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SYMMETRY BREAKING BIFURCATIONS IN A D_4 SYMMETRIC HAMILTONIAN SYSTEM

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Abstract. In this work we investigate a numerical method to locate periodic orbits in Hamiltonian systems of two degrees of freedom in a D_4 and time reversal symmetric Hamiltonian. The procedure to obtain the "skeleton" of periodic orbits is a combination of several methods such as continuation theory, systematic search algorithm, Poincaré surface of section and a fast chaos indicator, OFLI2. Those techniques are used to provide a complete study of symmetry breaking bifurcations in a particular Hamiltonian system. Moreover, we show in detail the evolution of some families of periodic orbits and an analysis of new bifurcations.

Keywords: skeleton of periodic orbits, bifurcations, Poincaré surfaces of section, OFLI2.

AMS classification: 37G15, 37G25.

§1. Introduction

Periodic orbits (PO) and their stabilities are powerful tool in understanding of dynamical systems. The studies of changes in the behavior of PO a can provide essential insights into nature of simple integrable dynamics and complicated, chaotic dynamics. These knowledge have considerably broad applications in physics (quantum eigen state studies [11]) and in astrophysics (numerous problems of stellar and celestial dynamics, e.g., satellite orbits stabilities, etc. Cf. [13]).

Bifurcation is nothing more than qualitative changes in the system's asymptotic behavior and the points where those changes appear are called Bifurcation Points (BP). Whereas, a bifurcation of PO is when those changes affects on the stability of a equilibria or a PO. For better understanding of the bifurcation we have to concentrate on the study of Periodic Orbits. For instance at the period-doubling bifurcation a PO of period T jumps from stable to unstable branch and simultaneously a new stable PO of period $2T$ is created.

A symmetry breaking bifurcation, appears when some perturbation with less symmetry is added to symmetric system. In this note we consider the quartic homogeneous potential system having a general form

$$\mathcal{H} = \frac{1}{2}(X^2 + Y^2) + \frac{1}{4}(x^4 + y^4) + \alpha x^2 y^2 + \beta(x^2 + y^2), \qquad (1)$$

in terms of the Cartesian coordinates x, y and their conjugate momenta X, Y, $\alpha, \beta \in \mathcal{R}$. This system is characterized by discrete symmetries and it is invariant under a rotation by $\pi/4$ (Fig. 1). This system was studied e.g. in [10] to find soliton solution in three space dimensions and also in [7], where a direct method to identify integrable N-degree of freedom Hamiltonian systems was described. The existence of large regions of chaotic orbits in parameter space in the neighborhood of the degenerate bifurcation point was reported in [1].

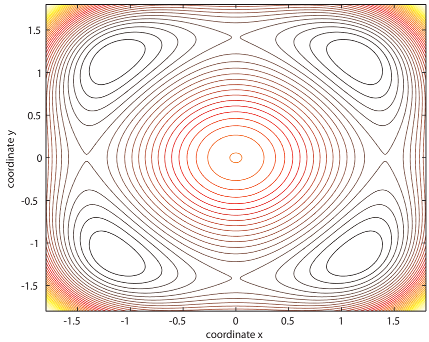

Figure 1: Contour plot of the potential

We choose this Hamiltonian system to create skeleton of periodic orbits and investigate connection between them. Therefore, we set $\alpha = 1/4, \beta = -1$, hence

$$\mathcal{H} = \frac{1}{2}(X^2 + Y^2) + \frac{1}{4}(x^2 + y^2)^2 - x^2 - y^2 - \frac{1}{4}x^2y^2. \tag{2}$$

The dynamics of the Takens-Bogdanov bifurcation with D_4 symmetry was studied by Rucklidge [15], and he founded that a symmetry-breaking, period-doubling bifurcation and chaotic sets with five symmetry types allows a quantitative description of the bifurcation sequence were stability is assigned from one subspace to the another.

This Hamiltonian system (2) can be explored for the largest number of orbits. For instance, at Poincaré surface of section our computations include up to $70 \times 70 = 4900$ orbits, however with chaos indicator we compute even with $700 \times 700 = 562500$ orbits for different energy E.

§2. Numerical techniques

Our goal was to find families of periodic orbits and to create the skeleton of periodic orbits in the Hamiltonian system with D_4, and time-reversal symmetries. For that we use set of numerical techniques that are introduced in this section.

First tool is based on continuation theory implemented in the software AUTO created by [9], that handles continuation and bifurcation problems in ordinary, differential equation [14]. Not only it prevents the continuation of the solution curve irrespective of the direction of this curve, but also it allows to detect and follow vertical solution branches. A disadvantage of this technique is that initial computation requires a well defined periodic orbit, without it we are not able to obtain the complete family of periodic orbits nor bifurcations points on it. For further studies two families were chosen (Fig. 5).

To define initial condition systematic search algorithms were used [5]. This technique was developed based on the Brent's method and the Taylor series method that permits to compute the orbits using extended precision. This technique contains several steps, starting with computing of the Poincaré map. The manifold was chosen to be transverse to all orbits, therefore we choose $y = 0, \dot{x} = 0$ and $\dot{y}$ was obtained from Hamiltonian constant. Considering an orbit which starts at position perpendicular to the x-axis

$$(x(0), y(0), \dot{x}(0), \dot{y}(0)) = (x_0, 0, 0, \dot{y}(0)), \tag{3}$$

and crosses the x-axis again perpendicularly, then the orbit is closed and symmetric. We define a new cross at the half period T of the orbit which is perpendicular to the x-axis. Next step in the method is giving a mesh in the parameter and variable space ($x - \mathcal{H}$ plane). Complete set of initial conditions is specified by a value of x and $\mathcal{H}$. By integrating numerically each set of initial conditions we obtain Poincaré map for a given multiplicity (for more details see, [5]).

Next technique that was used in this work is chaos indicator OFLI2, that is an interesting alternative to the standard Poincaré sections, to distinguish among periodic, regular and chaotic orbits [4]. With the second order variational equations, numerical ODE integrator and a specially developed Taylor method [3] gives a fast and accurate numerical integration. The OFLI2 is looking for a set of initial conditions where we may expect strong dependence on initial conditions. The OFLI2 indicator at the final time t_f is given by

$$\mathrm{OFLI2} := \sup_{0<t<t_f} \log \left\| \left\{ \delta \boldsymbol{y}(t) + \frac{1}{2}\, \delta^2 \boldsymbol{y}(t) \right\}^{\perp} \right\|, \tag{4}$$

where $\delta \boldsymbol{y}(t)$ and $\delta^2 \boldsymbol{y}(t)$ are the first and second order sensitivities with respect to carefully chosen initial vectors and $\boldsymbol{y}^{\perp}$ stands for the component of $\boldsymbol{y}$ orthogonal to the flow [4]. The above description gives us the value of the OFLI2 for a particular orbit for a given set of initial conditions. The OFLI2 picture is describing the global dynamical properties of the system when Poincaré section does only for local multiplicity.

In Fig. 3 we compare the evolution of the OFLI2 for the system with energy $E = 2.0$ and $E = 2.5$ on the surface $y = 0$, with Poincaré section. Note that OFLI2 gives much more information than the Poincaré section and locates the periodic orbits and the chain of regular islands inside the chaotic area (see magnification), where the Poincaré maps instead gives a cloud of points.

§3. Bifurcation

In this section we present a study of the bifurcation points of the dynamical system using the Monodromy Method ([2, 8]). 4×4 matrix (M) provides full information about periodic trajectories and can be represented as a first order variational equation

$$\dot{M} = K \cdot \mathrm{Hess}(\mathcal{H}(\boldsymbol{q}, \boldsymbol{p})) \cdot M. \tag{5}$$

For $T = 0$ we can simplify (5) to $M(0) = I_4$, which is four dimensional identity matrix, K is canonical sympletic matrix and $\mathrm{Hess}(\mathcal{H}(\boldsymbol{u}))$ is the Hessian matrix of $\mathcal{H}$ with respect to $\boldsymbol{u}$. Characteristic multipliers of the fixed point (eigenvalues of M), can be use to study linear stability of the system. From now on, the multipliers will be denoted by λ_i ($i = 1 \ldots 4$) and are in reciprocal pairs

$$\lambda_1 \lambda_2 = 1, \quad \lambda_3 \lambda_4 = 1. \tag{6}$$

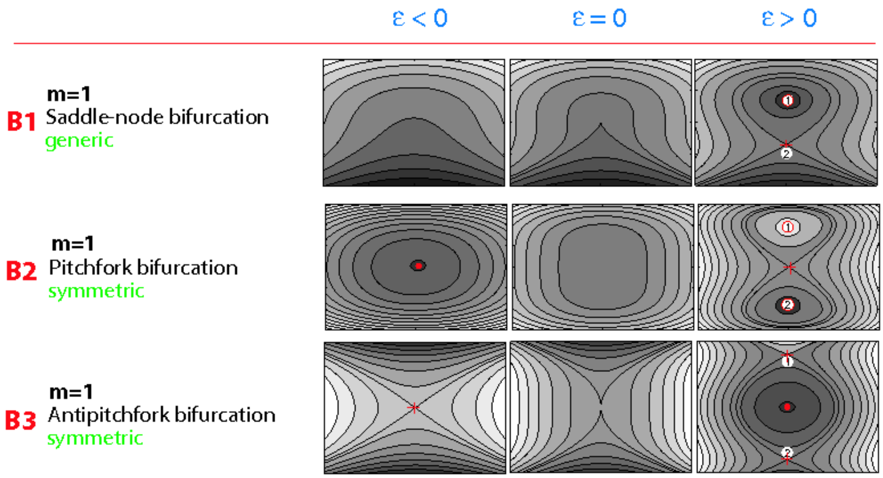

Figure 2: Typical bifurcations for local multiplicity $m = 1$.

That is possible, because system is Hamiltonian and the monodromy matrix M is a real symplectic matrix. Also, complex eigenvalues are in conjugated pairs. In the work we are using definition of stability index introduced by [12] in a form

$$\kappa := \kappa(M(T)) = \text{Tr }(M(T)) - 2, \tag{7}$$

were three cases can be distinguished:

- $|\kappa| < 0$, periodic orbit is stable,
- $|\kappa| > 0$, periodic orbit is unstable (λ_3, λ_4 are real),
- $|\kappa| = 2$, appear special point where stability may change.

The bifurcation point among the family of periodic orbits appears when $\kappa = \lambda_3 + \lambda_4 = 2\ \text{Re}(\lambda_{3,4}) = 2$.

The most typical bifurcation called *saddle-node* bifurcation (Fig. 2) is an example of creating new families of periodic orbits, (apart from the boundaries of the domain of definition of the Poincaré map). This special point is a place where two branches (stable and unstable) met and annihilate (or create).

Since our system have symmetries two more types of the bifurcation points can be detected *pitchfork* and *antipitchfork* (Fig. 2). The former appears when stable family changes to unstable branch and in the same point two new stable branches are created. For antipitchfork is opposite, basic family is unstable and jumps into stable branch and two unstable families are created. In all the cases of the bifurcated families a symmetry lost compare to main family.

A 4-islands chain of isochronous bifurcation was also detected in the system. In this case, the main family after bifurcation point remains in the stable branch and four new stable families are created (see [6]).

3.1. Bifurcation on the system

The focus of this study is pitchfork and 4-islands chain of isochronous bifurcation points. We compare two maps with different energy value $E_1 = 2.5$ (before BP, left) and $E_2 = 2.0$ (after BP, right); Fig. 3. From the maps we know that we start with a stable family and after bifurcation point the main family jumps into unstable branch and two new stable families are created, this can be seen on top of figure, where OFLI2 results are plotted. Those families are also in the skeleton of periodic orbits obtained from AUTO (fig. 4a). If we compare projected orbits from the main family (Fig. 4b) with orbits from the new families we can see that orbits projected into the xy plane, lose one symmetry with respect to y-axis.

The 4-islands chain of isochronous bifurcation is special bifurcation that appears in the symmetric systems Fig. 4c. We project five different periodic orbits from each family. One orbit represents orbit at BP and we see that it is symmetric with respect to x-axis and y-axis, and other orbits loose one of the symmetry. Plot on Fig. 4c contains two bifurcated families, each one consists of two branches. Notice that a orbits from opposite branches have the same shape and are shifted by 180° relative to each other.

§4. Connection symmetric and asymmetric families of periodic orbits

To study the evolution of a periodic orbits along a family we choose two different families (symmetric and asymmetric). In the symmetric family (Fig. 5), we start the evolution from a point close to extreme ($x = 0$, $E = 0$) and moving clockwise. The family that starts with highly eccentricity decreases until reaching the highest energy where eccentricity is the lowest. This family was found to have only one perpendicular intersection with y-axis, so the evolution runs symmetrically. Moreover, along the family we can see that stability changes several times, at those points we have bifurcation points (Fig. 5c). The plot presenting how the orbits change along the families are in the figures (5a, 5d).

From the main symmetric family we choose two bifurcation points and we found two new asymmetric families of periodic orbits (Fig. 5e). Those branches finished at the extreme ($x = 0$, $E = 0$) and are symmetric with respect to x-axis and y-axis. The study of the evolution of this family we start from BP and we decrease value of parameter y. The orbit begins with symmetry with respect to both axis, but the farther we are from bifurcation point, the more significant asymmetry is (Fig. 5f).

In conclusion, we have shown a procedure to obtain skeleton of PO. We have started with creation of a initial conditions using the systematic search with fixed multiplicity. Then those results were used to create the skeleton, which consists of symmetric and asymmetric families of PO. We found also some special bifurcations (the 4-islands chain of isochronous bifurcation) and shown in details the evolution of some families of PO.

Acknowledgements

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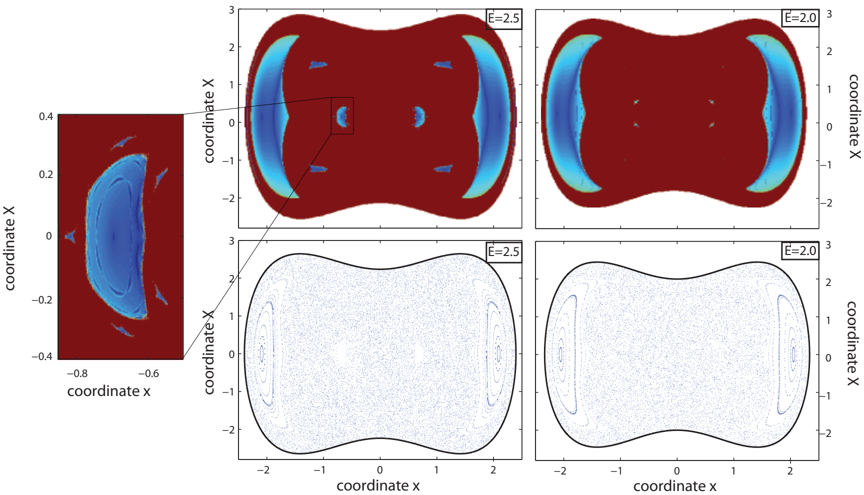

Figure 3: OFLI2 (top) and Poincaré surface of section (bottom), before (left) and after (right) bifurcation point (pitchfork bifurcation), projected on xX plane.

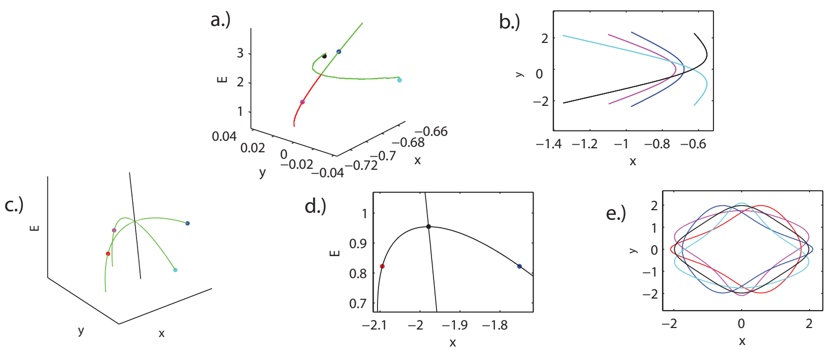

Figure 4: Skeleton of periodic orbits close to pitchfork bifurcation ((a) green and red correspond respectively to stable and unstable family), next to it there are orbits projected on plane xy (b). Outline of 4-islands chain of isochronous bifurcation (c), dots represents orbits projected on xy plane (e). In the middle (d) we got skeleton obtained from AUTO.

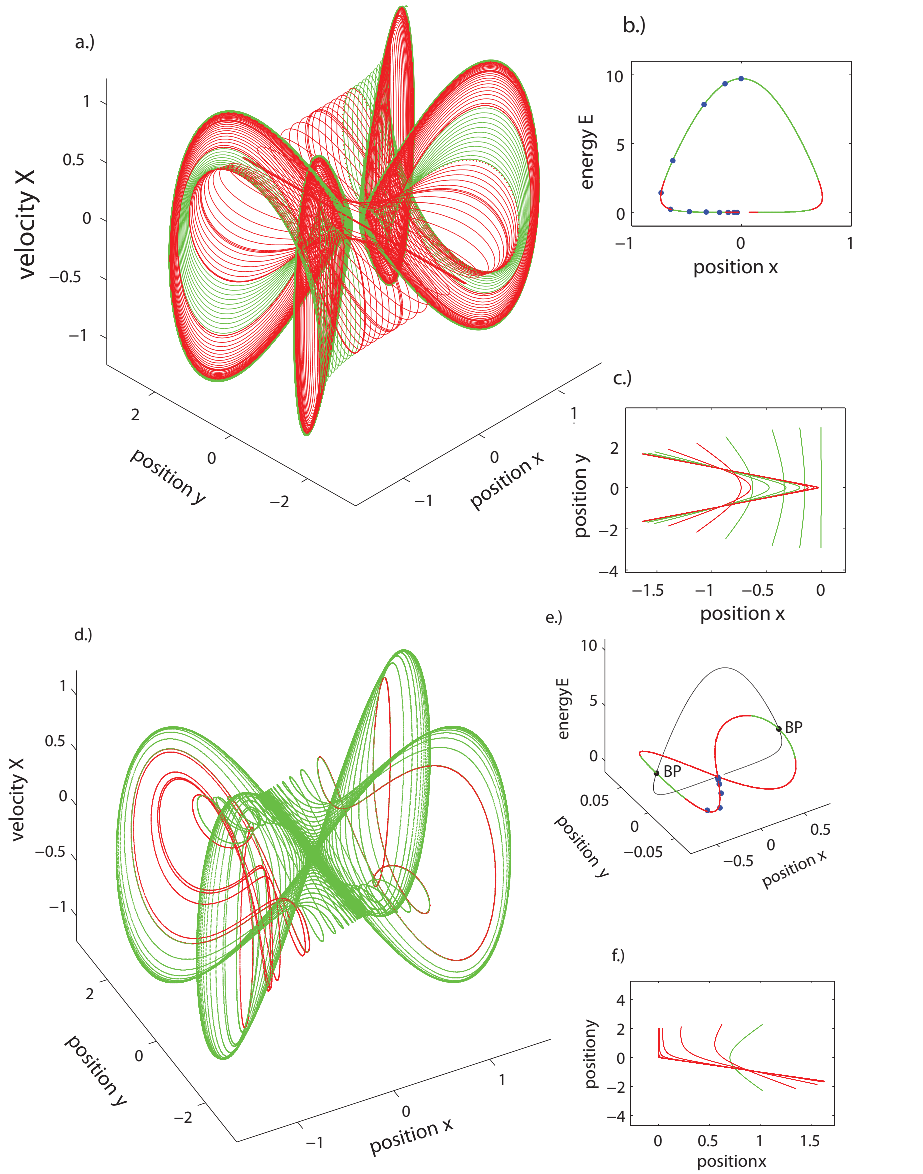

Figure 5: Evolution of symmetric (*a,b,c*) and asymmetric (*d,e,f*) periodic orbits. Main graphs shows (*a,d*), evolution of periodic orbits along the family in three dimension. The main family is plotted on figures *b* and *e* (black). Blue dots represent chosen orbits projected on the plane *xy* (*c, f*). Colors on plots corresponds to stability of orbits, red unstable and green stable.

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