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UNIQUENESS OF STRONG SOLUTIONS TO DOUBLY NONLINEAR EVOLUTION EQUATIONS

Jochen Merker

Abstract. In this article uniqueness of strong solutions to the abstract doubly nonlinear evolution equation

$$\frac{\partial Bu}{\partial t} + Au = f$$

is discussed under the main assumptions that B^{-1} is strongly monotone and there is a $C < \infty$ such that $\Phi_A + C\Phi_B$ is convex for the potentials Φ_A resp. Φ_B of A resp. B.

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§1. Introduction

The aim of this article is to discuss strong solutions of abstract doubly nonlinear evolution equations

$$\frac{\partial Bu}{\partial t} + Au = f\,, \tag{1}$$

and especially the uniqueness of strong solutions to an initial value. Hereby, $A : X \to X^*$ resp. $B : Y \to Y^*$ are operators on Banach spaces X resp. Y with a dense and separable intersection, and f is an inhomogeneity or nonlinearity.

Uniqueness of weak solutions to initial data with finite energy has been established for the concrete case of a degenerate elliptic-parabolic equation

$$\frac{\partial b(u)}{\partial t} + \operatorname{div}(a(b(u), \nabla u)) = f \tag{2}$$

by [11] via an L^1-contraction principle for $b(u)$. Uniqueness of entropy solutions to L^1-initial data has been shown by [3] (even in presence of transport terms and therefore for degenerate elliptic-parabolic-hyperbolic equations), and uniqueness of renormalized solutions has been proved by [4]. In literature uniqueness is also discussed for several variants of (2) like the anisotropic case ([9]), the so-called triply nonlinear case ([1]) or the case of variable exponents ([2]). All these articles have in common that uniqueness is proved via Kruzhkov's method of doubling the variables.

In this article, an elementary proof of the uniqueness of strong solutions to the abstract problem (1) along the lines of [7, 5, 6] is given, see also [12, Section 8.5 and 11.2.3]. While a discussion of the abstract problem is more general than a discussion of the concrete equation (2) (e.g. parts of B could be fractional derivatives or general convolution operators), it is a major restriction to prove uniqueness only for strong solutions and not for weak, entropy or renormalized solutions, because in general strong solutions may not exist. However, this is the price to pay for applying an elementary method instead of a more sophisticated method like Kruzhkov's doubling of variables.

1.1. Outline

In Section 2 existence of strong solutions to (1) is established for initial values $u_0 \in X \cap Y$ under the main additional assumption that

$$\langle v^*, dB^{-1}(u^*)v^* \rangle \geq c\|v^*\|_{H^*}^2 \tag{3}$$

holds for all $u^*, v^* \in Y^*$ with a constant $c > 0$, where $X \cap Y \subset H \subset Y$ is an interpolation triple with a Hilbert space H. This assumption is equivalent to strong monotonicity of B^{-1} as an operator $B^{-1} : Y^* \subset H^* \to H$. Note that there are also other situations which allow to prove the existence of certain types of strong solutions (see [10]), but here we concentrate on this situation.

For the concrete equation (2) existence of strong solutions can be guaranteed for regular initial data and potential $a = d\phi_a$, if b is not only assumed to be nondecreasing, but additionally b^{-1} is assumed to be differentiable with a nonvanishing derivative at 0. Thus, b must not be degenerate or singular at 0, but is still allowed to grow nonlinearly.

Uniqueness of strong solutions is shown in section 3 under the convexity assumption that there is a $C < \infty$ such that $\Phi_A + C\Phi_B$ is convex for the potentials Φ_A resp. Φ_B of A resp. B. Further, continuous dependence of strong solutions on the initial value and on the right hand side is established within this abstract framework. However, before we start our discussion let us mention two examples which illustrate that in general neither u nor Bu need to be unique.

1.2. Examples for non-uniqueness

The following examples illustrate in which way weak solutions of a doubly nonlinear reaction diffusion equation (2) to an initial value may not be unique.

Example 1. Let $A : W^{1,2}(\Omega) \to (W^{1,2}(\Omega))^*$ be the negative of the one-dimensional Laplacian on the interval $\Omega := (0, 1)$ under Neumann-boundary conditions $\partial u/\partial x = 0$ on $\partial\Omega$, and let $B : L^2(\Omega) \to L^2(\Omega)$ be the superposition operator $(Bu)(x) := b(u(x))$ induced by

$$b(u) := \begin{cases} u + 1, & \text{if } u \leq -1, \\ 0, & \text{if } -1 \leq u \leq 1, \\ u - 1, & \text{if } u \geq 1. \end{cases}$$

Obviously, B is a monotone potential operator, which is coercive, bounded and continuous. However, the equation $\partial Bu/\partial t + Au = 0$ does not have a unique solution u to the zero function as initial value of Bu. In fact, if $u(t, x)$ is an arbirary continuous function independent of x which attains values between -1 and 1, then $Au(t) = 0$ and $Bu(t) = 0$ for every t. Thus, there are many weak solution u to the the initial value 0 of Bu.

Non-uniqueness of u may not be considered as a problem if at least Bu is unique. However, in general it may even happen that Bu is not unique, as the following example shows, where B is multivalued (so that B^{-1} is not strictly monotone), see also [8, Remark 4].

Example 2. Let $\Omega := (0,1)$, let $B : L^2(\Omega) \to L^2(\Omega)$ be the superposition operator induced by the multivalued mapping

$$b(u) := \begin{cases} u-1, & \text{if } u < 0, \\ [-1,1], & \text{if } u = 0, \\ u+1, & \text{if } u > 0, \end{cases}$$

and let $A : W_0^{1,2}(\Omega) \to (W_0^{1,2}(\Omega))^*$ be the operator $\langle Au, w\rangle = \int_\Omega u_x w_x + b(u) w_x \, dx$, i.e. $Au = -u_{xx} - v_x$ on smooth functions with $v(t,x) \in b(u(t,x))$ under Dirichlet conditions $u = 0$ on $\partial\Omega$. Then one solution to the initial value 1 of Bu is given by $u := 0$ and $v := 1 \in Bu$, but for every $C \geq 1$ and every C^1-function h on $[0,\infty)$ with values between 0 and 2 also $u := 0$ and

$$v(t,x) := \begin{cases} 1, & \text{if } 0 \leq t+x \leq C, \\ 1 - h(t+x-C), & \text{if } t+x \geq C, \end{cases}$$

define a solution with $v(0) = 1$ due to $v_t - v_x = 0$ (and $u_{xx} = 0$).

§2. Existence of strong solutions

In this section the existence of strong solutions to the abstract equation (1) is discussed by energy methods for the case that B^{-1} exists and is strongly monotone as an operator on some intermediate Hilbert space. However, first let us formulate standard structural assumptions which allow to prove existence of weak solutions to (1) :

(A1) X and Y are reflexive Banach spaces with a dense and separable intersection $X \cap Y$ [1], which is compactly embedded into Y.

(A2) $B : Y \to Y^*$ is a continuous strictly monotone potential operator, which is coercive and satisfies the growth condition $\|Bu\|_{Y^*} \leq C(1 + \|u\|_Y^{m-1})$ with a constant $C < \infty$ and a parameter $1 < m < \infty$.

(A3) $A : X \to X^*$ is a pseudomonotone operator, which satisfies the semicoercivity condition $\langle Au, u\rangle \geq c_1\|u\|_X^p - c_2\|u\|_X - c_3\|Bu\|_{Y^*}^{m'}$ and has growth $\|Au\|_{X^*} \leq C(\|u\|_Y)(1 + \|u\|_X^{p-1})$ for a parameter $1 < p < \infty$ with constants $c_1 > 0$, c_2, c_3 and an increasing function $C : \mathbb{R}_0^+ \to \mathbb{R}_0^+$.

If $f \in L^{p'}(0,T;X^*) + L^1(0,T;Y^*)$ is an inhomogeneity, then under the assumptions (A1)-(A3) a weak solution u exists to an initial value $u_0 \in Y$ in the sense that $u \in L^p(0,T;X) \cap L^\infty(0,T;Y)$ is such that $Bu \in L^\infty(0,T;Y^*)$ has the initial value $Bu_0 \in Y^*$ and a weak derivative $\partial Bu/\partial t \in L^{p'}(0,T;X^*) + L^1(0,T;Y^*)$ satisfying (1) as an equation in $(X \cap Y)^*$ for a.e. $t \in (0,T)$, or equivalently as an equation in $L^{p'}(0,T;X^*) + L^1(0,T;Y^*)$.

[1] i.e. there are continuous linear embeddings of X and Y into a complete locally convex space Z such that the intersection $X \cap Y$ within Z is dense in X resp. Y w.r.t. the norms $\|\cdot\|_X$ resp. $\|\cdot\|_Y$, and that $X \cap Y$ is separable w.r.t. the norm $\|\cdot\|_X + \|\cdot\|_Y$.

Here we are interested in a slightly different case, where the inhomogeneity f satisfies $f \in L^{p'}(0,T;X^*)+L^2(0,T;H^*)$ for an intermediate Hilbert space H of the inclusion $X \cap Y \subset Y$ given by (A1). More precisley, we require that $X \cap Y \subset H \subset Y$ is an interpolation triple, i.e. there is a $\theta \in [0,1]$ and a constant $C < \infty$ such that $\|u\|_H \leq C\|u\|_X^\theta \|u\|_Y^{1-\theta}$ for every $u \in X \cap Y$. In this case, under the additional assumptions that B satisfies the coercivity condition $\|u\|_Y \leq C(1 + \|Bu\|_{Y^*}^{m'})$ with a constant $C < \infty$ and $p \geq 2$ or $1/2 \leq \theta \leq p/2$ hold, there exists a weak solution of (1) in the following sense:

Definition 1. A function $u \in L^p(0,T;X) \cap L^\infty(0,T;Y)$ is called a weak solution of equation (1) to the initial value $u_0 \in Y$, if $Bu \in L^\infty(0,T;Y^*)$ has the initial value $Bu_0 \in Y^*$ and a weak derivative $\partial Bu/\partial t \in L^{p'}(0,T;X^*) + L^2(0,T;H^*)$ satisfying equation (1) as an equation in $(X \cap H)^*$ for a.e. $t \in (0,T)$, or equivalently as an equation in $L^{p'}(0,T;X^*) + L^2(0,T;H^*)$.

The existence of weak solutions in the sense of Definition 1 can even be generalised to the case where $f = f(t,u)$ is a nonlinearity. In fact, if B satisfies the stronger coercivity condition $\|u\|_Y \leq C(1 + \|Bu\|_{Y^*}^{m'-1})$ with a constant $C < \infty$ and $f = f(t,u)$ is a nonlinearity which satisfies the growth condition $\|f(t,u)\|_{H^*} \leq C(\gamma(t) + \|u\|_Y^{(m-1)(1-\theta)})$ with a constant $C < \infty$ and a function $\gamma \in L^2(0,T)$, then there still exist weak solutions in the sense of Definition 1.

Now we are interested in assumptions, which guarantee that weak solutions even have better properties than those mentioned in Definition 1. The following theorem formulates such assumptions in the special case that $B^{-1} : Y^* \subset H^* \to H$ is strongly monotone, see [10].

Theorem 1. *Additionally to the structural assumptions (A1)-(A3) assume that H is a Hilbert space such that $X \cap Y \subset H \subset Y$ is an interpolation triple and $p \geq 2$ or $1/2 \leq \theta \leq p/2$ hold. Further, assume that*

- *$B^{-1} : Y^* \to Y$ is C^1, satisfies $\|u\|_Y \leq C(1 + \|Bu\|_{Y^*}^{m'-1})$ with a constant $C < \infty$, and is strongly monotone in the sense that $\langle v^*, dB^{-1}(u^*)v^*\rangle \geq c\|v^*\|_{H^*}^2$ for all $u^*, v^* \in Y^*$ with a constant $c > 0$* [2]*,*
- *$A : X \to X^*$ is a potential operator such that the intersection of Y and the domain $D(A) := \{u \in X \,|\, Au \in H^*\}$ of A w.r.t. H^* is dense in $X \cap Y$,*
- *f is an inhomogeneity in $L^2(0,T;H^*)$ or a nonlinearity $f = f(t,u)$ such that $g(t,u) := dB^{-1}(Bu)^* f(t,u)$ satisfies the growth condition $\|g(t,u)\|_H \leq C\left(\gamma(t) + \|u\|_Y^{(m-1)(1-\theta)}\right)$ with a constant $C < \infty$ and a function $\gamma \in L^2(0,T)$.*

Then there exists to every initial value $u_0 \in X \cap Y$ a strong solution u of equation (1) *in the sense that u is a weak solution which additionally satisfies $u \in L^\infty(0,T;X)$, and $Bu \in L^\infty(0,T;Y^*)$ and a weak derivative $\partial Bu/\partial t \in L^2(0,T;H^*)$.*

Let us shortly sketch the proof of this theorem given in [10].

Proof. Use a Faedo-Galerkin method and consider the restrictions

$$\frac{\partial B_k u_k}{\partial t} + A_k u_k = f_k \tag{4}$$

of equation (1) to an increasing sequence of finite-dimensional subspaces $W_k \subset D(A) \cap Y \subset X \cap Y$, where A_k, B_k are the restrictions of A, B to W_k and f_k is a continuous approximation

[2]This condition is equivalent to strong monotonicity of B^{-1} as an operator $B^{-1} : Y^* \subset H^* \to H \subset Y$, i.e. to $\langle u^* - v^*, B^{-1}u^* - B^{-1}v^*\rangle \geq c\|u^* - v^*\|_{H^*}^2$ for arbitrary $u^*, v^* \in Y^*$ with a constant $c > 0$.

of f with values in W_k. Due to (A1)-(A3) short-time existence of solutions u_k of this ODE to initial values $u_{0k} \in W_k$ can be guaranteed. Test (4) by u_k to obtain from the semicoercivity condition on A the a priori estimate

$$\hat{\Phi}_B(u_k(t)) + \left(c_1 - \frac{\epsilon^p}{p}\right) \int_0^T \|u_k(s)\|_X^p \, ds$$
$$\leq \hat{\Phi}_B(u_{0k}) + \frac{1}{p' \epsilon^{p'}} |c_2|^{p'} T + \int_0^t c_3 \|B_k u_k(s)\|_{Y^*}^{m'} \, ds + \int_0^t \|f_k(s)\|_{H^*} \|u\|_H \, ds \,,$$

where $\hat{\Phi}_B(u) = \Phi_B^*(Bu)$ denotes the Legendre transform of the convex potential Φ_B of B in dependence of Bu, $\epsilon > 0$ is sufficiently small and the energy identity $\frac{d}{dt}\hat{\Phi}_B(u) = \langle \frac{\partial Bu}{\partial t}, u \rangle$ was used. As a consequence of the growth condition $\|Bu\|_{Y^*} \leq C(1 + \|u\|_Y^{m-1})$ we have $\|Bu\|_{Y^*}^{m'} \leq C(1 + \hat{\Phi}_B(u))$, as a consequence of the coercivity condition $\|u\|_Y \leq C(1 + \|Bu\|_{Y^*}^{m'})$ we have $\|u\|_Y \leq C(1 + \hat{\Phi}_B(u))$, and the assumptions $p \geq 2$ or $1/2 \leq \theta \leq p/2$ allow to estimate the last term by $C \int_0^t (1 + \hat{\Phi}_B(u)) \, ds$ in the case that $f \in L^2(0, T; H^*)$ is an inhomogeneity. In the case that $f = f(t, u)$ is a nonlinearity apply inequality (3) to $u^* := Bu$, $v^* := f(t, u)$, to obtain

$$c\|f(t,u)\|_{H^*}^2 \leq \langle f(t,u), dB^{-1}(Bu) f(t,u) \rangle = \langle f(t,u), g(t,u) \rangle \leq \|f(t,u)\|_{H^*} \|g(t,u)\|_H$$

so that by the assumptions on g the growth condition

$$\|f(t,u)\|_{H^*} \leq \frac{1}{c} \|g(t,u)\|_H \leq \frac{C}{c} \left(\gamma(t) + \|u\|_Y^{(m-1)(1-\theta)} \right)$$

is valid and the last term can again be estimated by $C \int_0^t (1 + \hat{\Phi}_B(u)) \, ds$. Thus, Gronwall's lemma allows to deduce uniform bounds w.r.t. k of u_k in $L^\infty(0, T; Y) \cap L^p(0, T; X)$, Bu_k in $L^\infty(0, T; Y^*)$ and Au_k in $L^{p'}(0, T; X^*)$. Due to these bounds a weakly convergent subsequence $u_k \rightharpoonup u$ can be extracted. Finally, time-compactness and pseudomonotonicity allow to conclude that u is a weak solution of (1).

To obtain a strong solution we would like to test the approximate equation (4) by $\partial u_k/\partial t$, but (4) only guarantees the existence of $\partial B_k u_k/\partial t \in C(0, T; W^*)$ and not the existence of $\partial u_k/\partial t$. However, as B^{-1} is assumed to be continuously differentiable, the chain rule implies the existence of

$$\frac{\partial u}{\partial t} = dB^{-1}(Bu) \frac{\partial Bu}{\partial t} \,. \tag{5}$$

Due to $W_k \subset D(A)$ and $f_k \in L^2(0, T; H^*)$ a solution $u_k \in C^1(0, T; W_k)$ of the approximate equation (4) satisfies $\partial B_k u_k/\partial t \in H^*$ for a.e. t. Especially, inequality (3) can be applied to $u^* := Bu_k(t)$, $v^* = \partial Bu_k(t)/\partial t$, to obtain

$$\left\langle \frac{\partial B_k u_k}{\partial t}, \frac{\partial u_k}{\partial t} \right\rangle \geq c \left\| \frac{\partial Bu_k}{\partial t} \right\|_{H^*}^2 .$$

Further, as $g(t, u) := dB^{-1}(Bu)^* f(t, u)$ satisfies $\|g(t, u)\|_H \leq C(\gamma(t) + \|u\|_Y^{(m-1)(1-\theta)})$ with a constant $C < \infty$ and a function $\gamma \in L^2(0, T)$, and as a uniform bound of u_k in $L^\infty(0, T; Y)$

w.r.t. k has already been established, we can conclude that $g(\cdot, u_k(\cdot))$ is uniformly bounded in $L^2(0,T;H)$. Thus, a test of (4) by $\partial u_k/\partial t$ yields

$$\left(c - \frac{\epsilon^2}{2}\right)\left\|\frac{\partial Bu_k}{\partial t}\right\|_{H^*}^2 + \frac{d}{dt}\Phi_A(u_k) \leq \frac{1}{2\epsilon^2}\|g(\cdot, u_k(\cdot))\|_H^2 \leq C$$

with a constant $C < \infty$ for sufficiently small $\epsilon > 0$. Using this differential inequality uniform a priori estimates w.r.t. k of $\partial Bu_k/\partial t$ in $L^2(0,T;H^*)$ and u_k in $L^\infty(0,T;X)$ can be established. Therefore, additionally we are able to guarantee weak* convergence of a subsequence of the approximate solutions u_k in $L^\infty(0,T;X)$ and weak convergence of $\partial Bu_k/\partial t$ in $L^2(0,T;H^*)$, It is simple to verify that the weak limits of these sequences are identical with their expected values u and $\partial Bu/\partial t$, hence the proof of Theorem 1 is finished. □

As a consequence of Theorem 1 we have $Bu \in W^{1,2}(0,T;H^*) \subset C(0,T;H^*)$ for strong solutions due to $Bu \in L^\infty(0,T;Y^*) \subset L^2(0,T;H^*)$ and $\partial Bu/\partial t \in L^2(0,T;H^*)$. Further, as $\partial Bu/\partial t$ and f lie in $L^2(0,T;H^*)$, also $Au = f - \partial Bu/\partial t$ lies in $L^2(0,T;H^*)$. Therefore, equation (1) holds as an equation in H^* for a.e. $t \in [0,T]$, and thus $u(t) \in D(A)$ for a.e. $t \in [0,T]$. Let us explicitly mention this observation as a corollary.

Corollary 2. *Under the assumptions of Theorem 1 the relation $Au \in L^2(0,T;H^*)$ holds for a strong solution u, and equation* (1) *is valid as an equation in H^* for a.e. $t \in (0,T)$.*

The following example shows how Theorem 1 can be applied to the concrete problem (2).

Example 3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $1 < m < 2$. Consider the space $Y := L^m(\Omega)$ so that $H := L^2(\Omega)$ is continuously embedded into Y. Assume that $\phi_b : \mathbb{R} \to \mathbb{R}$ is a convex function which behaves like $(C_1/2)|u|^2 + o(|u|^2)$ as $|u| \to 0$ and like $(C_2/m)|u|^m + \omega(|u|^m)$ as $|u| \to \infty$. Denote by $b := d\phi_b$ the derivative of ϕ_b and by $B : Y \to Y^*$ the corresponding superposition operator. Then $b^{-1}(u)$ behaves like ${C_1}^{-1}u$ as $|u| \to 0$ and like $C_2^{1-m'}|u|^{m'-2}u$ as $|u| \to \infty$, so that $(b^{-1})'(u)$ behaves like ${C_1}^{-1}$ as $|u| \to 0$ and like $(m'-1)C_2^{1-m'}|u|^{((2-m)/(m-1))}$ as $|u| \to \infty$. Especially, pointwisely $(b^{-1})'(u) \geq c$ for a constant $c > 0$ so that

$$c\|v^*\|_2^2 = \int_\Omega c|v^*|^2\,dx \leq \int_\Omega (b^{-1})'(u^*)|v^*|^2\,dx$$

and as a consequence

$$\langle v^*, dB^{-1}(u^*)v^*\rangle \geq c\|v^*\|_2^2$$

for all $u^*, v^* \in Y^*$, i.e. inequality (3) is valid. Note that although b is not degenerate or singular at $u = 0$, the operator B can not be realized as an operator on H as b grows like $C|u|^{m-1}$ as $|u| \to \infty$. Thus (2) is not degenerate or singular at $u = 0$, but still should be considered as an equation for $u \in Y$ and not for $u \in H$.

Finally, assume that a has a p-coercive potential, $1 < p < \infty$, e.g. $a(\nabla u) = |\nabla u|^{p-2}\nabla u$, and consider the corresponding operator $A : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$, $\langle Au, v\rangle := \int_\Omega a(\nabla u)\cdot\nabla u\,dx$, so that (2) is solved under Dirichlet boundary conditions. For this choice $X := W_0^{1,p}(\Omega)$, and $m < p^*$ has to be required to have a compact embedding $X \cap Y \subset Y$. Now Gagliardo-Nirenberg inequalities

$$\|u\|_{L^2} \leq \|\nabla u\|_{L^{p^*}}^\theta \|u\|_{L^m}^{1-\theta}$$

are valid for $1/2 = \theta/p^* + (1-\theta)/m$, $1/p^* \le 1/2$, where the parameter θ of the interpolation triple $X \cap Y \subset H \subset Y$ is given by $\theta = ((2-m)p^*)/(2(p^*-m))$. Especially, in the case $p < 2$ the inequality $1/2 \le \theta \le p/2$ is valid iff $((2-p)p^*)/(p^*-p) \le m \le p^*/(p^*-1)$. For example, if $n = 3$ and p is slightly smaller than 2, then already $m < 6/5$ has to be required.

Further, the right hand side f of (2) and $g := dB^{-1}(Bu)^* f(u)$ are related by

$$f(t,x,u) = \frac{g(t,x,u)}{(b^{-1})'(u)}, \tag{6}$$

where $(b^{-1})'$ is bounded away from zero. Thus, if $g(t,x,u)$ is a pregiven nonlinearity such that $|g(t,x,u)| \le C\left(\gamma(t,x) + |u|^{(m-1)(1-\theta)}\right)$, then by (6) a corresponding right hand side f can be defined such that the assumptions of Theorem 1 are satisfied. Thus, under the former conditions there exists a strong solution of (2) to initial values $u_0 \in W_0^{1,p}(\Omega) \cap L^m(\Omega)$.

Finally, it can be shown that inhomogeneities

$$f \in L^{2\left(\left(m(m-1)^2(p^*-2)\right)/\left(2(2-m)(p^*-m)\right)\right)'}\left(0,T; L^{\left(2m(m-1)\right)/\left((m+1/2)^2-17/4\right)}(\Omega)\right)$$

can be represented via (6) by a function $g(t,x,u) = f(t,x)(b^{-1})'(u)$ satisfying the growth condition provided that

$$\frac{\sqrt{17}-1}{2} < m \le 2 \quad \text{and} \quad p^* > \frac{2m(m^2-m-1)}{m^3-2m^2+3m-4}$$

in the case $p < n$.

§3. Uniqueness of strong solutions

Under the assumptions of Theorem 1 equation (1) admits a strong solution to an initial value $u_0 \in X \cap Y$ in the sense that $u \in L^\infty(0,T; X \cap Y)$ is a weak solution such that $Bu \in L^\infty(0,T;Y^*)$ has a weak derivative $\partial Bu/\partial t \in L^2(0,T;H^*)$, and especially $Au \in L^2(0,T;H^*)$. The following theorem guarantees uniqueness of strong solutions and continuous dependence on the initial value and the right hand side.

Theorem 3. *Additionally to the assumptions of Theorem 1 suppose that there is a constant $C < \infty$ such that*

$$\langle Au - Av, u-v\rangle + C\langle Bu - Bv, u-v\rangle \ge 0 \; \textit{for all } u,v \in X \cap Y \textit{ and} \tag{7}$$

$$\langle Bu - Bv, dB^{-1}(Bu)Au - dB^{-1}(Bv)Av\rangle + C\langle Bu - Bv, u - v\rangle \ge 0 \; \textit{ for all } u,v \in D(A) \cap Y, \tag{8}$$

where $D(A) = \{u \in X \,|\, Au \in H^*\}$ denotes the domain of A w.r.t. H^*. Then the following statements are valid:

- *If $f = 0$, then strong solutions of equation (1) are unique.*
- *If $f \in L^1(0,T;Y^*)$ and $dB^{-1} : Y^* \subset H \to L(Y^*, H)$ is Lipschitz continuous, then strong solutions of equation (1) are unique and $Y \ni u_0 \mapsto Bu \in C(0,T;H^*)$ is continuous.*
- *If dB^{-1} and B^{-1} are Lipschitz continuous, then $Y \times L^1(0,T;Y^*) \ni (u_0, f) \mapsto Bu \in C(0,T;H^*)$ is continuous.*

Remark 1. Note that inequality (7) is equivalent to the convexity of $\Phi_A + C\Phi_B$ on $X \cap Y$, while inequality (8) is equivalent to the convexity of $\Phi_A \circ B^{-1} + C\Phi_B^*$ on $B(D(A) \cap Y)$, where Φ_B^* is the Legendre transform of Φ_B and hence a potential of B^{-1}.

Proof. Assume that u, v are strong solutions of

$$\frac{\partial Bu}{\partial t} + Au = f_1 \quad \text{resp.} \quad \frac{\partial Bv}{\partial t} + Av = f_2\,.$$

To prove uniqueness, test the difference of these equations by $u - v$ and integrate the resulting equation over $[0, t]$ to obtain

$$\int_0^t \left\langle \frac{\partial}{\partial s}(Bu - Bv), u - v \right\rangle ds + \int_0^t \langle Au - Av, u - v\rangle\, ds = \int_0^t \langle f_1 - f_2, u - v\rangle\, ds\,.$$

Now

$$\left\langle \frac{\partial}{\partial s}(Bu - Bv), u - v \right\rangle = \frac{d}{dt}\langle Bu - Bv, u - v\rangle - \left\langle Bu - Bv, \frac{\partial}{\partial s}(u - v) \right\rangle$$

and thus

$$\begin{aligned}\int_0^t \left\langle \frac{\partial}{\partial s}(Bu - Bv), u - v \right\rangle ds &= (\langle Bu - Bv, u - v\rangle)(t) - (\langle Bu - Bv, u - v\rangle)(0) \\ &\quad - \int_0^t \langle Bu - Bv, dB^{-1}(Bu)(f_1 - Au) - dB^{-1}(Bv)(f_2 - Av)\rangle\, ds\end{aligned}$$

due to $\partial u/\partial t = dB^{-1}(Bu)\partial Bu/\partial t = dB^{-1}(Bu)(f_1 - Au)$ and similar for v. Hence, if $f_1 = 0 = f_2$, then

$$\begin{aligned}&(\langle Bu - Bv, u - v\rangle)(t) \\ &= (\langle Bu - Bv, u - v\rangle)(0) - \int_0^t \langle Au - Av, u - v\rangle\, ds - \int_0^t \langle Bu - Bv, dB^{-1}(u)Au - dB^{-1}(v)Av\rangle\, ds \\ &\leq (\langle Bu - Bv, u - v\rangle)(0) + 2C \int_0^t \langle Bu - Bv, u - v\rangle\, ds\end{aligned}$$

due to the assumptions (7) and (8). By Gronwall's lemma

$$(\langle Bu - Bv, u - v\rangle)(t) \leq (\langle Bu - Bv, u - v\rangle)(0) \exp(2Ct)\,,$$

so that $u(0) = v(0)$ implies $(\langle Bu - Bv, u - v\rangle)(t) = 0$ for a.e. $t \in [0, T]$ and hence $u = v$ by strict monotonicity of B.

If $f_1 = f_2 =: f \in L^1(0, T; Y^*)$, then

$$\begin{aligned}&(\langle Bu - Bv, u - v\rangle)(t) \\ &= (\langle Bu - Bv, u - v\rangle)(0) - \int_0^t \langle Au - Av, u - v\rangle\, ds \\ &\quad - \int_0^t \langle Bu - Bv, dB^{-1}(u)Au - dB^{-1}(v)Av\rangle\, ds + \int_0^t \langle Bu - Bv, (dB^{-1}(Bu) - dB^{-1}(Bv))f\rangle\, ds \\ &\leq (\langle Bu - Bv, u - v\rangle)(0) + 2C \int_0^t \langle Bu - Bv, u - v\rangle\, ds + M \int_0^t \|Bu - Bv\|_{H^*}^2 \|f\|_{Y^*}\, ds\end{aligned}$$

with the Lipschitz constant M of $dB^{-1} : Y^* \subset H^* \to L(Y^*, H)$. By strong monotonicity of $B^{-1} : Y^* \subset H^* \to H$ the inequality $\|Bu - Bv\|_{H^*}^2 \le c^{-1}\langle Bu - Bv, u - v\rangle$ is valid, hence by Gronwall's lemma

$$(\langle Bu - Bv, u - v\rangle)(t) \le (\langle Bu - Bv, u - v\rangle)(0) \exp(2CT + \frac{M}{c}\int_0^T \|f\|_{Y^*}\, ds) .$$

Especially, again by strong monotonicity of B^{-1}

$$c\|Bu(t) - Bv(t)\|_{H^*}^2 \le \|Bu(0) - Bv(0)\|_{Y^*}\|u(0) - v(0)\|_Y \exp\Big(2CT + \frac{M}{c}\int_0^T \|f\|_{Y^*}\, ds\Big),$$

so that $Y \ni u(0) \mapsto Bu \in C(0, T; H^*)$ is continuous.

Finally, if $f_1, f_2 \in L^1(0, T; Y^*)$, then the additional terms may be estimated by

$$\int_0^t \langle f_1 - f_2, u - v\rangle\, ds \le \frac{L}{2}\int_0^t \|f_1 - f_2\|_{Y^*}(1 + \|Bu - Bv\|_{H^*}^2)\, ds ,$$

with the Lipschitz constant L of $B^{-1} : Y^* \subset H^* \to Y$ and by

$$\begin{aligned}\int_0^t &\langle Bu - Bv, dB^{-1}(Bu)f_1 - dB^{-1}(Bv)f_2\rangle\, ds\\ &\le M\int_0^t \|Bu - Bv\|_{H^*}^2\|f_1\|_{Y^*}\, ds + \frac{MK}{2}\int_0^t (1 + \|Bu - Bv\|_{H^*}^2)\|f_1 - f_2\|_{Y^*}\, ds ,\end{aligned}$$

with a bound K of $dB^{-1}(Bv)$ in $C(0, T; L(Y^*, H))$. Thus,

$$\begin{aligned}(\langle Bu - Bv, u - v\rangle)(t) \le &\left((\langle Bu - Bv, u - v\rangle)(0) + \frac{MK + L}{2}\int_0^T \|f_1 - f_2\|_{Y^*}\, ds\right)\\ &\exp\left(2CT + \frac{M}{c}\int_0^T \|f_1\|_{Y^*}\, ds + \frac{MK + L}{2c}\int_0^T \|f_1 - f_2\|_{Y^*}\, ds\right),\end{aligned}$$

and especially

$$\begin{aligned}c^2\|Bu(t) - Bv(t)\|_{H^*}^2 \le &\left(\|Bu(0) - Bv(0)\|_{Y^*}\|u(0) - v(0)\|_Y + \frac{MK + L}{2}\int_0^T \|f_1 - f_2\|_{Y^*}\, ds\right)\\ &\exp\left(2CT + \frac{M}{c}\int_0^T \|f_1\|_{Y^*}\, ds + \frac{MK + L}{2c}\int_0^T \|f_1 - f_2\|_{Y^*}\, ds\right),\end{aligned}$$

so that $Y \times L^1(0, T; Y^*) \ni (u(0), f) \mapsto Bu \in C(0, T; H^*)$ is continuous. □

§4. Conclusion

In this article strong solutions to abstract doubly nonlinear evolution equations were discussed under the assumption that B^{-1} is strongly monotone on some intermediate Hilbert space. In this case, strong solutions behave similar as strong solutions to nonlinear evolution equations $\partial u/\partial t + Au = f$. Particularly, under the two convexity conditions (7) and (8) it is possible to give an elementary proof of uniqueness and to obtain continuous dependence on the data. However, for degenerate resp. singular problems where merely weak solutions exist it does not seem possible to avoid more sophisticated methods like Kruzhkov's doubling of variables.

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Jochen Merker
University of Rostock - Institute of Mathematics
Ulmenstr. 69 (Haus 3)
18057 Rostock
jochen.merker@uni-rostock.de