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Stationary Stokes equations with friction slip boundary conditions

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Abstract. The main purpose of this work is to study the existence and uniqueness of weak and strong solutions of the stationary Stokes system with the Navier boundary condition in the Hilbert case.

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§1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary Γ. In this paper, we consider the stationary Stokes equations:

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{1.1}$$

where $\boldsymbol{u}$ denotes the velocity and π the pressure and $\boldsymbol{f}$ is the external forces.
To study this problem, it is necessary to add some boundary conditions. Note that this system is often studied with the Dirichlet boundary condition, also called no-slip boundary condition, which is applicable in the case where the boundary of the flow is solid. However, from a point of view of physical applications, we often encounter situations where this condition is not quite feasible. In this case, it is really important to introduce other boundary conditions to describe the behavior of the fluid on the wall. For example, when a part of the flow on the boundary is the air, it is convenient to use a slip boundary condition. In literature, Navier [5], in 1827, was the first to propose a slip-friction boundary condition, in which there is a stagnant layer of the fluid close to the wall allowing the fluid to slip, and the tangential component of the strain tensor should be proportional to the tangential component of the fluid velocity on the boundary:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad 2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha \boldsymbol{u}_{\boldsymbol{\tau}} = \mathbf{0}, \tag{1.2}$$

where $\mathbf{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\top})$ denotes the deformation tensor associated with the velocity field $\boldsymbol{u}$, α is a scalar friction function, $\boldsymbol{n}$ is the exterior unit normal and $\boldsymbol{\tau}$ the corresponding tangent vector. System (1.1) with (1.2) has been studied by many authors. Note that, the first paper is due to Solonnikov and Scadilov [7] in 1973 without friction function ($\alpha = 0$) in the Hilbert case. We also refer to the paper of Beir*ã*o da Veiga [2]. We can cite the work of Clopeau, Mikelic and Robert in two dimensions [3]. In this paper, we investigate, on the first hand, the existence and uniqueness of weak solution and on the second hand, we prove the regularity of these solutions. Our method consists to used the Lax-Milgram theorem and such Korn's inequality to prove the existence and uniqueness of weak solutions and exploit the relationship between slip-Navier boundary condition and the the following boundary condition:

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0}, \tag{1.3}$$

to prove the regularity.
The outline of this paper is as follows. In the first paragraph, we will recall some preliminary results and introduce an important framework. The second section is devoted to prove the existence and uniqueness of weak solution for Stokes problem (1.1) with (1.2). In the paragraph 3, we prove the regularity of solution.

§2. Preliminary results and functional framework

In this section we review such basic notations and definitions and collect many known results, that will be useful for our studies.
We note that the vector-valued Laplace operator of a vector field $\boldsymbol{v} = (v_1, v_2, v_3)$ is equivalently defined by

$$\Delta\,\boldsymbol{v} = 2\mathrm{div}\,\mathbf{D}(\boldsymbol{v}) - \mathbf{grad}\,(\mathrm{div}\,\boldsymbol{v}). \tag{2.1}$$

Now, we define some functional spaces.

$$\boldsymbol{H}(\mathrm{div},\Omega) = \{\boldsymbol{v}\in\boldsymbol{L}^2(\Omega);\ \mathrm{div}\,\boldsymbol{v}\in L^2(\Omega)\},$$

$$\boldsymbol{H}(\mathbf{curl},\Omega) = \{\boldsymbol{v}\in\boldsymbol{L}^2(\Omega);\ \mathbf{curl}\,\boldsymbol{v}\in\boldsymbol{L}^2(\Omega)\},$$

which equipped with the graph norm. The closure of $\mathcal{D}(\Omega)$ in $\boldsymbol{H}(\mathrm{div},\Omega)$ and in $\boldsymbol{H}(\mathbf{curl},\Omega$ are denoted respectively by $\boldsymbol{H}_0(\mathrm{div},\Omega)$ and $\boldsymbol{H}_0(\mathbf{curl},\Omega)$ and can be respectively characterized by:

$$\boldsymbol{H}_0(\mathrm{div},\Omega) = \{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},\Omega); \boldsymbol{v}\cdot\boldsymbol{n} = 0\,\mathrm{on}\,\Gamma\},$$

$$\boldsymbol{H}_0(\mathbf{curl},\Omega) = \{\boldsymbol{v}\in\boldsymbol{H}(\mathbf{curl},\Omega);\ \boldsymbol{v}\times\boldsymbol{n} = 0\,\mathrm{on}\,\Gamma\}.$$

In the sequel, we need the following Korn inequality, whose proof is given in [4].

Lemma 1. *Let Ω be a bounded connected open set of $\mathbb{R}^3$ of class $C^{1,1}$. Then there exists a constant $C = C(\Omega)$ such that*

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leq C\{\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{D}(\boldsymbol{v})\|^2_{\boldsymbol{L}^2(\Omega)}\}^{\frac{1}{2}} \quad \text{for all} \quad \boldsymbol{v} \quad \text{in} \quad \boldsymbol{H}^1(\Omega). \tag{2.2}$$

Let's now introduce some notation to describe a boundary Γ. We consider any point P on Γ and choose an open neighborhood W of P on Γ small enough to allow the existence of two families of C^2-curves on W with these properties: a curve of each family passes through every point of W and the unit tangent vectors to these curves form an orthonormal system (which we assume to have the direct orientation) at every point of W. The lengths s_1, s_2 along each family of curves, respectively, are a possible coordinate system in W. We denote by $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$ the unit tangent vectors to each family of these curves, respectively.
With this notation, we have $\boldsymbol{v} = \sum_{k=1}^{2} v_k\boldsymbol{\tau}_k + (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}$ where $\boldsymbol{\tau}_k^T = (\tau_{k1}, \tau_{k2}, \tau_{k3})$ and $v_k = \boldsymbol{v}\cdot\boldsymbol{\tau}_k$.
The lemma below give a relationship between the slip-Navier boundary condition and the boundary condition given by (1.3). It will be used in the proof of existence and uniqueness of strong solutions.

Lemma 2. *We suppose that* Γ *is of class* C^2. *Then, for any* $\boldsymbol{v} \in \boldsymbol{H}^2(\Omega)$, *we have*

$$[2\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_\tau = \nabla_\tau(\boldsymbol{v}\cdot\boldsymbol{n}) + (\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}})_\tau - \sum_{k=1}^{2}(\boldsymbol{v}_\tau \cdot \frac{\partial \boldsymbol{n}}{\partial s_k})\boldsymbol{\tau}_k \tag{2.3}$$

and

$$\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n} = \nabla_\tau(\boldsymbol{v}\cdot\boldsymbol{n}) - (\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}})_\tau - \sum_{k=1}^{2}(\boldsymbol{v}_\tau \cdot \frac{\partial \boldsymbol{n}}{\partial s_k})\boldsymbol{\tau}_k. \tag{2.4}$$

Remark 1. In the particular case $\boldsymbol{v}\cdot\boldsymbol{n} = 0$, we have the following equality:

$$[2\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_\tau = (\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}})_\tau - \sum_{k=1}^{2}(\boldsymbol{v}_\tau \cdot \frac{\partial \boldsymbol{n}}{\partial s_k})\boldsymbol{\tau}_k$$

and

$$\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n} = -(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}})_\tau - \sum_{k=1}^{2}(\boldsymbol{v}_\tau \cdot \frac{\partial \boldsymbol{n}}{\partial s_k})\boldsymbol{\tau}_k$$

implying

$$[2\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_\tau = -\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n} - 2\sum_{k=1}^{2}(\boldsymbol{v}_\tau \cdot \frac{\partial \boldsymbol{n}}{\partial s_k})\boldsymbol{\tau}_k. \tag{2.5}$$

Remark 2. We note that, if Ω is of class $C^{2,1}$, then slip-Navier boundary condition differs from (1.3) only by the term $-2\sum_{k=1}^{2}(\boldsymbol{v}_\tau \cdot \dfrac{\partial \boldsymbol{n}}{\partial s_k})\boldsymbol{\tau}_k$. This term vanishes on the flat boundary, consequently, we have (1.2) and (1.3) are identical.

Now, we define the following space:

$$\boldsymbol{K}_N(\Omega) = \{\boldsymbol{v}\in \boldsymbol{L}^2(\Omega), \quad \operatorname{div}\boldsymbol{v} = 0, \quad \mathbf{curl}\,\boldsymbol{v} = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \boldsymbol{v}\times\boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Gamma\}.$$

We also recall the following result which will be useful in sequel.(cf. [1]).

Lemma 3. *Let* $\boldsymbol{\beta}$ *such that* $\boldsymbol{\beta}\times\boldsymbol{n} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ *and* $\boldsymbol{f} \in [\boldsymbol{H}_0(\mathbf{curl},\Omega)]'$ *with* $\operatorname{div}\boldsymbol{f} = 0$ *in* Ω *and satisfying the compatibility condition:*

$$\forall \boldsymbol{v}\in \boldsymbol{K}_N(\Omega), \quad \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{[\boldsymbol{H}_0(\mathbf{curl},\Omega)]'\times \boldsymbol{H}_0(\mathbf{curl},\Omega)} = 0. \tag{2.6}$$

Then, the following problem

$$\begin{cases} -\Delta \boldsymbol{u} &= \boldsymbol{f} & \text{in} \quad \Omega, \\ \operatorname{div}\boldsymbol{u} &= 0 & \text{in} \quad \Omega, \\ \boldsymbol{u}\times\boldsymbol{n} &= \boldsymbol{\beta}\times\boldsymbol{n} & \text{on} \quad \Gamma, \end{cases}$$

has a unique solution in $\boldsymbol{H}^1(\Omega)$ *and we have:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \le C(\|f\|_{[\boldsymbol{H}_0(\mathbf{curl},\Omega)]'} + \|\boldsymbol{\beta}\times\boldsymbol{n}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}).$$

§3. Weak solutions

In this section we will study the following Stokes problem.

$$(\mathcal{S}_T)\begin{cases} -\Delta\boldsymbol{u}+\nabla\pi &= \boldsymbol{f} \quad \text{in} \quad \Omega,\\ \operatorname{div}\boldsymbol{u} &= \chi \quad \text{in} \quad \Omega,\\ \boldsymbol{u}\cdot\boldsymbol{n} &= g \quad \text{on} \quad \Gamma,\\ 2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}+\alpha\boldsymbol{u}_{\boldsymbol{\tau}} &= \boldsymbol{h} \quad \text{on} \quad \Gamma. \end{cases}$$

The aim of this section is to give a variational formulation of the problem $(\mathcal{S}_T)$ and prove a theorem of existence and uniqueness of weak solutions. We set α by a positive function such that α belongs to $L^\infty(\Gamma)$.
Let us introduce the following space:

$$\boldsymbol{V}=\{\boldsymbol{v}\in\boldsymbol{H}^1(\Omega);\quad \operatorname{div}\boldsymbol{v}=0 \quad \text{in}\,\Omega \quad \text{and} \quad \boldsymbol{v}\cdot\boldsymbol{n}=0 \quad \text{on} \quad \Gamma\},$$

$$\boldsymbol{E}(\Omega)=\{\boldsymbol{v}\in\boldsymbol{H}^1(\Omega);\quad \Delta\boldsymbol{v}\in[\boldsymbol{H}_0(\operatorname{div},\Omega)]'\}.$$

$\boldsymbol{E}(\Omega)$ is a Banach space for the norm

$$||\boldsymbol{v}||_{\boldsymbol{E}(\Omega)}=(||\boldsymbol{v}||^2_{\boldsymbol{H}^1(\Omega)}+||\Delta\boldsymbol{v}||^2_{[\boldsymbol{H}_0(\operatorname{div},\Omega)]'})^{\frac{1}{2}}.$$

We give the following result where the proof can be found in [6].

Lemma 4. *The space $\mathcal{D}(\overline{\Omega})$ is dense in $\boldsymbol{E}(\Omega)$.*

We can now deduce the following result.

Corollary 5. *The linear mapping $\Theta:\boldsymbol{v}\longrightarrow[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}|\Gamma}$ defined on $\mathcal{D}(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping*

$$\Theta:\boldsymbol{E}(\Omega)\longrightarrow\boldsymbol{H}^{-\frac{1}{2}}(\Gamma).$$

Moreover, we have the Green formula: for any $\boldsymbol{v}\in\boldsymbol{E}(\Omega)$ and $\boldsymbol{\varphi}\in\boldsymbol{V}$,

$$-\langle\Delta\boldsymbol{v},\varphi\rangle_{[\boldsymbol{H}_0(\operatorname{div},\Omega)]'\times\boldsymbol{H}_0(\operatorname{div},\Omega)}=2\int_\Omega\mathbf{D}(\boldsymbol{v}):\mathbf{D}(\boldsymbol{\varphi})\,d\boldsymbol{x}-2\langle[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}},\varphi\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}. \tag{3.1}$$

Proof. For any $\boldsymbol{\varphi}$ in $\boldsymbol{V}$ and for any $\boldsymbol{v}\in\mathcal{D}(\overline{\Omega})$ we have

$$-\langle\Delta\boldsymbol{v},\boldsymbol{\varphi}\rangle_{[\boldsymbol{H}_0(\operatorname{div},\Omega)]'\times\boldsymbol{H}_0(\operatorname{div},\Omega)}=2\int_\Omega\mathbf{D}(\boldsymbol{v}):\mathbf{D}(\boldsymbol{\varphi})\,d\boldsymbol{x}-2\int_\Gamma\boldsymbol{\varphi}\cdot\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\,d\boldsymbol{\sigma}. \tag{3.2}$$

We know also, for all $\boldsymbol{\varphi}\in\boldsymbol{V}$ we have

$$\begin{aligned}\int_\Gamma\boldsymbol{\varphi}\cdot\mathbf{D}(\boldsymbol{v})\boldsymbol{n}\,d\sigma&=\int_\Gamma\boldsymbol{\varphi}\cdot\{[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_n\,\boldsymbol{n}+[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}\}\,d\sigma\\&=\int_\Gamma\boldsymbol{\varphi}\cdot[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\boldsymbol{\tau}}\,d\sigma,\end{aligned}$$

where $[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_n$ is the component of $\mathbf{D}(\boldsymbol{u})\boldsymbol{n}$ in the direction of $\boldsymbol{n}$ and $[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_\tau$ is the projection of $\mathbf{D}(\boldsymbol{u})\boldsymbol{n}$ on the tangent hyperplane of Γ.
Now, let $\boldsymbol{\mu}$ be any element of $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$, then there exists an element $\boldsymbol{\varphi}$ of $\boldsymbol{H}^1(\Omega)$ such that $\operatorname{div}\boldsymbol{\varphi} = 0$ in Ω and $\boldsymbol{\varphi} = \boldsymbol{\mu}_\tau$ on Γ with

$$||\boldsymbol{\varphi}||_{\boldsymbol{H}^1(\Omega)} \leq C||\boldsymbol{\mu}_\tau||_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)} \leq C||\boldsymbol{\mu}||_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}.$$

Consequently,

$$\begin{aligned}
\left|\langle[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_\tau, \boldsymbol{\mu}\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}\right| &= \left|\langle[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_\tau, \boldsymbol{\mu}_\tau\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}\right| \\
&= \left|\langle[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_\tau, \boldsymbol{\varphi}\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}\right| \\
&\leq ||\Delta\boldsymbol{v}||_{[\boldsymbol{H}_0(\operatorname{div},\Omega)]'}||\boldsymbol{\varphi}||_{\boldsymbol{H}_0(\operatorname{div},\Omega)} + ||\mathbf{D}(\boldsymbol{v})||_{\boldsymbol{L}^2(\Omega)}||\mathbf{D}(\boldsymbol{\varphi})||_{\boldsymbol{L}^2(\Omega)} \\
\left|\langle[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_\tau, \boldsymbol{\mu}\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}\right| &\leq \{||\Delta\boldsymbol{v}||^2_{[\boldsymbol{H}_0(\operatorname{div},\Omega)]'} + ||\mathbf{D}(\boldsymbol{v})||^2_{\boldsymbol{L}^2(\Omega)}\}^{\frac{1}{2}}\{||\boldsymbol{\varphi}||^2_{\boldsymbol{L}^2(\Omega)} + ||\mathbf{D}(\boldsymbol{\varphi})||^2_{\boldsymbol{L}^2(\Omega)}\}^{\frac{1}{2}}.
\end{aligned}$$

It follows from Korn's Inequality, we have

$$\left|\langle[\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_\tau, \boldsymbol{\mu}\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}\right| \leq C||\boldsymbol{v}||_{\boldsymbol{E}(\Omega)}||\boldsymbol{\varphi}||_{\boldsymbol{H}^1(\Omega)}.$$

Thus,

$$||\, [\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_\tau \,||_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)} \leq C||\boldsymbol{v}||_{\boldsymbol{E}(\Omega)}.$$

Therefore, the linear mapping $\Theta : \boldsymbol{v} \longrightarrow [\mathbf{D}(\boldsymbol{v})\boldsymbol{n}]_{\tau|\Gamma}$ defined on $\mathcal{D}(\overline{\Omega})$ is continuous for the norm of $\boldsymbol{E}(\Omega)$. Since $\mathcal{D}(\overline{\Omega})$ is dense in $\boldsymbol{E}(\Omega)$, Θ can be extended by continuity to a mapping still called $\Theta \in \mathfrak{L}(\boldsymbol{E}(\Omega), \boldsymbol{H}^{-\frac{1}{2}}(\Gamma))$ and Formula (3.2) holds for all $\boldsymbol{v} \in \boldsymbol{E}(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{V}$. □

The following proposition will help us to solve the problem $(\mathcal{S}_T)$.

Proposition 6. *We suppose that* $\chi = 0$, $g = 0$. *Let* $\boldsymbol{f} \in [\boldsymbol{H}_0(\operatorname{div}, \Omega)]'$, $\boldsymbol{h} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$ *such that*

$$\boldsymbol{h} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma.$$

Then, the problem: Find $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega)$ *satisfying* $(\mathcal{S}_T)$ *is equivalent to the following variational problem:*

$$\begin{cases}
\text{Find} \quad \boldsymbol{u} \in \boldsymbol{V} \quad \text{such that,} \\
\forall \boldsymbol{\varphi} \in \boldsymbol{V},\ 2\int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{\varphi})\, d\boldsymbol{x} + \int_\Gamma \alpha(x)\boldsymbol{\varphi}_\tau \boldsymbol{u}_\tau\, d\sigma = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{[\boldsymbol{H}_0(\operatorname{div},\Omega)]'\times\boldsymbol{H}_0(\operatorname{div},\Omega)} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}.
\end{cases} \tag{3.3}$$

Proof. Using the Green formula (3.1), we deduce that every solution of $(\mathcal{S}_T)$ also solves (3.3). Conversely, let $\boldsymbol{u}$ be a solution of the problem (3.1). Let us take a function $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$ such that $\operatorname{div}\boldsymbol{\varphi} = 0$ as a test function in (3.3). It is clear that a second term in the left-hand side of (3.3) vanishes because $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$. Now the first term in the left-hand side is

$$2\int_{\Omega}\mathbf{D}(\boldsymbol{u}):\mathbf{D}(\boldsymbol{\varphi})\,d\boldsymbol{x}=\langle-\Delta u,\boldsymbol{\varphi}\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)}. \tag{3.4}$$

As a consequence,

$$\forall\boldsymbol{\varphi}\in\mathcal{D}_{\sigma}(\Omega)\qquad\langle-\Delta\boldsymbol{u}-\boldsymbol{f},\boldsymbol{\varphi}\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)}=0.$$

So, by the De Rham's Theorem, there exists a distribution π in $\mathcal{D}^{'}(\Omega)$ defined uniquely up to an additive constant such that

$$-\Delta\boldsymbol{u}+\nabla\pi=\boldsymbol{f}. \tag{3.5}$$

Moreover, by the fact that $\boldsymbol{u}$ belonging to the space $\boldsymbol{V}$, we have $\operatorname{div}\boldsymbol{u}=0$ in Ω and $\boldsymbol{u}\cdot\boldsymbol{n}=0$ on Γ. It remains to prove the Navier boundary condition $2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}+\alpha\boldsymbol{u}_{\boldsymbol{\tau}}=\boldsymbol{h}$ on Γ. We multiply the equation (3.5) by $\boldsymbol{\varphi}\in\boldsymbol{V}$ and we integrate on Ω we obtain

$$2\int_{\Omega}\mathbf{D}(\boldsymbol{u}):\mathbf{D}(\boldsymbol{\varphi})\,d\boldsymbol{x}-2\int_{\Gamma}\boldsymbol{\varphi}\cdot[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}\,d\sigma=\langle\boldsymbol{f},\boldsymbol{\varphi}\rangle_{[\boldsymbol{H}_0(\mathrm{div},\Omega)]'\times\boldsymbol{H}_0(\mathrm{div},\Omega)}\,. \tag{3.6}$$

Using (3.3) and (3.6), we deduce that

$$\forall\boldsymbol{\varphi}\in\boldsymbol{V},\qquad\int_{\Gamma}\{2\,[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}+\alpha\boldsymbol{u}_{\boldsymbol{\tau}}-\boldsymbol{h}\}\cdot\boldsymbol{\varphi}\,d\sigma=0. \tag{3.7}$$

Let now $\boldsymbol{\mu}$ be any element of the space $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$. So, there exists $\boldsymbol{\varphi}\in\boldsymbol{H}^1(\Omega)$ such that $\operatorname{div}\boldsymbol{\varphi}=0$ in Ω and $\boldsymbol{\varphi}=\boldsymbol{\mu}_{\boldsymbol{\tau}}$ on Γ. It is clear that $\boldsymbol{\varphi}\in\boldsymbol{V}$ and

$$\begin{aligned}\big\langle[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}+\alpha\boldsymbol{u}_{\boldsymbol{\tau}}-\boldsymbol{h},\,\boldsymbol{\mu}\big\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}&=\big\langle[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}+\alpha\boldsymbol{u}_{\boldsymbol{\tau}}-\boldsymbol{h},\,\boldsymbol{\mu}_{\boldsymbol{\tau}}\big\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}\\&=\big\langle[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}+\alpha\boldsymbol{u}_{\boldsymbol{\tau}}-\boldsymbol{h},\,\boldsymbol{\varphi}\big\rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)\times\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}\\&=0.\end{aligned}$$

This implies that

$$[\mathbf{D}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}}+\alpha\boldsymbol{u}_{\boldsymbol{\tau}}=\boldsymbol{h}\quad\text{on}\quad\Gamma.$$

□

Now, we show the following inequality which will be useful in the proof of coercivity.

Lemma 7. *Let Ω be a bounded open set of $\mathbb{R}^3$ of class $C^{1,1}$. Then, there exists a constant $C=C(\Omega)>0$ such that*

$$\int_{\Omega}|\mathbf{D}(\boldsymbol{u})|^2d\boldsymbol{x}+\int_{\Gamma}\alpha|\boldsymbol{u}_{\boldsymbol{\tau}}|^2d\boldsymbol{\sigma}\geq C\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)},\quad\forall\,\boldsymbol{u}\in\boldsymbol{V}. \tag{3.8}$$

Proof. Assume that the inequality (3.8) is not true. Then there is a sequence $(\boldsymbol{u}_k)_k$ of $\boldsymbol{V}$ such that

$$\|\boldsymbol{u}_k\|_{\boldsymbol{H}^1(\Omega)}=1$$

and

$$\int_\Omega |\mathbf{D}(\boldsymbol{u}_k)|^2 d\boldsymbol{x} + \int_\Gamma \alpha |(\boldsymbol{u}_k)_{\boldsymbol{\tau}}|^2 d\boldsymbol{\sigma} < \frac{1}{n}, \quad \forall k \geq 1. \tag{3.9}$$

Hence, there is a subsequence, again denoted by $(\boldsymbol{u}_k)_k$ and $\boldsymbol{u} \in \boldsymbol{V}$ such that

$$(\boldsymbol{u}_k)_k \text{converge to } \boldsymbol{u} \text{ weakly in } \boldsymbol{H}^1(\Omega), \text{ strongly in } \boldsymbol{L}^2(\Omega),$$

and

$$\mathbf{D}(\boldsymbol{u}_k) \longrightarrow 0 \quad \text{in} \quad \boldsymbol{L}^2(\Omega)$$

It follows from the Korn inequality that for all $\varepsilon > 0$ we have

$$\begin{aligned} \|\boldsymbol{u}_p - \boldsymbol{u}_k\|^2_{\boldsymbol{H}^1(\Omega)} &\leqslant C\{\|\boldsymbol{u}_p - \boldsymbol{u}_k\|^2_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{D}(\boldsymbol{u}_p) - \mathbf{D}(\boldsymbol{u}_k)\|^2_{\boldsymbol{L}^2(\Omega)}\} \\ &\leqslant \varepsilon. \end{aligned}$$

Thus, the subsequence $(\boldsymbol{u}_k)_k$ is a Cauchy sequence in $\boldsymbol{H}^1(\Omega)$ and then

$$\boldsymbol{u}_k \longrightarrow \boldsymbol{u} \quad \text{strongly in } \boldsymbol{H}^1(\Omega).$$

Therefore, $\mathbf{D}(\boldsymbol{u}) = 0$, which means that $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}$, $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$. But by (3.9) we have $\boldsymbol{u}_{\boldsymbol{\tau}} = 0$ on Γ, then $\boldsymbol{u} = 0$ on Γ. Using the second left-hand side of (3.9) , we deduce that $\boldsymbol{u} = 0$, that is a contradiction with $\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} = 1$. □

Now, we can solve the Stokes problem with Navier boundary condition.

Theorem 8. *(**Weak solution for** $(\mathcal{S}_T)$)*
Suppose that $\chi = 0$ *and* $g = 0$. *Let* $\boldsymbol{f} \in [\boldsymbol{H}_0(\text{div}, \Omega)]'$ *and* $\boldsymbol{h} \in \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$, *satisfying*

$$\boldsymbol{h} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma.$$

Then, the Stokes problem $(\mathcal{S}_T)$ *has a unique solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ *and we have the following estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\pi\|_{L^2(\Omega)/\mathbb{R}} \leq \|\boldsymbol{f}\|_{[\boldsymbol{H}_0(\text{div},\Omega)]'} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}. \tag{3.10}$$

Proof. Thanks to the Proposition 6, the problem $(\mathcal{S}_T)$ is equivalent to the following variational formulation:

$$\begin{cases} \text{Find} \quad \boldsymbol{u} \in \boldsymbol{V} \quad \text{such that,} \\ \\ \forall \boldsymbol{\varphi} \in \boldsymbol{V}, \quad 2\int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{\varphi})\, d\boldsymbol{x} + \int_\Gamma \alpha(x) \boldsymbol{\varphi}_{\boldsymbol{\tau}} \boldsymbol{u}_{\boldsymbol{\tau}}\, d\sigma = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{[\boldsymbol{H}_0(\text{div},\Omega)]' \times \boldsymbol{H}_0(\text{div},\Omega)} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \langle \boldsymbol{h}, \boldsymbol{\varphi} \rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \times \boldsymbol{H}^{\frac{1}{2}}(\Gamma)}. \end{cases} \tag{3.11}$$

We consider the Hilbert space $\boldsymbol{V}$ endowed with the norm $\|\boldsymbol{v}\|_{\boldsymbol{V}} = \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|D(\boldsymbol{v})\|_{\boldsymbol{L}^2(\Omega)}$,

$$E(\boldsymbol{u}, \boldsymbol{\varphi}) = 2\int_\Omega \mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{\varphi})\, d\boldsymbol{x} + \int_\Gamma \alpha(x) \boldsymbol{\varphi}_{\boldsymbol{\tau}} \boldsymbol{u}_{\boldsymbol{\tau}}\, d\sigma \quad for,\ \boldsymbol{u},, \boldsymbol{\varphi} \in \boldsymbol{V},$$

$$l(\boldsymbol{\varphi}) = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{[\boldsymbol{H}_0(\mathrm{div},\Omega)]' \times \boldsymbol{H}_0(\mathrm{div},\Omega)} - \langle \boldsymbol{h}, \boldsymbol{\varphi} \rangle_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \times \boldsymbol{H}^{\frac{1}{2}}(\Gamma)} .$$

Its easy to see that the bilinear form $E(.,.)$ (resp. the linear form $l(.)$)is continuous on $\boldsymbol{V} \times \boldsymbol{V}$(resp. on $\boldsymbol{V}$) since we have

$$E(\boldsymbol{u}, \boldsymbol{\varphi}) \leq C(\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \|\boldsymbol{\varphi}\|_{\boldsymbol{H}^1(\Omega)}) \leq C(\|\boldsymbol{u}\|_{\boldsymbol{V}} \|\boldsymbol{\varphi}\|_{\boldsymbol{V}}),$$

$$l(\boldsymbol{\varphi}) \leq (\|\boldsymbol{f}\|_{[\boldsymbol{H}_0(\mathrm{div},\Omega)]'}\| + \|\boldsymbol{h}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\Gamma)}) \|\boldsymbol{\varphi}\|_{\boldsymbol{V}}.$$

Furthermore, the bilinear form $E(.,.)$ is $\boldsymbol{V}-$elliptic, as we have

$$\forall \boldsymbol{u} \in \boldsymbol{V}, \quad E(\boldsymbol{u}, \boldsymbol{u}) \geq C\|\boldsymbol{u}\|_{\boldsymbol{V}}.$$

thanks to the Lemma 7. According to Lax-Milgram theorem, the problem $(\mathcal{S}_T)$ has a unique function $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ and $\pi \in L^2(\Omega)$ solution of the Stokes problem. □

Remark 3. We can also solve the Stokes problem $(\mathcal{S}_T)$ when the divergence operator does not vanish and $g \neq 0$. To do this, we consider the following Neumann problem:

$$\Delta\theta = \chi \quad \text{in}\Omega \quad \text{and} \quad \frac{\partial\theta}{\partial \boldsymbol{n}} = g \quad \text{on} \quad \Gamma. \tag{3.12}$$

and setting $\boldsymbol{z} = \boldsymbol{u} - \nabla\theta$.

§4. Strong Solution and regularity for the Stokes system $(\mathcal{S}_T)$

We prove now the existence of strong solution $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^2(\Omega) \times H^1(\Omega)$ for the Stokes problem $(\mathcal{S}_T)$. The proof of the following theorem is based on results given in Lemma2 and Lemma3.

Theorem 9. *Assume that* Ω *is of class* $C^{2,1}$*. Suppose that* $\chi = 0$*. Let* $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$*,* $g \in H^{\frac{3}{2}}(\Omega)$ *and* $\boldsymbol{h} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$*, satisfying*

$$\int_\Gamma g \, d\sigma = 0 \quad \text{and} \quad \boldsymbol{h} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma.$$

Then, the solution $(\boldsymbol{u}, \pi)$ *given by Theorem 8 belongs to* $\boldsymbol{H}^2(\Omega) \times H^1(\Omega)/\mathbb{R}$ *and satisfies the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} + \|\pi\|_{H^1(\Omega)} \leq C(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|g\|_{H^{\frac{3}{2}}(\Gamma)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}). \tag{4.1}$$

Proof. Note that under the hypothesis of Theorem 9, the data $\boldsymbol{f}$ satisfies also the hypothesis of Theorem 8. So, this implies that problem $(\mathcal{S}_T)$ has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{H}^1(\Omega) \times L^2(\Omega)/\mathbb{R}$. To prove the regularity of the velocity, we set $\boldsymbol{z} = \mathbf{curl}\, \boldsymbol{u}$. Using Remark 1, we deduce that $\boldsymbol{z}$ satisfies the following problem:

$$\begin{cases} -\Delta \boldsymbol{z} &= \mathbf{curl} \boldsymbol{f} \quad \text{in} \quad \Omega, \\ \mathrm{div}\, \boldsymbol{z} &= 0 \qquad \text{in} \quad \Omega, \\ \boldsymbol{z} \times \boldsymbol{n} &= \boldsymbol{H} \qquad \text{on} \quad \Gamma. \end{cases}$$

where $\boldsymbol{H} = \alpha \boldsymbol{u}_{\boldsymbol{\tau}} - 2\sum_{k=1}^{2}(\boldsymbol{u}_{\boldsymbol{\tau}} \cdot \dfrac{\partial \boldsymbol{n}}{\partial s_k})\boldsymbol{\tau}_k - \boldsymbol{h}$.

Because Ω is of class $C^{2,1}$ and $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, we have $\sum_{k=1}^{2}(\boldsymbol{u}_{\boldsymbol{\tau}} \cdot \dfrac{\partial \boldsymbol{n}}{\partial s_k})\boldsymbol{\tau}_k \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$, it follows that $\boldsymbol{H} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ and $\boldsymbol{H} \cdot \boldsymbol{n} = 0$. Since $\mathbf{curl} \boldsymbol{f}$ belongs to $[\boldsymbol{H}_0(\mathbf{curl}, \Omega)]'$ and satisfies the compatibility condition (2.6), as a result of Lemma 3, we have $z \in \boldsymbol{H}^1(\Omega)$. Then $\boldsymbol{u} \in \boldsymbol{X}^{2,2}(\Omega)$. As a consequence, thanks to the imbedding of $\boldsymbol{X}^{2,2}(\Omega)$ in $\boldsymbol{H}^2(\Omega)$ (see[1]), the solution $\boldsymbol{u}$ of the problem $(\mathcal{S}_T)$ belongs to $\boldsymbol{H}^2(\Omega)$,
where

$$\boldsymbol{X}^{2,2}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega); \operatorname{div} \boldsymbol{v} \in H^1(\Omega), \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \text{ and } \boldsymbol{v} \cdot \boldsymbol{n} \in H^{\frac{3}{2}}(\Gamma)\}.$$

Moreover, since

$$\nabla \pi = \Delta \boldsymbol{u} + \boldsymbol{f} \in \boldsymbol{L}^2(\Omega),$$

we deduce that $\pi \in H^1(\Omega)$. □

We have the following result concerning strong solution of the Stokes problem $(\mathcal{S}_T)$, where the divergence operator does not vanish.

Corollary 10. *Assume that Ω is of class $C^{2,1}$. For every $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, $\chi \in H^1(\Omega)$, $g \in H^{\frac{3}{2}}(\Omega)$ and $\boldsymbol{h} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$, satisfying*

$$\int_\Gamma g \, d\sigma = 0 \quad \text{and} \quad \boldsymbol{h} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma.$$

The solution $(\boldsymbol{u}, \pi)$ given by Theorem8 belongs to $\boldsymbol{H}^2(\Omega) \times H^1(\Omega)/\mathbb{R}$ and satisfies the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} + \|\pi\|_{H^1(\Omega)} \leq C(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} + \|\chi\|_{H^1(\Omega)} + \|g\|_{H^{\frac{3}{2}}(\Gamma)} + \|\boldsymbol{h}\|_{\boldsymbol{H}^{\frac{1}{2}}(\Gamma)}). \tag{4.2}$$

Proof. We solve the following Neumann problem:

$$\Delta \theta = \chi \quad \text{in}\Omega \quad \text{and} \quad \frac{\partial \theta}{\partial \boldsymbol{n}} = g \quad \text{on} \quad \Gamma. \tag{4.3}$$

This problem has a unique solution this problem has a unique solution $\theta \in H^3(\Omega)/\mathbb{R}$ satisfying the estimate:

$$\|\theta\|_{H^3(\Omega)} \leq C\{\|\chi\|_{H^1(\Omega)} + \|g\|_{H^{\frac{3}{2}}(\Omega)}\}. \tag{4.4}$$

Setting $z = \boldsymbol{u} - \nabla \theta$, then $(\mathcal{S}_T)$ becomes: Find $(z, \pi) \in \boldsymbol{H}^2(\Omega) \times H^1(\Omega)/\mathbb{R}$ such that

$$\begin{cases} -\Delta z + \nabla \pi & = \boldsymbol{f} + \nabla \chi \quad \text{in} \quad \Omega, \\ \operatorname{div} z & = 0 \quad \text{in} \quad \Omega, \\ z \cdot \boldsymbol{n} & = g \quad \text{on} \quad \Gamma, \\ 2\,[\mathbf{D}(z)\boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha z_{\boldsymbol{\tau}} & = \boldsymbol{H} \quad \text{on} \quad \Gamma. \end{cases}$$

where $\boldsymbol{H} = \boldsymbol{h} - 2\,[\mathbf{D}(\nabla\theta)\boldsymbol{n}]_{\boldsymbol{\tau}} - \alpha(\nabla\theta)_{\boldsymbol{\tau}}$ and belongs to $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$. Observe that $\boldsymbol{f} + \nabla\chi$ belongs to $\boldsymbol{L}^2(\Omega)$. Thus, due to Theorem 9, this problem has a unique solution $(z, \pi) \in \boldsymbol{H}^2(\Omega) \times H^1(\Omega)/\mathbb{R}$. Therefore $\boldsymbol{u} = z + \nabla\theta$ and belongs to $\boldsymbol{H}^2(\Omega)$.

□

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