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MONOTONICITY OF BIFURCATING BRANCHES FOR THE RADIAL p-LAPLACIAN

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Abstract. We prove the monotonicity of the curve of positive solutions for a p-Laplacian problem in the unit ball. The solution curve bifurcates from the line of trivial solutions at the first eigenvalue of a related p-linear problem.

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§1. Setting and main result

Consider the Dirichlet boundary value problem

$$\begin{cases} -\Delta_p(u) = \lambda f(|x|, u) & \text{in } \Omega, \\ \qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p(u) := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $p > 1$, $\lambda \geqslant 0$, and Ω is the unit ball in $\mathbb{R}^N$, with $N \geqslant 1$. The function f is continuous, such that $f(r, 0) = 0$ for all $r \in [0, 1]$.

We are interested in C^1 radial solutions of this problem, which satisfy

$$\begin{cases} -(r^{N-1}\phi_p(u'))' = \lambda r^{N-1} f(r, u), \quad 0 < r < 1, \\ u'(0) = u(1) = 0, \end{cases} \tag{1.2}$$

where $\phi_p(\xi) := |\xi|^{p-2}\xi$, $\xi \in \mathbb{R}$, $r = |x|$ and $'$ denotes differentiation with respect to r. By a solution of (1.2) will be meant a couple (λ, u), with $\lambda \in \mathbb{R}$ and $u \in C^1[0, 1]$, such that $\phi_p(u') \in C^1[0, 1]$, which satisfies (1.2). Note that, since $f(r, 0) = 0$ for all $r \in [0, 1]$, we have a line of trivial solutions, $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$.

Problem (1.2) was studied in [4] by the author, who proved that, under appropriate assumptions on f, there exist two smooth curves of respectively positive and negative solutions, bifurcating from the line of trivial solutions at the first eigenvalue $\lambda_0 > 0$ of a related p-linear problem (problem (E_0) below). The aim of this note is to prove that the bifurcating branches are monotonous. We will focus here on the branch of positive solutions, the negative branch being handled similarly. The exact hypotheses made in [4] are the following:

(H1) $f(r, \cdot) \in C^1(\mathbb{R})$ for all $r \in [0, 1]$ and $\partial_2 f \in C^0([0, 1] \times \mathbb{R})$;

(H2) $f(r, \xi) > 0$ for $(r, \xi) \in [0, 1] \times \mathbb{R}^*$ and $f(r, 0) \equiv 0$;

(H3) $(p - 1)f(r, \xi) \geqslant \partial_2 f(r, \xi)\xi$ for $(r, \xi) \in [0, 1] \times [0, \infty)$, and there exist $\delta, \epsilon > 0$ such that $(p - 1)f(r, \xi) > \partial_2 f(r, \xi)\xi$ for all $(r, \xi) \in (1 - \delta, 1] \times (0, \epsilon)$.

It follows from (H3) that, for any fixed $r \in [0,1]$, the mapping $\xi \mapsto f(r,\xi)/\phi_p(\xi)$ is decreasing on $(0,\infty)$. Therefore, for each $r \in [0,1]$ there exist $f_0(r)$ and $f_\infty(r)$ such that

$$f(r,\xi)/\phi_p(\xi) \to f_{0/\infty}(r) \quad \text{as } \xi \to 0^+/+\infty,$$

with

$$0 \leqslant f_\infty(r) \leqslant f(r,\xi)/\phi_p(\xi) \leqslant f_0(r), \quad \text{for all } (r,\xi) \in [0,1]\times(0,\infty). \tag{1.3}$$

We will further suppose that $f_0, f_\infty \in C^0[0,1]$, and that (for $p > 2$):

(H4) $\lim_{\xi\to 0^+} |f(\cdot,\xi)/\phi_p(\xi) - f_0|_0 = \lim_{\xi\to 0^+} |\partial_2 f(\cdot,\xi)/\xi^{p-2} - (p-1)f_0|_0 = 0$;

(H5) $\lim_{\xi\to+\infty} |f(\cdot,\xi)/\phi_p(\xi) - f_\infty|_0 = 0$,

where $|\cdot|_0$ is the usual norm of $C^0[0,1]$. In order to state the main result of [4], we need some properties of the p-linear eigenvalue problems

$$\begin{cases} -(r^{N-1}\phi_p(v'))' = \lambda r^{N-1} f_{0/\infty}(r)\phi_p(v), & 0 < r < 1, \\ v'(0) = v(1) = 0, \end{cases} \tag{$\mathrm{E}_{0/\infty}$}$$

which govern the behaviour of small/large solutions of (1.2). The following result follows readily from [6, Sec. 5].

Lemma 1. *If $f_{0/\infty} > 0$ on $[0,1]$ then problem $(\mathrm{E}_{0/\infty})$ has a simple eigenvalue $\lambda_{0/\infty} > 0$ with a corresponding eigenfunction $v_{0/\infty} > 0$ in $[0,1)$, and no other eigenvalue having a positive eigenfunction. Furthermore, $f_\infty \lneqq f_0$ implies $\lambda_0 < \lambda_\infty$.*

It follows from the above hypotheses that $f_0 > 0$ and $0 \leqslant f_\infty \lneqq f_0$. For λ_∞ to be well-defined, we make the following additional assumption.

(H6) Either
(a) $N \geqslant 1$ is arbitrary and $f_\infty > 0$ on $[0,1]$, or
(b) $N = 1$ and $f_\infty \equiv 0$ on $[0,1]$.
If (a) holds, $\lambda_\infty > 0$ is defined in Lemma 1; if (b) holds, we set $\lambda_\infty = \infty$.

We are now in a position to state the main result of [4]. We will use the following short-hand notations throughout the paper:

$$X_p = \{u \in C^1[0,1] : \phi_p(u') \in C^1[0,1],\ u'(0) = u(1) = 0\} \quad \text{and} \quad Y = C^0[0,1].$$

We equip Y with its usual sup-norm, which we denote by $|\cdot|_0$.

Theorem 2 (Theorem 2.3 of [4])**.** *Suppose that $p > 2$. If (H1) to (H6) hold then there exists $u \in C^1((\lambda_0,\lambda_\infty), Y)$ such that $u(\lambda) \in X_p$, $u(\lambda) > 0$ on $[0,1)$ and, for any $\lambda \in (\lambda_0,\lambda_\infty)$, $(\lambda, u(\lambda))$ is the unique non-trivial solution of (1.2). Furthermore,*

$$\lim_{\lambda\to\lambda_0} |u(\lambda)|_0 = 0 \quad \textit{and} \quad \lim_{\lambda\to\lambda_\infty} |u(\lambda)|_0 = \infty. \tag{1.4}$$

Assuming that $f(r,\xi)\xi > 0$ for $(r,\xi) \in [0,1]\times\mathbb{R}^*$ instead of $f(r,\xi) > 0$ in (H2), and with additional assumptions on the behaviour of f for $\xi < 0$, a more general result is proved in [4], providing a second branch, of negative solutions — see [4, Theorem 2.4].

The requirement that $p > 2$ is due to differentiability issues, as mentioned in the proof of Lemma 5 below. (Note that the result is well known for $p = 2$.)

The bifurcation theory for quasilinear boundary value problems has a long-standing history, part of which was recently reviewed in [5], where the results of [4] were put in a broader perspective. We refer the interested reader to [4, 5] and the references in these papers for more details. Let us just emphasize here that most previous bifurcation results on problems similar to (1.1) were obtained by topological methods, yielding only connected sets of solutions. In contrast, our method is purely analytical and provides smooth curves of solutions (of course, under stronger assumptions).

The aim of the present note is to exploit the differentiability of the mapping u given by Theorem 2 in order to prove the following monotonicity result.

Theorem 3. *Under the hypotheses of Theorem 3, the mapping*

$$\lambda \mapsto |u(\lambda)|_0, \quad \lambda \in (\lambda_0, \lambda_\infty),$$

is strictly increasing.

Under the hypotheses of Theorem 2.4 in [4], an analogous result can be proved for the branch of negative solutions given by this theorem. The proof being very similar, we shall focus here on positive solutions only.

For the sake of completeness, before proving Theorem 3, we will briefly sketch the proof of Theorem 2.

§2. Proof of Theorem 2

Let us denote by $f : Y \to Y$ the Nemitskii mapping induced by the function f, that is, $f(u)(r) := f(r, u(r))$, $r \in [0, 1]$. Then $f : Y \to Y$ is bounded and continuous. We will use a similar notation for other Nemitskii mappings below.

To prove Theorem 2, it is convenient to consider the integral form of (1.2),

$$u = S_p(\lambda f(u)), \quad (\lambda, u) \in \mathbb{R} \times Y, \tag{2.1}$$

where $S_p : C^0[0, 1] \to C^1[0, 1]$ is the inverse of (minus) the radial p-Laplacian, explicitly given by

$$S_p(h)(r) = \int_r^1 \phi_{p'}\Big(\int_0^s \Big(\frac{t}{s}\Big)^{N-1} h(t)\, dt\Big)\, ds, \quad h \in C^0[0, 1], \tag{2.2}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. This operator is continuous, bounded and compact. Before we can explain the proof of Theorem 2, we need a result about the differentiability of S_p, which depends on the value of $p > 1$. Theorem 4 relies on the related work [1], and we borrow the following notation from there:

$$B_p = \begin{cases} C^1[0, 1], & 1 < p \leqslant 2, \\ W^{1,1}(0, 1), & p > 2. \end{cases} \tag{2.3}$$

Theorem 4 (Theorem 3.5 of [4]).

(i) Suppose $1 < p < 2$. Then $S_p : C^0[0,1] \to B_p$ is C^1, and for all $h, \bar{h} \in C^0[0,1]$,

$$DS_p(h)\bar{h}(s) = \frac{1}{p-1}\int_r^1 |u(h)'(s)|^{2-p} \int_0^s \left(\frac{t}{s}\right)^{N-1} \bar{h}(t)\,dt\,ds, \tag{2.4}$$

where $u(h) = S_p(h)$. Furthermore,

$$\begin{aligned} &v = DS_p(h)\bar{h} \iff v \in B_p \text{ and } v \text{ satisfies}\\ &-(p-1)(r^{N-1}|u(h)'(r)|^{p-2}v'(r))' = r^{N-1}\bar{h}(r), \quad 0 < r < 1, \quad v'(0) = v(1) = 0. \end{aligned} \tag{2.5}$$

(ii) Suppose $p > 2$ and let $h_0 \in C^0[0,1]$ be such that $u(h_0)'(r) = 0 \Rightarrow h_0(r) \neq 0$. Then there exists a neighbourhood V_0 of h_0 in $C^0[0,1]$ such that the mapping

$$h \mapsto |u(h)'|^{2-p} : V_0 \to L^1(0,1)$$

is continuous, $S_p : V_0 \to B_p$ is C^1, and DS_p satisfies (2.4) and (2.5), for all $h \in V_0$, $\bar{h} \in C^0[0,1]$.

Proof. The proof follows closely that of Theorem 3.4 in Binding and Rynne [1]. In view of the definition of S_p in (2.2), the main difficulty is that, for $1 < p' < 2$, the Nemistkii mapping $u \mapsto \phi_{p'}(u)$ does not map $C^1[0,1]$ into itself — this is due to the lack of differentiability of $\phi_{p'}(s)$ at $s = 0$. Nevertheless, if $g \in C^1[0,1]$ has only simple zeros, then $\phi_{p'}$ maps a neighbourhood of g in $C^1[0,1]$ continuously into $L^1(0,1)$. This result [1, Lemma 2.1] is the key ingredient in the proof of Theorem 4. □

Note that a similar result was stated by García-Melián and Sabina de Lis [3, Theorem 5] but their proof seems to be incomplete (see [5, Remark 5.4] for more details).

The proof of Theorem 2 can now be carried out in two steps:

(1) local bifurcation from $(\lambda_0, 0)$ in $\mathbb{R} \times Y$;
(2) global continuation and asymptotic analysis.

Step 1. The first step is a continuous local bifurcation result à la Crandall-Rabinowitz, which is essentially due to García-Melián and Sabina de Lis [3]. However, we consider a slightly more general setting, where bifurcation from the line of trivial solutions occurs at the first eigenvalue of the weighted problem (E_0) (they have $f_0 \equiv 1$ in [3]). Nevertheless, our proof follows closely the arguments in [3].

We normalize the eigenvector v_0 of (E_0) so that $\int_0^1 r^{N-1} f_0 |v_0|^p\,dr = 1$ and we define the subspace

$$Z = \Big\{ z \in Y : \int_0^1 r^{N-1} f_0 |v_0|^{p-2} v_0 z\,dr = 0 \Big\}.$$

Note that

$$Y = \operatorname{span}\{v_0\} \oplus Z. \tag{2.6}$$

The local bifurcation from $(\lambda_0, 0)$ now follows by applying the implicit function theorem as stated in [2, Appendix A] to the function $G : \mathbb{R}^2 \times Z \to Y$ defined by

$$G(s, \lambda, z) = \begin{cases} v_0 + z - S_p(\lambda f(sv_0 + sz)/\phi_p(s)), & s \neq 0, \\ v_0 + z - S_p(\lambda f_0 \phi_p(v_0 + z)), & s = 0. \end{cases}$$

Lemma 5 (Lemma 5.1 of [4]). *There exist $\varepsilon > 0$, a neighbourhood U of $(\lambda_0, 0)$ in $\mathbb{R} \times Z$ and a continuous mapping $s \mapsto (\lambda(s), z(s)) : (-\varepsilon, \varepsilon) \to U$ such that $(\lambda(0), z(0)) = (\lambda_0, 0)$ and*

$$\{(s, \lambda, z) \in (-\varepsilon, \varepsilon) \times U : G(s, \lambda, z) = 0\} = \{(s, \lambda(s), z(s)) : s \in (-\varepsilon, \varepsilon)\}. \tag{2.7}$$

Proof. Let us first remark that we need $p \geqslant 2$ here for the Nemitskii mapping $z \mapsto \phi_p(v_0 + z)$ to be differentiable. We are thus in case (ii) of Theorem 4, from which follows that S_p is C^1 in a neighbourhood of $\lambda_0 f_0 \phi_p(v_0)$ in Y. Indeed, (E_0) is equivalent to

$$v_0 = S_p(\lambda_0 f_0 \phi_p(v_0)),$$

and we have $\lambda_0 f_0(0)\phi_p(v_0(0)) > 0$, where $r = 0$ is the only zero of v_0' by Lemma 1. This enables one to verify the regularity properties required by the implicit function theorem [2, Theorem A]. To apply this theorem, one needs to check the usual non-degeneracy condition, namely that the linear mapping $D_{(\lambda,z)}G(0, \lambda_0, 0) : \mathbb{R} \times Z \to Y$ be an isomorphism. In view of (2.6), an inspection of the Fréchet derivative $D_{(\lambda,z)}G(0, \lambda_0, 0)$ shows that this condition is equivalent to the invariance of the subspace Z under the mapping

$$\bar{z} \mapsto L\bar{z} := \lambda_0 (p^*)^{-1} DS_p(\lambda f_0 \phi_p(v_0)) f_0 |v_0|^{p-2} \bar{z},$$

where $p^* = p' - 1$, with $\frac{1}{p} + \frac{1}{p'} = 1$. Using the properties of the derivative DS_p, the relation $L\bar{z} = z$ can be expressed as

$$\begin{cases} -(r^{N-1}|v_0'|^{p-2}z')' = \lambda_0 r^{N-1} f_0 |v_0|^{p-2} \bar{z}, & 0 < r < 1, \\ \quad z'(0) = z(1) = 0. \end{cases} \tag{2.8}$$

Then, multiplying both sides of the equation by v_0 and integrating by parts easily shows that $z \in Z$, and the proof of Lemma 5 can be completed. □

Step 2. Let us denote by $\mathcal{S} \subset \mathbb{R} \times Y$ the set of positive solutions of (2.1). We define a function $F : [0, \infty) \times Y \to Y$ by

$$F(\lambda, u) := u - S_p(\lambda f(u)) = 0, \quad (\lambda, u) \in [0, \infty) \times Y,$$

so that (1.2) is now equivalent to $F(\lambda, u) = 0$. It follows from hypothesis (H2) that any non-trivial solution (λ, u) of (1.2) satisfies $u > 0$ in $[0, 1)$ and $u' < 0$ in $(0, 1]$ — see Proposition 7 below. Therefore, Theorem 4 implies that F is C^1 in a neighbourhood of any $(\lambda, u) \in \mathcal{S}$, with

$$D_u F(\lambda, u)v = v - \lambda DS_p(\lambda f(u))\partial_2 f(u)v, \quad v \in Y.$$

Furthermore, we have the following result.

Lemma 6 (Lemma 6.1 of [4]). *For any $(\lambda, u) \in \mathcal{S}$, the linear mapping $D_uF(\lambda, u) : Y \to Y$ is an isomorphism.*

Proof. By the compactness of S_p, the linear operator $D_uF(\lambda, u) : Y \to Y$ is a compact perturbation of the identity, for any $(\lambda, u) \in \mathcal{S}$. Therefore, we need only show that the kernel $N(D_uF(\lambda, u)) = \{0\}$. Using (H3), this follows by inspecting the Lagrange identity derived from (1.2) and the kernel equation, $D_uF(\lambda, u)v = 0$. □

Hence, through each solution $(\lambda, u) \in \mathcal{S}$ passes a unique local C^1 curve, that can be parametrized by λ. It then follows by standard compactness arguments that any of these curves can be extended smoothly to the whole interval $(\lambda_0, \lambda_\infty)$, and that the solutions along these curves satisfy

$$\lambda \to \lambda_0 \text{ if and only if } |u|_0 \to 0 \quad \text{and} \quad \lambda \to \lambda_\infty \text{ if and only if } |u|_0 \to \infty.$$

Consequently, the uniqueness statement in Theorem 2 follows from the local uniqueness in (2.7), which concludes the proof of Theorem 2. □

§3. Proof of Theorem 3

We start with the following result, giving basic properties of non-trivial solutions of (1.2). We will use the notation $p^* = p' - 1$, with $\frac{1}{p} + \frac{1}{p'} = 1$. Note that $p > 2 \implies 0 < p^* < 1$.

Proposition 7. *Let $(\lambda, u) \in \mathcal{S}$. Then $u > 0$ on $[0, 1)$, u is decreasing and satisfies $u'(1) < 0$.*

Proof. Equation (2.1) yields

$$u(r) = \lambda^{p^*} \int_r^1 \phi_{p'}\Big(\int_0^s \Big(\frac{t}{s}\Big)^{N-1} f(t, u(t))\, dt \Big)\, ds.$$

Since $u \not\equiv 0$ is continuous, it follows from (H2) that $u(0) > 0$. Furthermore,

$$\phi_p(u'(r)) = -\lambda \int_0^r \Big(\frac{t}{r}\Big)^{N-1} f(t, u(t))\, dt \leqslant 0, \quad r \in [0, 1],$$

showing that $u'(r) \leqslant 0$ for all $r \in [0, 1]$, so u is decreasing on $[0, 1]$. Finally,

$$\phi_p(u'(1)) = -\lambda \int_0^1 t^{N-1} f(t, u(t))\, dt < 0.$$

This implies $u'(1) < 0$, from which $u > 0$ on $[0, 1)$ now follows. □

Now, let $u \in C^1((\lambda_0, \lambda_\infty), Y)$ be the solution curve given by Theorem 2. We define

$$\zeta(\lambda) = \frac{\mathrm{d}u}{\mathrm{d}\lambda}(\lambda), \quad \lambda \in (\lambda_0, \lambda_\infty).$$

Hence, $\zeta(\lambda) \in Y = C^0[0, 1]$, for all $\lambda \in (\lambda_0, \lambda_\infty)$. Furthermore, it follows from Proposition 7 that

$$|u(\lambda)|_0 = u(\lambda)(0), \quad \lambda \in (\lambda_0, \lambda_\infty).$$

Therefore, we need only show that $\zeta(\lambda)(0) > 0$, for all $\lambda \in (\lambda_0, \lambda_\infty)$. In fact, the bifurcation at $\lambda = \lambda_0$ readily implies that $\zeta(\lambda)(0) > 0$ for some $\lambda > \lambda_0$, with $\lambda - \lambda_0$ small enough. Thus, it suffices to prove that $\zeta(\lambda)(0) \neq 0$, for all $\lambda \in (\lambda_0, \lambda_\infty)$.

We will suppose by contradiction that there exists $\lambda_1 \in (\lambda_0, \lambda_\infty)$ such that $\zeta(\lambda_1)(0) = 0$. Differentiating

$$u(\lambda)(r) = \lambda^{p^*} \int_r^1 \phi_{p'}\Big(\int_0^s \Big(\frac{t}{s}\Big)^{N-1} f(t, u(\lambda)(t))\, dt \Big)\, ds \tag{3.1}$$

with respect to r and using $p^* = \frac{1}{p-1}$ yields

$$|u(\lambda)'(r)|^{2-p} = \lambda^{p^*-1} \Big| \int_0^r \Big(\frac{t}{r}\Big)^{N-1} f(t, u(\lambda)(t))\, dt \Big|^{p^*-1}. \tag{3.2}$$

Then, differentiating (3.1) with respect to λ and using (3.2) yields

$$\zeta(\lambda)(r) = p^* \lambda^{-1} u(\lambda)(r) + p^* \lambda \rho(\lambda)(r),$$

where

$$\rho(\lambda)(r) = \int_r^1 |u(\lambda)'(s)|^{2-p} \int_0^s \Big(\frac{t}{s}\Big)^{N-1} \partial_2 f(t, u(\lambda)(t))\, \zeta(\lambda)(t)\, dt\, ds.$$

Furthermore,

$$\rho(\lambda)'(r) = -|u(\lambda)'(r)|^{2-p} \int_0^r \Big(\frac{t}{r}\Big)^{N-1} \partial_2 f(t, u(\lambda)(t))\, \zeta(\lambda)(t)\, dt.$$

Since f is bounded, (3.2) yields a constant $C(\lambda) > 0$ such that

$$|u(\lambda)'(r)|^{2-p} \leqslant C(\lambda) r^{p^*-1}, \quad r \in [0,1]. \tag{3.3}$$

(For the remainder of the proof, we shall use the symbol $C(\lambda)$ to denote various positive constants, the value of which may change from line to line, but is not essential to the analysis.) In view of (H1), a first consequence of (3.3) is that

$$|\rho(\lambda)'(r)| \leqslant C(\lambda) r^{p^*} \to 0 \quad \text{as } r \to 0.$$

In particular, $\zeta(\lambda_1) \in C^1[0,1]$, and satisfies the linear initial value problem

$$\begin{cases} -(r^{N-1}|u(\lambda_1)'(r)|^{p-2}\zeta')' = p^* \lambda_1 r^{N-1} \partial_2 f(r, u(\lambda_1)(r))\, \zeta - p^* \lambda_1^{-1} (r^{N-1} \phi_p(u(\lambda_1)'(r)))', \\ \zeta(0) = \zeta'(0) = 0. \end{cases} \tag{3.4}$$

Due to the singular behaviour of the coefficients at $r = 0$, we cannot apply standard ODE theory to (3.4), so we shall first establish the following result.

Lemma 8. *The initial value problem* (3.4) *has a unique solution, defined on* $[0,1]$.

Proof. We first remark that (3.4) is equivalent to the integral equation

$$\zeta = T\zeta + \xi,$$

where

$$\xi(r) = p^* \lambda_1^{-1} \int_0^r u(\lambda_1)'(s)\, ds = p^* \lambda_1^{-1}(u(\lambda_1)(r) - u(\lambda_1)(0)),$$

and $T : C[0,1] \to C[0,1]$ is the linear operator defined by

$$(T\zeta)(r) = -p^* \lambda_1 \int_0^r |u(\lambda_1)'(s)|^{2-p} \int_0^s \left(\frac{t}{s}\right)^{N-1} \partial_2 f(t, u(\lambda_1)(t))\, \zeta(t)\, dt\, ds.$$

Existence and uniqueness of a local (near $r = 0$) solution of (3.4) will follow from the contraction mapping principle, if we can show that, for $\delta > 0$ small enough, $T : C[0,\delta] \to C[0,\delta]$ is a contraction. Since there are no further singularities in $(0, 1]$, this solution can then be extended uniquely to the whole of $[0, 1]$ by standard ODE theory.

For $\zeta, \varphi \in C[0, 1]$, it follows from (3.3) and the boundedness of $\partial_2 f$ that

$$\begin{aligned}|(T\zeta - T\varphi)(r)| &\leqslant p^* \lambda_1 \int_0^r |u(\lambda_1)'(s)|^{2-p} \int_0^s |\partial_2 f(t, u(\lambda_1)(t))|\, |\zeta(t) - \varphi(t)|\, dt\, ds \\ &\leqslant C(\lambda_1) \int_0^r s^{p^*-1} \int_0^s |\zeta(t) - \varphi(t)|\, dt\, ds.\end{aligned}$$

Hence,

$$\sup_{[0,\delta]} |(T\zeta - T\varphi)| \leqslant C(\lambda_1) \frac{\delta^{p^*+1}}{p^* + 1} \sup_{[0,\delta]} |\zeta - \varphi|,$$

showing that $T : C[0,\delta] \to C[0,\delta]$ is a contraction, provided $C(\lambda_1)\frac{\delta^{p^*+1}}{p^*+1} < 1$. □

Now, by uniqueness of the solution of (3.4), we must have $\zeta(\lambda_1) \equiv 0$ on $[0, 1]$. On the other hand, since $(\lambda_1, u(\lambda_1))$ is a solution of (1.2), we have $F(\lambda_1, u(\lambda_1)) = 0$, where

$$F(\lambda, u) = u - S_p(\lambda f(u)) = u - \lambda^{p^*} S_p(f(u))$$

was introduced in Step 2 of the proof of Theorem 2. It follows that

$$\begin{aligned}\zeta(\lambda_1) &= -[D_u F(\lambda_1, u(\lambda_1))]^{-1} D_\lambda F(\lambda_1, u(\lambda_1)) \\ &= p^* \lambda_1^{-1} [D_u F(\lambda_1, u(\lambda_1))]^{-1} \lambda_1^{p^*} S_p(f(u(\lambda_1))) \\ &= p^* \lambda_1^{-1} [D_u F(\lambda_1, u(\lambda_1))]^{-1} u(\lambda_1) \neq 0.\end{aligned}$$

This contradiction completes the proof of Theorem 3. □

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