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Legendre Transform of sampled signals by fractal methods

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Abstract. The fractal interpolation functions provide an alternative to the classical methods of study of experimental variables. They have been proved useful in many applications, from image compression to signal processing.

The spectral methods (in terms of trigonometric polynomials) are suitable to model periodic or near periodic phenomena. However some experimental variables are far from periodicity. In this paper we present a method to compute Legendre Transform and series expansions for sampled signals by means of fractal methods.

The periodic Fourier case is generalized considering polynomial orthogonal series. Pointwise, uniform and mean-square convergences of the sums are studied and weak sufficient conditions for these types of approximation are found. The procedures ensure a good approach whenever the sampling frequency and the order of the sums are properly chosen.

Keywords: Fractal interpolation functions, orthogonal expansions, Legendre series.

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§1. Introduction

We present a method of computing a Legendre expansion for a sampled signal, with the single hypothesis of continuity. The calculus is made via an affine fractal interpolation of the experimental variable. For a suitable election of the scale vector, the pointwise, uniform and mean-square convergences of the expansion obtained are proved. The Legendre Transform provides a formula for the power of the signal, where the hypothesis of periodicity is not needed.

§2. Affine fractal interpolation functions

Let $t_0 < t_1 < \cdots < t_N$ be real numbers, and $I = [t_0, t_N]$ the closed interval that contains them. Let a set of data points $\{(t_n, x_n) \in I \times \mathbb{R} : \; n = 0, 1, 2, \dots, N\}$ be given. Set $I_n = [t_{n-1}, t_n]$ and let $L_n : I \to I_n, \; n \in \{1, 2, \dots, N\}$ be contractive homeomorphisms such that:

$$L_n(t_0) = t_{n-1}, \; L_n(t_N) = t_n \tag{1}$$

$$|L_n(c_1) - L_n(c_2)| \leq l \, |c_1 - c_2| \quad \forall \, c_1, c_2 \; \in I \tag{2}$$

for some $0 \leq l < 1$.

Let $-1 < \alpha_n < 1$, for $n = 1, 2, ..., N$, $F = I \times \mathbb{R}$ and N continuous mappings $F_n : F \to \mathbb{R}$ be given satisfying:

$$F_n(t_0, x_0) = x_{n-1}, \quad F_n(t_N, x_N) = x_n, \tag{3}$$

where $n = 1, 2, ..., N$ and

$$|F_n(t, x) - F_n(t, y)| \leq |\alpha_n| \, |x - y| \tag{4}$$

with $t \in I$, $x, y \in \mathbb{R}$.

Now define functions

$$w_n(t, x) = (L_n(t), F_n(t, x))$$

for $n = 1, 2, \ldots, N$.

Theorem 1 (Cf. [1])**.** *The iterated function system (IFS) $\{F, \ w_n \ : \ n = 1, 2, ..., N\}$ defined above admits a unique attractor G. G is the graph of a continuous function $f : I \to \mathbb{R}$ which obeys $f(t_n) = x_n$ for $n = 0, 1, 2, \ldots, N$.*

The previous function is called a fractal interpolation function (FIF) corresponding to $\{(L_n(t), F_n(t, x))\}_{n=1}^{N}$ and it is unique satisfying the functional equation [1]:

$$f(t) = F_n(L_n^{-1}(t), f \circ L_n^{-1}(t)) \tag{5}$$

for $n = 1, 2, ..., N$, $t \in I_n = [t_{n-1}, t_n]$.

The most widely studied fractal interpolation functions so far are defined by the IFS

$$\begin{cases} L_n(t) = a_n t + b_n, \\ F_n(t, x) = \alpha_n x + q_n(t), \end{cases} \tag{6}$$

where

$$a_n = \frac{t_n - t_{n-1}}{t_N - t_0} \quad \text{and} \quad b_n = \frac{t_N t_{n-1} - t_0 t_n}{t_N - t_0}; \tag{7}$$

α_n is called a vertical scaling factor of the transformation w_n and $\bar{\alpha}$ is the scale vector of the IFS, $\bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N)$. In this case, the equation (5) becomes

$$f(t) = \alpha_n \, f \circ L_n^{-1}(t) + q_n \circ L_n^{-1}(t) \tag{8}$$

for $n = 1, 2, \ldots, N$, $t \in I_n = [t_{n-1}, t_n]$.

If $q_n(t)$ is a line, the FIF is termed affine (AFIF). In this case, by Eq. (3), $q_n(t) = q_{n1}t + q_{n0}$, where

$$q_{n1} = \frac{x_n - x_{n-1}}{t_N - t_0} - \alpha_n \frac{x_N - x_0}{t_N - t_0}, \tag{9}$$

$$q_{n0} = \frac{t_N x_{n-1} - t_0 x_n}{t_N - t_0} - \alpha_n \frac{t_N x_0 - t_0 x_N}{t_N - t_0}. \tag{10}$$

These approximants are discussed in the references [4], [5], [6] and [7]. In [4] and [7], several ways of obtaining the scaling factors from the data are presented.

2.1. Rate of approximation

We consider the following notation, for a continuous function g defined on a compact interval I,

$$||g||_\infty = \max\{|g(t)| : t \in I\}$$

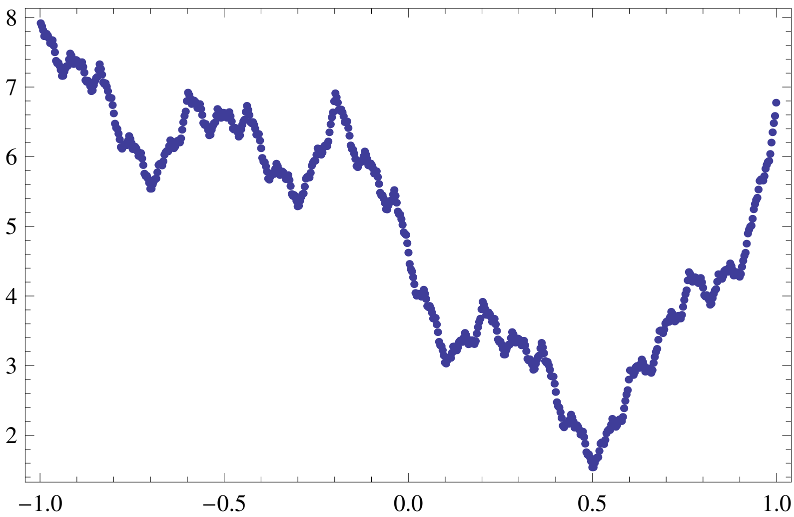

Figure 1: Graph of an affine fractal interpolation function for the set of data points $\{(-1, 8), (-3/5, 7), (-1/5, 7), (1/5, 4), (3/5, 3), (1, 7)\}$ and scale factors $\alpha_n = 0.3$ for $n = 1, 2, \ldots, 5$

The modulus of continuity of g is defined as

$$\omega_g(\delta) = \sup\{|g(t) - g(t')| \,;\, |t - t'| \leq \delta, t, t' \in I\}$$

By $g \in Lip\,\beta$ (g is Hölder-continuous with exponent β) we mean that there exists $M \geq 0$ such that, for all $t, t' \in I$,

$$|g(t) - g(t')| \leq M|t - t'|^{\beta}.$$

Lemma 2. *$g \in Lip\,\beta$ if and only if $\omega_g(\delta) \leq K\delta^{\beta}$.*

Proof. See [3]. □

Proposition 3. *If x is a continuous function providing the data $\{(t_n, x_n)\}_{n=0}^{N}$ with a constant step $h = t_n - t_{n-1}$, and f is the corresponding AFIF with scale vector $\bar{\alpha}$,*

$$\|x - f\|_{\infty} \leq \; w_x(h) + \frac{2|\bar{\alpha}|_{\infty}}{1 - |\bar{\alpha}|_{\infty}} \, \|x\|_{\infty}, \tag{11}$$

where $w_x(h)$ is the modulus of continuity of $x(t)$.

Proof. Let g_0 be the polygonal with vertices $\{(t_n, x_n)\}_{n=0}^{N}$. One has

$$\|x - f\|_{\infty} \leq \|x - g_0\|_{\infty} + \|g_0 - f\|_{\infty}.$$

The first term is bounded in Lemma 3.9 of [7] and the second in Proposition 5.1 of [6]. Thus

$$\|x - g_0\|_{\infty} \leq w_x(h), \tag{12}$$

$$\|g_0 - f\|_\infty \leq \frac{2|\bar{\alpha}|_\infty}{1 - |\bar{\alpha}|_\infty} X_{\max},$$

where $X_{\max} = \max_{0 \leq n \leq N}\{|x_n|\}$ and the result is deduced. □

§3. Legendre Transform

In the article [1], a recurrence formula for the computation of the moments M_m,

$$M_m = \int_I t^m f(t)\, dt \tag{13}$$

was given, for a function f defined by the general iterated function system (6). The formula is expressed as

$$M_m = \frac{1}{(1 - \sum_{n=1}^N a_n^{m+1}\alpha_n)} \left(\sum_{k=0}^{m-1} \binom{m}{k} M_k \sum_{n=1}^N a_n^{k+1}\alpha_n b_n^{m-k} + Q_m \right) \tag{14}$$

where

$$Q_m = \int_I t^m Q(t)\, dt \tag{15}$$

and

$$Q(t) = q_n \circ L_n^{-1}(t) \quad \text{if} \quad t \in I_n \tag{16}$$

Without loss of generality, we consider here the interval $I = [-1, 1]$. Let $\{p_n\}_{n=0}^\infty$ be the system of normalized polynomials of Legendre. These functions are orthonormal with respect to the inner product

$$(f, g) = \int_I f(t)g(t)\, dt. \tag{17}$$

To compute the Fourier-Legendre coefficients of a FIF f with respect to this complete system, we can proceed in the following way; if the n-th Legendre polynomial p_n is

$$p_n(t) = \sum_{m=0}^n d_m t^m,$$

the coefficients of f are

$$c_n = (f, p_n) = \int_I f(t)p_n(t)\, dt = \sum_{m=0}^n d_m \int_I t^m f(t)\, dt = \sum_{m=0}^n d_m M_m, \tag{18}$$

where M_m are the moments defined in (13). The expansion of f in terms of Legendre polynomials is

$$\sum_{n=0}^{+\infty} c_n p_n$$

and the sequence (c_n) is the Legendre transform of f.

§4. Power of the signal

The scalars c_n enable the construction of the expansion

$$f \sim \sum_{n=0}^{\infty} c_n p_n,$$

which is convergent in quadratic mean to f, that is to say, it is convergent with respect to the $\mathcal{L}^2$-norm:

$$\|f\|_2 = \left(\int_I |f(t)|^2 \, dt \right)^{1/2}$$

To compute the convolution (in a wide sense) of two FIFs, we may use the Parseval's identity:

$$(f, g) = \int_I f(t) g(t) \, dt = \sum_{n=0}^{\infty} c_n^f \, c_n^g, \tag{19}$$

where c_n^f and c_n^g are the Fourier coefficients of f and g respect to Legendre polynomials. The power (or energy) of a signal is given by the Parseval's equality as

$$P = (f, f) = \int_I |f(t)|^2 dt = \sum_{n=0}^{+\infty} |c_n|^2,$$

where c_n are the coefficients of f.

Proposition 4. *The error in the computation of the square root of the power is bounded by the expression*

$$\left| P_x^{1/2} - P_f^{1/2} \right| \leq \left(w_x(h) + \frac{2|\bar{\alpha}|_\infty}{1 - |\bar{\alpha}|_\infty} \|x\|_\infty \right) \left(\text{length}(I) \right)^{1/2},$$

where P_x is the power of the original continuous function $x(t)$, P_f is the power computed by means of an AFIF f with scale vector $\overline{\alpha}$, and $\text{length}(I) = (b - a)$ *if $I = [a, b]$.*

Proof. The error in the square root of the power is given by

$$\left| P_x^{1/2} - P_f^{1/2} \right| = \left| \|x\|_2 - \|f\|_2 \right| \leq \|x - f\|_2,$$

where $\|g\|_2 = \left(\int_I |g(t)|^2 \, dt \right)^{1/2}$. Moreover,

$$\|x - f\|_2 = \left(\int_I |x(t) - f(t)|^2 \, dt \right)^{1/2} \leq \|x - f\|_\infty \left(\text{length}(I) \right)^{1/2}. \tag{20}$$

Proposition 3 provides then the estimation of the statement. □

§5. Convergence of the Legendre expansion

The next result proves the validity of using AFIFs to construct Legendre series expansions of a real sampled signal, according to the procedure described in the Section 3.

Theorem 5. *Let $x \in C(I)$ be the original function providing the data. If we choose a fractal f with scale vector $\bar{\alpha}_h$ tending to zero as $h \to 0$, then the Legendre expansion defined by means of f converges in quadratic mean to x as $m \to \infty$ and $h \to 0$.*

Proof. Let $S_m f$ be the m-th partial sum of the Legendre series of f. Let us consider

$$\|x - S_m f\|_2 \leq \|x - f\|_2 + \|f - S_m f\|_2. \tag{21}$$

By (20),

$$\|x - S_m f\|_2 \leq \|x - f\|_\infty (\mathrm{length}(I))^{1/2} + \|f - S_m f\|_2$$

and, by (11),

$$\|x - S_m f\|_2 \leq (\mathrm{length}(I))^{1/2} \left(w_x(h) + \frac{2|\bar{\alpha}_h|_\infty}{1 - |\bar{\alpha}_h|_\infty} \|x\|_\infty \right) + \|f - S_m f\|_2.$$

The uniform continuity of x on I implies that $\lim \omega_x(h) = 0$ as h tends to zero ([3]).

The second adding of (21) goes to zero as m tends to infinity due to the convergence in quadratic mean of the Legendre series of f. □

Remark 1. The former theorem ensures the goodness of the procedure to obtain the power whenever the step and the expansion order are suitably chosen.

In the following we study the pointwise and uniform convergence of the Legendre series. We need two previous lemmas.

Lemma 6. *Let f be a FIF defined by (6) with equally spaced t_n and q_n arbitrary satisfying $q_n(t) \in Lip\, \delta_n$, $0 < \delta_n \leq 1$. Let $\delta = \min\{\delta_n : n = 1, 2, \ldots, N\}$. Then, if $|\bar{\alpha}|_\infty < h^\delta$, $f(t) \in Lip\, \delta$.*

Proof. ([2]) □

Lemma 7. *If $f \in C^p[-1, 1]$ is such that $f^{(p)} \in Lip\, \delta$, then the m-th Legendre sum of f satisfies the inequality*

$$\|f - \sum_{n=0}^{m} c_n p_n\|_\infty \leq \frac{K \ln m}{m^{p+\delta-1/2}} \tag{22}$$

for $p + \delta \geq 1/2$.

Proof. ([9]) □

Theorem 8. *The Legendre expansion of any affine fractal interpolation function f converges pointwisely to f almost everywhere. If the scale vector of f is such that $|\bar{\alpha}|_\infty < h$ then the Legendre expansion of f converges pointwise and uniformly to f on the interval $I = [-1, 1]$.*

Proof. In the reference [8], the author proves that the Legendre series of any function $f \in \mathcal{L}^p(I)$ such that $p > 4/3$ converges pointwisely to f almost everywhere. This fact assures the pointwise convergence for any AFIF a.e. (due to its continuity on I).

The mappings q_n defined in the Section 2 are linear and, consequently, $q_n \in Lip\,1$. If $|\bar{\alpha}|_\infty < h$ according to the Lemma 6, $f(t) \in Lip\,1$. Now, we apply the Lemma 7 for $p = 0$ and $\delta = 1$ obtaining

$$\left\| f - \sum_{n=0}^{m} c_n p_n \right\|_\infty \leq \frac{K \ln m}{m^{1/2}}. \tag{23}$$

As m tends to infinity the Legendre sum tends to f and the uniform convergence is satisfied on the interval $I = [-1, 1]$. □

Remark 2. This result is true for any step h.

Theorem 9. *Let $x(t) \in C(I)$ be the original function providing the data. If we choose $|\bar{\alpha}|_\infty < h$, then the Legendre expansion defined by means of an AFIF converges uniformly to x as $m \to \infty$ and $h \to 0$.*

Proof. The uniform continuity of $x(t)$ on I implies that $\lim \omega_x(h) = 0$ as h tends to zero ([3]). Let $S_m f$ be the m-th partial sum of the Legendre series of f. Let us consider

$$\|x - S_m f\|_\infty \leq \|x - f\|_\infty + \|f - S_m f\|_\infty.$$

The first term goes to zero if $h \to 0$ due to Proposition 3. The second term goes to zero as well when $m \to \infty$ according to the previous theorem,

$$\lim_{m\to\infty} \|f - S_m f\|_\infty = 0$$

and the result is obtained. □

Remark 3. The former theorem ensures the goodness of the procedure to represent and evaluate the signal whenever the step and the expansion order are suitably chosen.

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