Monografías Matemáticas *García de Galdeano* **39**, 155–166 (2014)

On the stability of Hamiltonian Dynamical Systems

Víctor Lanchares

Abstract. One of the key questions in the analysis of a dynamical system is the characterization of stability properties of equilibrium solutions. From these properties it is possible to deduce some aspects of local dynamics that, in some cases, can be extended to understand global dynamics. The goal of this work is to give a summary of well known results about stability of equilibrium points in Hamiltonian systems as well as some open problems related to degenerate cases in the presence of resonances.

Keywords: equilibria, stability, resonances.

AMS classification: 70H08, 70H14, 37J25.

§1. Introduction

The scientific revolution that took place after Newton's work brought the development of Celestial Mechanics, a branch of Astronomy dealing with the motion of celestial bodies subject to Newton's gravitational law. It is in this context where a very old human concern could be posed in mathematical terms: *Do the Earth and the rest of the planets remain in their orbits or they experience collisions or escape away in the future?* In fact it is asked about the stability of the solar system and, from a mathematical point of view, about the stability of the solutions of a system of differential equations. Since then, the theory of stability of differential equations has experience a great development and can be considered the origin of the modern theory of dynamical systems [5].

To begin with, we need a precise definition of the term stability, because it can have different meanings, depending on what properties of a system we are interested on. For instance, in the pioneering works of Poisson, Lagrange and Laplace a system of particles is considered stable if its configuration returns close to its initial position over and over again or also, a system is stable if their solutions are bounded [5]. We will use the definition of stability given by Lyapunov [10], which is the most frequently used and, for the special case of equilibrium solutions, it reads as

Definition 1. Let be $\mathbf{x}^*$ an equilibrium point of an autonomous system of differential equations

$$\dot{\mathbf{x}} = f(\mathbf{x}), \quad \mathbf{x} \in \Omega \subset \mathbb{R}^n,$$

that is $f(\mathbf{x}^*) = 0$. Denote by $\phi(t, \mathbf{x})$ a solution verifying $\phi(0, \mathbf{x}) = \mathbf{x}$. We say that the equilibrium solution $\mathbf{x}^*$ is stable (in the future) in the sense of Lyapunov if, given $\epsilon > 0$, there exists $\delta > 0$ such that for all $\mathbf{x} \in B_\delta(\mathbf{x}^*)$ the solution $\phi(t, \mathbf{x})$ is defined for all $t \geq 0$ and $\phi(t, \mathbf{x}) \in B_\epsilon(\mathbf{x}^*)$ for all $t \geq 0$. Otherwise, the equilibrium is unstable. Moreover, if $\mathbf{x}^*$ is stable and

$$\lim_{t\to\infty} \phi(t, \mathbf{x}) = \mathbf{x}^*$$

for all $\mathbf{x} \in B_\delta(\mathbf{x}^*)$, the equilibrium is said to be asymptotically stable.

It was Lyapunov [10] who gave the first results about the stability of equilibrium points for a system of differential equations. In fact, he provided us with two different approaches. The first one is based on the construction of appropriate functions, which are positive definite, in a neighbourhood of the equilibrium position. Then, we have the following theorem

Theorem 1. *If there exists a positive definite function ψ in a neighbourhood U of $\mathbf{x}^*$, an equilibrium point of the system of differential equations $\dot{\mathbf{x}} = f(\mathbf{x})$, and $d\psi/dt$, computed along the solutions, satisfies $d\psi(\mathbf{x})/dt \leq 0$ if $\mathbf{x} \in U$, then $\mathbf{x}^*$ is stable. If the previous inequality is in the strict sense, then $\mathbf{x}^*$ is asymptotically stable.*

This theorem is an extension of a previous result of Dirichlet [6, 13] that also considers appropriate functions in a domain containing the equilibrium point and it is connected with the existence of positive definite integrals

Theorem 2. *If there exists a positive definite integral ψ of the system of differential equations $\dot{\mathbf{x}} = f(\mathbf{x})$ in a neighborhood of the equilibrium point $\mathbf{x}^*$, then $\mathbf{x}^*$ is stable.*

It is worth noting that Theorem 2 is easier to apply than Theorem 1, as it provides a way to construct a suitable function that yields the stability of the equilibrium position. Indeed, it is very useful when dealing with systems with conserved quantities, as they are Hamiltonian systems.

A collection of theorems exists establishing sufficient conditions for the instability of an equilibrium point, by a similar approach of the previous theorems. These conditions are weaker, as it is enough that a trajectory leaves a small neighbourhood of $\mathbf{x}^*$ to ensure instability. We mention here a result due to Chetaev [4], may be the most used to prove instability

Theorem 3. *If there exists a function ψ defined in a neighborhood of $\mathbf{x}^*$ such that there is a domain $\psi > 0$ in an arbitrarily small vicinity of $\mathbf{x}^*$ and $d\psi/dt$, along the solutions, is positive in the domain $\psi > 0$, then $\mathbf{x}^*$ is unstable.*

The second approach of Lyapunov rests on the idea of linearization, that is, consider the first order power series expansion of the differential equations in a neighbourhood of the equilibrium point $\mathbf{x}^*$. As a linear system can be solved in closed form, its stability properties can be readily deduced. The next question is to see how the stability properties of the linear system are connected with those of the full system. In this way, Lyapunov proved the following result

Theorem 4. *Let us consider the system of differential equations $\dot{\mathbf{x}} = f(\mathbf{x})$, and $\mathbf{x}^*$ an equilibrium point. If f is twice differentiable in a neighbourhood of $\mathbf{x}^*$, consider the linear system*

$$\dot{\mathbf{x}} = Df(\mathbf{x}^*)x$$

If all the eigenvalues of $Df(\mathbf{x}^*)$ have negative real part, then $\mathbf{x}^*$ is asymptotically stable. If, at least, one of the eigenvalues of $Df(\mathbf{x}^*)$ has positive real part, then $\mathbf{x}^*$ is unstable.

There is a critical case, in the terminology of Lyapunov, when all the eigenvalues have zero real part. In this case, Theorem 4 does not apply and the first approach of Lyapunov or new appropriate results are necessary. To this critical case belongs a wide class of dynamical systems, the Hamiltonian systems, which are the subject of the rest of the paper.

§2. Hamiltonian systems

A Hamiltonian system is a system of differential equations of the form

$$\dot{x}_j = \frac{\partial H}{\partial y_j}, \quad \dot{y}_j = -\frac{\partial H}{\partial x_j}, \quad j = 1, \ldots, n, \tag{2.1}$$

with $H = H(x_1, \ldots, x_n, y_1, \ldots, y_n)$ a well behaved function, called the *Hamiltonian function*, we suppose to be analytic. The variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are said to be conjugate to each other. The former are the *positions* and the later are the *momenta*, and n, the number of positions and momenta, is referred as the number of degrees of freedom.

The Hamiltonian system (2.1) can be written in a compact form, through the $2n \times 2n$ skew symmetric matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Indeed, let be $\mathbf{z} \in \mathbb{R}^{2n}$ defined as $\mathbf{z} = (x_j, y_j)$, then

$$\dot{\mathbf{z}} = J \cdot \nabla H, \tag{2.2}$$

where ∇H is the gradient of H at $\mathbf{z}$. From (2.1) or (2.2) if follows that H is a first integral and, therefore, it remains constant along the solutions

$$\frac{dH(\mathbf{z}(t))}{dt} = \nabla H(\mathbf{z}(t))^T J \, \nabla H(\mathbf{z}(t)) = 0.$$

To find another first integrals, it is interesting to perform changes of variables in order to simplify as much as possible the resulting equations (2.2). However, not every change of variables is admissible, it is necessary to preserve the Hamiltonian structure. In this way, let us consider the change of variables given by

$$\zeta = \phi(\mathbf{z}),$$

where $\zeta = (\xi_i, \eta_i)$. It must be

$$\dot{\zeta} = J \, \nabla H(\zeta).$$

Set $\bar{H}(\zeta) = H(\mathbf{z})$ and let be $Q = Q(\mathbf{z})$ the Jacobian matrix of $\phi(\mathbf{z})$, then

$$\nabla H(\mathbf{z}) = Q^T \nabla \bar{H}(\zeta),$$

so that

$$\dot{\zeta} = QJQ^T \nabla \bar{H}(\zeta).$$

Thus, if $Q \, J \, Q^T = \mu J$ for some $\mu \neq 0$ the new system is Hamiltonian with Hamiltonian function

$$H(\zeta) = \mu H(\phi^{-1}(\zeta)).$$

If Q satisfies the previous relation we say that the transformation is μ-symplectic or canonical with multiplier μ. For example, the mapping

$$(I_1, \ldots, I_n, \theta_1, \ldots \theta_n) \mapsto (x_1, \ldots, x_n, y_1, \ldots, y_n)$$

defined by

$$x_i = \sqrt{2I_i}\cos\theta_i, \qquad y_i = \sqrt{2I_i}\sin\theta_i, \qquad (i = 1, \ldots, n)$$

is a symplectic or canonical transformation with multiplier 1, and the new variables (I_j, θ_j) are usually named *action-angle* variables.

Action-angle variables, as defined above or in a similar way, are very useful, as they help to describe the structure of solutions of a Hamiltonian system. Indeed, let us suppose that we are able to find a set of action-angle variables such that the Hamiltonian function depends only on the actions, that is

$$H = H(I_1, \ldots, I_n). \tag{2.3}$$

Thus, we have, from (2.1),

$$I_j = I_{j0}, \quad \theta_j = \omega_j(I_1, \ldots, I_n)t + \theta_{j0}, \quad j = 1, \ldots, n.$$

As a consequence, the phase space is foliated by invariant tori $I = const.$, where solutions are located. The type of motion in each torus is defined by the actions and, in general, is conditionally periodic, provided that the frequencies ω_j are rational independent. In such case we say that the corresponding torus is nonresonant. Let us introduce some definitions about a Hamiltonian system defined by (2.3).

Definition 2. The Hamiltonian system, defined by (2.3), is called nondegenerate if the frequencies ω_j are functionally independent, that is if

$$\det\left(\frac{\partial\omega}{\partial I}\right) = \det\left(\frac{\partial^2 H}{\partial I^2}\right) \neq 0.$$

Definition 3. We say that (2.3) is isoenergetically nondegenerate if

$$\det\begin{bmatrix} \left(\frac{\partial^2 H}{\partial I^2}\right) & \left(\frac{\partial H}{\partial I}\right) \\ \left(\frac{\partial H}{\partial I}\right) & 0 \end{bmatrix} \neq 0$$

These nondegeneracy definitions are the base of the celebrated KAM theory about the conservation of conditionally periodic motions in perturbed Hamiltonian systems [1]. The main result can be stated as follows

Theorem 5. *Let us consider the perturbed system*

$$H(\mathbf{I}, \boldsymbol{\theta}) = H_0(\mathbf{I}) + \epsilon H_1(\mathbf{I}, \boldsymbol{\theta}).$$

If the unperturbed system $H(\mathbf{I}, \boldsymbol{\theta}) = H_0(\mathbf{I})$ is nondegenerate or isoenergetically nondegenerate, then, for a sufficiently small perturbation, most nonresonant invariant tori do not vanish, but are only slightly deformed. So, in the phase space of the perturbed system there are invariant tori densely filled by conditionally periodic phase orbits, winding around them with a number of independent frequencies equal to the number of degrees of freedom.

§3. Stability of Hamiltonian systems

The most simple Hamiltonian system is a linear one, defined as

$$\dot{\mathbf{z}} = J\nabla H(\mathbf{z}) = A\mathbf{z}, \tag{3.1}$$

where A is a $2n \times 2n$ real matrix. In this case, there exists a real symmetric matrix S such that $A = JS$ and H is the quadratic form

$$H(\mathbf{z}) = \frac{1}{2}\mathbf{z}^T S\,\mathbf{z}.$$

For such systems, if λ is an eigenvalue of matrix A, then $-\lambda$, $\bar{\lambda}$ and $-\bar{\lambda}$ also are eigenvalues. As $\mathbf{z} = 0$ is an equilibrium solution of (3.1), it follows, from theorem 4, that a necessary condition for stability is that all the eigenvalues of A have zero real part, otherwise the equilibrium is unstable.

Now, let us consider a nonlinear Hamiltonian system and let us suppose, without lost of generality, that $\mathbf{z} = 0$ is an equilibrium point. Let us suppose $H(\mathbf{z})$ to be analytic in a vicinity of $\mathbf{z} = 0$, so that

$$H(\mathbf{z}) = H_2(\mathbf{z}) + H_3(\mathbf{z}) + \dots,$$

where $H_k(\mathbf{z})$ is a homogeneous polynomial of k degree in $\mathbf{z}$. In particular

$$H_2(\mathbf{z}) = \frac{1}{2}\mathbf{z}^T S\,\mathbf{z}, \qquad S = D^2 H(0),$$

then, if all the eigenvalues of $A = JS$ have zero real part and S is positive (or negative) definite, from theorem 2, it follows that $\mathbf{z} = 0$ is stable.

A critical situation takes place when all the eigenvalues of A have zero real part and S is not positive (or negative) definite. For this case, new stability results, derived from KAM theory (Theorem 5), apply for two degrees of freedom Hamiltonian systems [1, 13, 15].

Theorem 6 (Arnold-Moser)**.** *Let us consider a two degrees of freedom Hamiltonian system*

$$H = H_2 + H_4 + \cdots + H_{2N} + H^*,$$

where $H_{2k}(z)$ ($k = 1, 2, \dots, N$) is a homogeneous polynomial of degree k in the action variables

$$I_j = \frac{1}{2}(x_j^2 + y_j^2), \quad j = 1, 2.$$

Moreover, suppose that the quadratic part is

$$H_2 = \omega_1 I_1 - \omega_2 I_2.$$

If for some $k = 2, \dots N$, $D_{2k} = H_{2k}(\omega_2, \omega_1) \neq 0$, then the origin is stable, provided ω_1 and ω_2 are rationally independent.

It is worth noting that in the hypothesis of Theorem 6 the structure of the Hamiltonian function is very particular. This indicates that it has been brought to *normal form*. Indeed, it satisfies

$$\{H_{2k}; H_2\} = 0, \quad 1 \le k \le N,$$

where $\{-;-\}$ stands for the Poisson bracket, defined as

$$\{F;G\} = \frac{\partial F}{\partial x_1}\frac{\partial G}{\partial y_1} + \frac{\partial F}{\partial x_2}\frac{\partial G}{\partial y_2} - \frac{\partial F}{\partial y_1}\frac{\partial G}{\partial x_1} - \frac{\partial F}{\partial y_2}\frac{\partial G}{\partial x_2}.$$

In addition, we note that Theorem 6 does not apply if there are integers numbers m_1 and m_2 such that

$$m_1\omega_1 + m_2\omega_2 = 0, \qquad |m_1| + |m_2| > 0.$$

If the above relation is fulfilled we say that there is a resonance of order $r = |m_1| + |m_2|$.

To handle the stability of an equilibrium position of a Hamiltonian system in the presence of resonances, a case study is necessary. In this way, Markeev established appropriate results for third and fourth order resonances [12] and Sokolskii gave conditions for the case of first and second order resonances [16, 17]. However, the case of second order resonance was not properly solved and recently Lerman [9] and Meyer et al. [14] gave a complete proof.

In the search for a general result including resonant and non resonant cases, Cabral and Meyer [3] formulated a theorem covering a wide range of cases, excluding resonances of first and second order and also some special cases. To formulate this theorem, let us suppose that the linear part of the Hamiltonian function is expressed as

$$H_2 = \omega_1 I_1 - \omega_2 I_2.$$

In addition, let us suppose that a resonance of order $r > 2$ take place. In this way, there exist m and n positive integers, such that

$$n\omega_1 = m\omega_2, \qquad n + m = r.$$

Under these conditions, the normal form of the Hamiltonian function, up to order r, is

$$H = \omega_1 I_1 - \omega_2 I_2 + H_4(I_1, I_2) + \cdots + H_{2l}(I_1, I_2) + H_r(I_1, I_2, n\theta_1 + m\theta_2), \tag{3.2}$$

where $2l = r - 1$ or $2l = r - 2$, if r is odd or even respectively, and H_r is a homogeneous polynomial of degree r in $\sqrt{I_1}$ and $\sqrt{I_2}$ which coefficients are finite Fourier series in the single angle $n\theta_1 + m\theta_2$. Let us introduce the function

$$\Psi(\psi) = H_s(\omega_2, \omega_1, \psi),$$

where $\psi = n\theta_1 + m\theta_2$. Then we have the following result

Theorem 7 (Cabral and Meyer)**.** *Let us suppose that for the Hamiltonian function* (3.2) $D_{2k} = H_{2k}(\omega_2, \omega_1) = 0$*, for* $2 \leq k \leq l$*, otherwise Arnold's theorem guarantees the stability of the origin. Then, if* $\Psi(\psi) \neq 0$ *for all* ψ*, the origin is stable and if* $\Psi(\psi)$ *has a simple zero, the origin is unstable.*

This theorem has a geometric counterpart, a little bit more general, based on the orbits of the truncated Hamiltonian system defined by (3.2). In fact, after expressing the Hamiltonian in normal form, we obtain a one degree of freedom Hamiltonian system, whose orbits live in a phase space made of a collection of open surfaces labelled by the value of H_2. In [7], it is proven that it is enough to analyse the orbits in the surface of the phase space labelled by

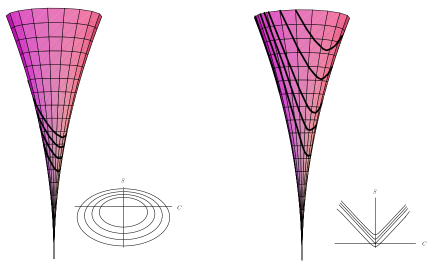

Figure 1: Orbits on the surface of the reduced phase space labelled by $H_2 = 0$. In the left the origin is stable and unstable in the right.

$H_2 = 0$, where the equilibrium position lies. If the orbits are closed in a vicinity of the origin, the equilibrium point is stable, otherwise it is unstable, provided there exists at least one asymptotic trajectory (see Figure 1). This idea can be extended to uncover first and second order resonances [8], but giving only necessary conditions of stability.

Theorem 7, and its geometric counterpart, can be reduced to the following lemma given by Sokolskii [17]

Lemma 8. *Let* $K(s, \varphi, t) = \Phi(\varphi)s^n + O(s^{n+\frac{1}{2}})$, $n = m/2$ *and* $m \geq 3$. *Assume* K *is an analytic function of* $\sqrt{s}, \varphi, t$, τ*-periodic in* φ *and* T*-periodic in* t. *If* $\Phi(\varphi) \neq 0$, *for all* φ, *then the origin* $s = 0$ *is a stable equilibrium for the Hamiltonian system*

$$\dot{s} = \frac{\partial K}{\partial \varphi}, \qquad \dot{\varphi} = -\frac{\partial K}{\partial s}.$$

If $\Phi(\varphi)$ *has a simple zero, i.e., if there exists* φ^* *such that* $\Phi(\varphi^*) = 0$ *and* $\Phi'(\varphi^*) \neq 0$, *then the equilibrium* $s = 0$ *is unstable.*

Indeed, the Hamiltonian function (3.2), in Theorem 7, can be written as the Hamiltonian function K in Lemma 8, where $\Phi(\varphi)$ is replaced by $\Psi(\psi)$ and s by $\omega_1 I_1 + \omega_2 I_2$. Thus, the stability problem comes down to determine the roots of a given function.

§4. Degenerate cases

When all the roots of the function $\Phi(\varphi)$, in Lemma 8, are multiple, we say that there is a degeneracy and new results are needed to determine the stability properties of the origin. The first attempt to solve the problem of degeneracies is to consider the next term in the normal form of the Hamiltonian function in Lemma 8 and try to prove the following.

Lemma 9. *Let* $K(\varphi, s, t) = \Phi_0(\varphi)s^{\alpha_0} + \Phi_1(\varphi)s^{\alpha_1} + O(s^{\alpha_2})$, *where* $\alpha_0 = m/2 \geq 3/2$, $\alpha_1 = n/2 > \alpha_0$, $\alpha_2 = p/2 > \alpha_1$ *and* m, n, p *positive integers. Assume* K *is an analytic function of* $\sqrt{s}, \varphi, t$, τ*-periodic in* φ *and* T*-periodic in* t *and all the zeroes of* $\Phi_0(\varphi)$ *are multiple. Then, if there exists* $\epsilon > 0$ *such that*

- $\Phi_0(\varphi)+\Phi_1(\varphi)r^{\alpha_1-\alpha_0} \neq 0$, *for all* φ, *and* $0 < s < \epsilon$, *then the origin is a stable equilibrium.*

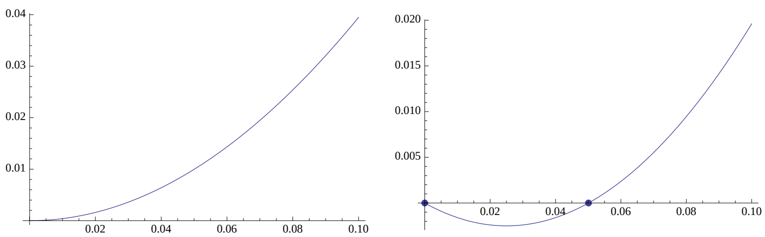

Figure 2: Splitting roots of the function $\sin^2 2\varphi$ when the small perturbation $-s\sin 2\varphi$ is added. In the left, the double root at $\varphi = 0$ and, in the right, the two splitting real roots when $s = 0.1$.

- $\Phi_0(\varphi) + \Phi_1(\varphi)r^{\alpha_1-\alpha_0}$ *has a simple zero,* $\varphi^*(s)$ *for* $0 < s < \epsilon$*, then the origin is unstable.*

Unfortunately this Lemma is not true. More insight is needed about how a multiple root is modified by higher order terms. To this end, let us consider

$$K(s,\varphi,t) = s^2(\sin^2 2\varphi - s\sin 2\varphi) + O(s^4). \tag{4.1}$$

The function $\Phi_0(\varphi) = \sin^2 2\varphi$ has double roots at $\varphi = k\pi/2$, $k \in \mathbb{Z}$ and, for s small enough,

$$F(\varphi, s) = \sin^2 2\varphi - s\sin 2\varphi$$

has simple roots. In Figure 2 it is shown the splitting roots from $\varphi^* = 0$, namely

$$\varphi_1^* = 0, \qquad \varphi_2^* = \frac{1}{2}\arcsin s.$$

However, a close look at the function $F(\varphi, s)$ reveals that the value of the function at the minimum is given by

$$\min F(\varphi, s) = (\sin y - s)\sin y, \quad y = \frac{1}{2}\arcsin\frac{s}{2}.$$

As a consequence,

$$\min F(\varphi, s) = -\frac{s^2}{4}.$$

Thus, the value attained by $F(\varphi, s)$ at the minimum is of the same order as the tail of the Hamiltonian function (4.1). Consequently, next terms in the normal form can destroy the real nature of the splitting roots. For instance, a proper choice of the next term produce either real or complex roots and even multiple roots. This situation is depicted in Figure 3.

This example shows that it is not so easy to decide when a multiple root has been properly split into simple roots, either real or complex. In order to solve this problem we focus on the series expansion of the roots as power series of s. For the case of the above example, we look for roots in the form

$$\varphi = a_0 + a_1 s^{1/2} + a_2 s + \dots \tag{4.2}$$

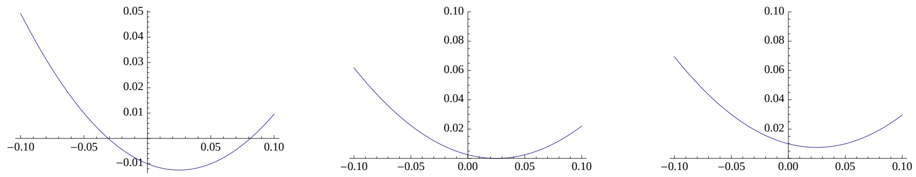

Figure 3: The roots of the function $F(\varphi, s) = \sin^2 2\varphi - s\sin 2\varphi$ when the perturbations $-s^2$, $s^2/4$ and s^2 are added (from left to right). On the left panel there are two real roots, a double root in the central panel and no real roots in the right panel.

where a_0, a_1, a_2, and subsequent coefficients, are determined by substituting (4.2) in $F(\varphi, s)$ and equating to zero. By doing so, up to the first power of s, we obtain the following equations

$$\sin^2 2a_0 = 0, \quad a_1 \sin 2a_0 \cos 2a_0 = 0,$$
$$4a_1^2 \cos^2 2a_0 + 4a_2 \sin 2a_0 \cos 2a_0 - 4a_1^2 \sin^2 2a_0 - \sin 2a_0 = 0,$$

with solutions $a_0 = k\pi/2$, $k \in \mathbb{Z}$, $a_1 = 0$. As it can be seen, up to the resolution allowed by the function $F(\varphi, s)$, that is up to the first power of s, roots have the same power expansion by pairs, so that they are not properly split. However if we add a new term of the form γs^2, and consider the series

$$\varphi = a_0 + a_1 s^{1/2} + a_2 s + a_3 s^{3/2} + s^2 \dots \tag{4.3}$$

we obtain

$$a_0 = k\pi/2,\ k \in \mathbb{Z}, \quad a_1 = 0, \quad a_2 = \frac{(-1)^k \pm \sqrt{1-4\gamma}}{4}$$

and, unless $\gamma = 1/4$, roots are properly split.

Now we are in position to give conditions to generalize Lemma 8. In this way, let us consider a one degree of freedom Hamiltonian system, periodic in time, given by

$$H(s, \phi, t) = s^\alpha \Phi(\varphi, s) + O(s^{\alpha+\gamma+\frac{1}{2}}), \tag{4.4}$$

where

$$\Phi(\varphi, s) = \sum_{i=0}^{2\gamma} \Phi_i(\varphi) s^{\frac{i}{2}}.$$

The functions $\Phi_i(\varphi)$ are 2π-periodic in φ and $\alpha = n/2$, $\gamma = k/2$, where n, k ($n \geq 3$) are integers. Let us assume that $\Phi_0(\varphi)$ has real roots but all of them are multiple. To establish when the multiple roots are properly split, we introduce the next definitions

Definition 4. Let us consider the finite series

$$\varphi_0(s) = \sum_{i=1}^{q} a_j s^{\frac{j}{m}},$$

where q is the maximal integer number such that the equation for a_q is obtained by substitution in $\Phi(\varphi, s) = 0$ and equating terms which order with respect to s is less or equal to γ. We call this finite series *main part* of the root.

Definition 5. We say that a root has a *simple main part* if among roots of equation $\Phi(\varphi, s) = 0$ there is not another one with the same main part.

Remark 1. It is worth noticing that a root with a simple main part is a simple root of equation $\Phi(\varphi, s) = 0$. The converse proposition is not true.

With the help of these definitions, Bardin and Lanchares [2] stated the following instability theorem

Theorem 10. *Let us consider the Hamiltonian system given by* (4.4)*. Suppose that all the real roots of the function* $\Phi_0(\varphi)$ *are multiple. If there exists* s_0 *such that for all* $0 < s < s_0$ *the function* $\Phi(\varphi, s)$ *has a real root* $\varphi^*(s)$ *which has a simple main part and the inequality* $\frac{\partial\Phi}{\partial\varphi}(\varphi^*, s) < 0$ *is hold, then the equilibrium* $s = 0$ *is unstable.*

Proof. The proof is based on Chetaev's theorem. First we construct a function $V(\varphi, s)$ in such a way that there exists a domain D in any arbitrarily small neighbourhood of the origin where $V > 0$ in D and $V = 0$ in the boundary of D. Then, we prove that the derivative of V along the trajectories of the Hamiltonian system is positive in the domain D.

The function $V(\varphi, s)$ is chosen to be

$$V(\varphi, s) = s\left(s^{\frac{2p}{m}} - (\varphi - \varphi^*(s))^2\right),$$

where $p = q + \delta$, being q the highest power of $s^{1/m}$ in the main part of the root $\varphi^*(s)$ and $\delta > 0$ to be fixed in order to have $dV/dt > 0$ in D. On the other hand, the domain D is defined as

$$D = \left\{(\varphi, s, t) \,\middle|\, |\varphi - \varphi^*(s)| < r^{\frac{p}{m}}\right\}.$$

It is clear that $V > 0$ in the domain D and $V = 0$ on the boundary. Moreover,

$$\frac{dV}{dt} = \frac{\partial V}{\partial\varphi}\frac{d\varphi}{dt} + \frac{\partial V}{\partial s}\frac{ds}{dt} + \frac{\partial V}{\partial t} = \frac{\partial V}{\partial\varphi}\frac{\partial H}{\partial s} - \frac{\partial V}{\partial s}\frac{\partial H}{\partial\varphi},$$

and H given by (4.4). Finally,

$$\begin{aligned}\frac{dV}{dt} =& -2(\varphi - \varphi^*(s))s\left[\alpha s^{\alpha-1}\Phi(\varphi, s) + s^{\alpha}\frac{\partial\Phi}{\partial s}(\varphi, s) + O\left(s^{\alpha+\gamma+1/2}\right)\right] \\ &+ \left[\frac{2p+m}{m}s^{\frac{2p}{m}} - (\varphi - \varphi^*(s))^2 + 2s(\varphi - \varphi^*(s))\frac{\partial\varphi^*}{\partial s}\right]\left[-s^{\alpha}\frac{\partial\Phi}{\partial\varphi}(\varphi, s) + O\left(s^{\alpha+\gamma+1/2}\right)\right].\end{aligned} \tag{4.5}$$

Now, it can be shown that it is always possible to choose δ in such a way that the expression (4.5) is positive in the domain D. □

Note that this is an instability theorem, easier to prove than a stability one. Also note that the proof is based on the existence of a simple root, that guaranties the function $\Phi(\varphi, s)$ is not a definite function and attains either positive or negative values.

§5. Conclusions

The stability of equilibrium positions of Hamiltonian systems is an interesting question, not completely solved. In particular, degenerate cases, in the presence of resonances, need further investigation, not only for their mathematical interest but also for their application to real situations, like the stability of rotations and oscillations of a satellite [11]. Moreover, we have deal with two degrees of freedom Hamiltonian systems, for which KAM theory ensures that invariant tori *separate* the phase space. All these facts leave us with several interesting open questions in the search for a complete solution of the stability problem. Here, we state some of these questions that follow from the discussion in Section 4

1. If all the roots of $\Phi(\varphi, s)$ have different main part and all of them are complex, does this imply the stability of the equilibrium point $s = 0$?
2. The role played by odd multiple roots is similar to that of simple roots. Does the existence of an odd multiple root imply instability of the equilibrium point?
3. What happens in a transcendental case, when higher order terms cannot destroy the multiple roots of Φ_0?

Acknowledgements

The author thanks financial support from Ministerio de Ciencia e Innovación of Spain (Project MTM2011–28227-C02.02).

References

[1] Arnold, V., Kozlov, V., and Neishtadt, A. *Mathematical Aspects of Classical and Celestial Mechanics*, 3rd ed., vol. 3 of *Encyclopaedia of Mathematical Sciences*. Springer, New York, 2006.

[2] Bardin, B., and Lanchares, V. Conditions of instability of hamiltonian system in case of degeneration. In *Materialen zum wissenschaftlichen Seminar der Stipendiaten des "Michail Lomonosov"-Programms 2007/2008* (2008), pp. 22–24.

[3] Cabral, H., and Meyer, K. Stability of equilibria and fixed points of conservative systems. *Nonlinearity 12* (1999), 1351–1362.

[4] Chetaev, N. *The Stability of Motion*, 2nd ed. Pergamon Press, New York, 1961.

[5] Diacu, F., and Holmes, P. *Celestial Encounters. The Origins of Chaos and Stability*. Princeton University Press, New Jersey, 1996.

[6] Dirichlet, G. L. *G. Lejeune Dirichlet's Werke*. Cambridge University Press, Cambridge, 2012.

[7] Elipe, A., Lanchares, V., and Pascual, A. On the stability of equilibria in two-degrees-of-freedom hamiltonian systems under resonances. *J. Nonlinear Sci. 19* (2005), 305–319.

[8] Lanchares, V., Pascual, A., and Elipe, A. Determination of nonlinear stability for low order resonances by a geometric criterion. *Regul. Chaotic Dyn. 17* (2012), 307–317.

[9] Lerman, L., and Markova, A. On stability at the hamiltonian hopf bifurcation. *Regul. Chaotic Dyn. 14* (2009), 148–162.

[10] Lyapunov, A. M. *General Problem of the Stability of Motion*. Taylor and Francis Ltd, London, 1992.

[11] Markeev, A., and Bardin, B. On the stability of planar oscillations and rotations of a satellite in a circular orbit. *Celest. Mech. Dyn. Astron. 85*, 1 (2003), 51–66.

[12] Markeev, A. P. Stability of a canonical system with two degrees of freedom in the presence of resonance. *J. Appl. Math. Mech. 39*, 3 (1968), 442–450.

[13] Meyer, K., Hall, G., and Offin, D. *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem*, 2nd ed., vol. 60 of *Applied Mathematical Sciences*. Springer, New York, 2009.

[14] Meyer, K., Palacián, J., and Yanguas, P. Stability of a hamiltonian system in a limiting case. *Regul. Chaotic Dyn. 17* (2012), 24–35.

[15] Siegel, C. L., and Moser, J. K. *Lectures on Celestial Mechanics*. Classics in Mathematics. Springer, New York, 1995.

[16] Sokol'skii, A. On the stability of an autonomous hamiltonian system with two degrees of freedom in the case of equal frequencies. *J. Appl. Math. Mech. 38*, 5 (1974), 791–799.

[17] Sokol'skii, A. G. On the stability of an autonomous hamiltonian system with two degrees of freedom under first order resonance. *J. Appl. Math. Mech. 41* (1977), 20–28.

Víctor Lanchares.
Grupo de Dinámica No Lineal.
Universidad de La Rioja.
vlancha@unirioja.es