D}(A_p^\alpha)$ for all $0 < t < T_0$. Moreover as in the proof of [7, Theorem 2.3] one can show the existence of $T_* > 0$ such that $\boldsymbol{u} \in C((0,T_*];\ \mathbf{D}(A_p^\alpha))$.

To prove estimate (15) observe that since $(I+A_p)^\alpha \boldsymbol{u}_m(t)$ converges weakly to $(I+A_p)^\alpha \boldsymbol{u}(t)$ in $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, then

$$\|(I+A_p)^\alpha \boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \ \leq\ \liminf_m \|(I+A_p)^\alpha \boldsymbol{u}_m(t)\|_{\boldsymbol{L}^p(\Omega)} \ \leq\ K_\alpha t^{-\alpha}.$$

Thus one has estimate (15). □

The next step is to prove that the solution $\boldsymbol{u}$ of Problem (11) is in $C((0,T_*];\ \mathbf{D}(A_p))$. Since the Stokes operator generates a bounded analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ then $\boldsymbol{u}_0(t)$ defined in (17) is in $\mathbf{D}(A_p)$ for all $t > 0$. It remains to prove that $\boldsymbol{Su}(t)$ defined in (17) is in $\mathbf{D}(A_p)$ for all $0 < t \leq T_*$. The proof is done in three steps. First we prove that $(I+A_p)^\alpha \boldsymbol{u}$, $0 < \alpha < 1$, is Hölder continuous on every interval $[\varepsilon,\ T_*]$. This gives us that the non-linear term $\boldsymbol{Fu}$ is also Hölder continuous on every interval $[\varepsilon,\ T_*]$ and thus $\boldsymbol{u} \in \mathbf{D}(A_p)$ for all $0 < t \leq T_*$.

Proposition 6. *Let* $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ *with* $p \geq 3$ *and let* $\boldsymbol{u}(t)$ *be the unique solution of Problem* (11). *Then :*
(i) *For all* $0 < \alpha < 1$, *the function* $(I+A_p)^\alpha \boldsymbol{u}$ *is Hölder continuous on every interval* $[\varepsilon, T_*]$ *with* $0 < \varepsilon < T_*$.
(ii) *The nonlinear function* $\boldsymbol{Fu}$ *given by* (12) *is Hölder continuous on every interval* $[\varepsilon, T_*]$ *with* $0 < \varepsilon < T_*$.

Proof. To prove the Hölder continuity of $(I+A_p)^\alpha \boldsymbol{u}$ and $\boldsymbol{Fu}$ we use Theorem 5 and proceed in the same way as in [7] and in [5]. □

Now we can prove our main result in this paper

Proof of Theorem 4. First we recall that the solution $\boldsymbol{u}$ is given explicitly by (16). We recall also that since e^{-tA_p} is an analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ then $\boldsymbol{u}_0(t) \in \mathbf{D}(A_p)$ for all $t > 0$. It suffices to verify that $\boldsymbol{Su}(t) \in \mathbf{D}(A_p)$ for all $t \in (0,\, T_*]$ which is a consequence of Proposition 6 and [10, Chapter 4, Corollary 3.3]. Moreover, thanks to [5, Lemma 2.14] one has $\boldsymbol{u} \in C^1((0,T];\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega))$ this ends the proof. □

References

[1] Al Baba, H., Amrouche, C., and Escobedo, M. Analyticity of semi-group generated by the stokes operator with navier-type boundary condition in L^p-space. *Contemporary Mathematics*.

[2] Al Baba, H., Amrouche, C., and Escobedo, M. Navier-stokes problem with navier-type boundary condition in L^p-spaces. *In preparation*.

[3] Al Baba, H., Amrouche, C., and Escobedo, M. Semi-group theory for the stokes operator with navier-type boundary condition in L^p-spaces. *Submitted*.

[4] Engel, K. J., and Nagel, R. *One-parameter semigroups for linear evolution equations*, vol. 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[5] Fujita, H., and Kato, T. On the Navier-Stokes initial value problem. I. *Arch. Rational Mech. Anal. 16* (1964), 269–315.

[6] Giga, Y. Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system. *J. Differential Equations 62*, 2 (1986), 186–212.

[7] Giga, Y., and Miyakawa, T. Solutions in L_r of the Navier-Stokes initial value problem. *Arch. Rational Mech. Anal. 89*, 3 (1985), 267–281.

[8] Hopf, E. Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen. *Math. Nachr. 4* (1951), 213–231.

[9] Kato, T., and Fujita, H. On the nonstationary Navier-Stokes system. *Rend. Sem. Mat. Univ. Padova 32* (1962), 243–260.

[10] Pazy, A. *Semi-groups of linear operators and applications to partial differential equations*. Department of Mathematics, University of Maryland, College Park, Md., 1974. Department of Mathematics, University of Maryland, Lecture Note, No. 10.

H. Al Baba and C. Amrouche
Laboratoire de Mathématiques et de leurs Applications de Pau, CNRS UMR 5142
Université de Pau et des Pays de l'Adour
Av. de l'Université, 64013 Pau Cedex, France
{hind.albaba, Cherif.amrouche} @univ-pau.fr

H. Al Baba and M. Escobedo
Departamento de Matemáticas Facultad de Ciencias y Tecnología
Universidad del País Vasco Barrio Sarriena s/n,
48940 Lejona (Vizcaya), Spain
miguel.escobedo@ehu.es