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1D NUMERICAL SIMULATION FOR NONLINEAR PSEUDOPARABOLIC PROBLEMS

Robert Luce, Ngonn Seam and Guy Vallet

Abstract. In this paper, we are interested in the numerical simulation of a pseudo-parabolic fully-nonlinear equation with a nonlinear term of Barenblatt's type. We are exactly interested in the illustrations of the solution of the boundary-value problem: find u such that

$$f(u_t) - \operatorname{div}\{a(u)\nabla u + b(u)\nabla u_t\} = g.$$

The mathematical analysis of a close problem and its simulation have recently been studied by S. N. Antontsev *et al.* [3] when $f = Id_R$ and the existence result has been generalized by N. Seam and G. Vallet in [8]. We propose in particular simulations of the nonlinear problem of the Barenblatt's type: $f(u_t) - \Delta u - \epsilon\Delta u_t = g$ (see [1]).

Keywords: Pseudoparabolic problems, numerical simulations, Barenblatt's problem.

AMS classification: 35K65, 35K70.

§1. Introduction

In this paper, we deal with the 1D numerical simulation to the fully-nonlinear pseudoparabolic problem:

$$f\left(\frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial x}\left\{a(u)\frac{\partial u}{\partial x} + b(u)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)\right\} = g \text{ in } Q,\ u|_\Gamma = 0 \text{ and } u(0,\cdot) = u_0, \tag{1}$$

where f is a Lipschitz-continuous and increasing function, a is Lipschitz-continuous and bounded and b is a positive Lipschitz-continuous and bounded function.

Problems close to that one have been previously studied by S. N. Antontsev *et al.* [3] for stratigraphic models by the way of an implicit time-discretization, and has recently been generalized by N. Seam and G. Vallet in [8] by the same way (see [6, 7] too). The existence of the solution at each step of the discretized scheme is based on Schauder-Tikhonov's fixed point theorem and the convergence of the scheme on an adapted compactness argument.

Our aim is then to illustrate the solution of the above problem by a standard P_1-finite element method in space and an implicit time discretization. In particular, we are interested in the pseudoparabolic singular perturbation when the molecular diffusion changes sign. To do this, we have modified the codes developed by Alberty [2] for the diffusion-reaction problem.

Let us denote by $\Omega =]x_l, x_r[$ a bounded interval of $\mathbb{R}$, T a positive number and assume the following assumptions:

(H_1) a and b are Lipschitz continuous functions over $\mathbb{R}$ such that

$$\exists\beta, M > 0,\ \forall u \in \mathbb{R},\ |a(u)| \le M,\ \beta \le b(u) \le M.$$

(H$_2$) f is a Lipschitz continuous and nondecreasing function over $\mathbb{R}$.

(H$_3$) $g \in L^2(Q)$ and $u_0 \in H_0^1(\Omega)$.

Then, one would say that

Definition 1. A solution of the problem (1) is $u \in H^1\left(0, T; H_0^1(\Omega)\right)$ such that for all $v \in H_0^1(\Omega)$ and $t \in]0, T[$ *a.e.*,

$$\int_{x_l}^{x_r} \left\{ f\left(\frac{\partial u}{\partial t}\right) v + a(u)\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + b(u)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)\frac{\partial u}{\partial x} \right\} dx = \int_{x_l}^{x_r} gv\, dx$$

with the initial condition $u(0, \cdot) = u_0$.

Let us recall a theorem concerning the existence and uniqueness:

Theorem 1 (N. Seam and G. Vallet [8]). *Under hypotheses (H$_1$), (H$_2$) and (H$_3$), there exists u in $H^1\left(0, T, H_0^1(\Omega)\right)$ such that for all v in $H_0^1(\Omega)$ and t almost everywhere in* $]0, T[$,

$$\int_{x_l}^{x_r} \left\{ f\left(\frac{\partial u}{\partial t}\right) v + a(u)\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + b(u)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)\frac{\partial u}{\partial x} \right\} dx = \int_{x_l}^{x_r} gv\, dx \;\; \textit{with} \;\; u(0,\cdot) = u_0 \qquad (2)$$

§2. 1D finite elements formulation

Let us remark that the problem can be strongly non linear and generally the explicit formulation fails because of very restrictive conditions of C.F.L type. So, an implicit formulation has been chosen to obtain solutions with reasonable time steps.

For any $N_t \in \mathbb{N}^*$ and all $k \in [0, N_t]$, let us denote by $\Delta t = T/N_t$ and $t_k = k\Delta t$. Thus, the implicit time discretization of the problem (2) is: find u^{k+1} in $H_0^1(\Omega)$ for a given u^k in $H_0^1(\Omega)$ such that, for any $v \in H_0^1(\Omega)$,

$$\begin{aligned} &\int_{x_l}^{x_r} f\left(\frac{u^{k+1} - u^k}{\Delta t}\right) v\, dx + \int_{x_l}^{x_r} a\left(u^{k+1}\right)\frac{\partial u^{k+1}}{\partial x}\frac{\partial v}{\partial x}\, dx \\ &\qquad + \int_{x_l}^{x_r} b\left(u^{k+1}\right)\frac{\partial}{\partial x}\left(\frac{u^{k+1} - u^k}{\Delta t}\right)\frac{\partial v}{\partial x}\, dx = \int_{x_l}^{x_r} g^{k+1} v dx, \quad k \in [0, N_t - 1], \end{aligned}$$

where g^{k+1} is an approximation of g at time t_{k+1}.
The formulation can be written

$$\begin{aligned} &\int_{x_l}^{x_r} f\left(\frac{u^{k+1} - u^k}{\Delta t}\right) v\, dx + \int_{x_l}^{x_r} \left[a\left(u^{k+1}\right) + \frac{1}{\Delta t} b\left(u^{k+1}\right)\right]\frac{\partial u^{k+1}}{\partial x}\frac{\partial v}{\partial x}\, dx \\ &\qquad - \frac{1}{\Delta t}\int_{x_l}^{x_r} b\left(u^{k+1}\right)\frac{\partial u^k}{\partial x}\frac{\partial v}{\partial x}\, dx - \int_{x_l}^{x_r} g^{k+1} v\, dx = 0, \quad k \in [0, N_t - 1]. \qquad (3) \end{aligned}$$

Now, for any $N_x \in \mathbb{N}$, denote by $h = \Delta x = (x_r - x_l)/(N_x + 1)$ for a uniform mesh with $x_0 = x_l$, and $x_{N_x+1} = x_r$. Thus $x_i = x_0 + ih$ for $i \in [0, N_x + 1]$. We construct the finite dimensional space V_h formed of linear piecewise polynomials:

$$V_h = \left\{ v_h \in H_0^1(\Omega);\; v_h|_{[x_i, x_{i+1}]} \in \mathbb{P}_1,\; 0 \le i \le N_x;\; v_h(x_l) = v_h(x_r) = 0 \right\}.$$

Clearly, $V_h = \text{span}\{\phi_1, \phi_2, \ldots, \phi_{N_x}\}$, where the ϕ_i's are the hat functions, and $\dim V_h = N_x$. By using V_h in place of $H_0^1(\Omega)$, the approximation by the finite element of the problem (3) can be written: find $u_h^{k+1} \in V_h$ for a giving $u_h^k \in V_h$ such that for all $v_h \in V_h$

$$\int_{x_l}^{x_r} f\left(\frac{u_h^{k+1} - u_h^k}{\Delta t}\right) v_h \, dx + \int_{x_l}^{x_r} \left[a\left(u_h^{k+1}\right) + \frac{b\left(u_h^{k+1}\right)}{\Delta t}\right] \frac{\partial u_h^{k+1}}{\partial x} \frac{\partial v_h}{\partial x} \, dx$$
$$- \int_{x_l}^{x_r} \frac{b\left(u_h^{k+1}\right)}{\Delta t} \frac{\partial u_h^k}{\partial x} \frac{\partial v_h}{\partial x} \, dx - \int_{x_l}^{x_r} g^{k+1} v_h \, dx = 0, \quad k = 0, 1, \ldots, N_t - 1,$$

For $k \in [0, N_t]$, inserting $u_h^{k+1} = \sum_{j=1}^{N_x} u_j^{k+1} \phi_j$ with a given approximation $u_h^0 = \sum_{j=1}^{N_x} u_j^0 \phi_j$ of u_0 and using, for $i \in [1, N_x]$, that ϕ_i as an admissible test function, we get the nonlinear system

$$\int_{x_l}^{x_r} f\left(\frac{u_h^{k+1} - u_h^k}{\Delta t}\right) \phi_i \, dx - \frac{1}{\Delta t} \int_{x_l}^{x_r} b\left(u_h^{k+1}\right) \left(u_h^k\right)' \phi_i' \, dx$$
$$+ \int_{x_l}^{x_r} \left[a\left(u_h^{k+1}\right) + \frac{b\left(u_h^{k+1}\right)}{\Delta t}\right] \left(u_h^{k+1}\right)' \phi_i' \, dx - \int_{x_l}^{x_r} g^{k+1} \phi_i \, dx = 0, \quad k \in [0, N_t], \ i \in [1, N_x].$$

The nonlinear system can be usually solved by the Newton Raphson method (cf. [4, 9]). In this case, for $k \in [0, N_t]$, we denote by $U_h^{k+1} = \left(u_1^{k+1}, u_2^{k+1}, \ldots, u_{N_x}^{k+1}\right)^T$ and we introduce the function $F : \mathbb{R}^{N_x} \to \mathbb{R}^{N_x}$, $U_h^{k+1} \mapsto F_i\left(U_h^{k+1}\right)$ for $[1, N_x]$, defined by the formula

$$F_i\left(U_h^{k+1}\right) = \int_{x_l}^{x_r} f\left(\frac{u_h^{k+1} - u_h^k}{\Delta t}\right) \phi_i \, dx - \frac{1}{\Delta t} \int_{x_l}^{x_r} b\left(u_h^{k+1}\right) \left(u_h^k\right)' \phi_i' \, dx$$
$$+ \int_{x_l}^{x_r} \left[a\left(u_h^{k+1}\right) + \frac{b\left(u_h^{k+1}\right)}{\Delta t}\right] \left(u_h^{k+1}\right)' \phi_i' \, dx - \int_{x_l}^{x_r} g^{k+1} \phi_i \, dx.$$

Thus, we have to solve the nonlinear system $F\left(U_h^{k+1}\right) = \mathbf{0}$, where $\mathbf{0} \in \mathbb{R}^{N_x}$, by the Newton Raphson algorithm (see [4], [5] and [9] for the details):

1. For $k \in [0, N_t]$, we initialize the vector U_h^k,
2. then, we compute U_h^{k+1}, solution to the linear system in the Newton method,
3. we give a initial estimation $U_h^{k+1,0}$ of U_h^{k+1},
4. for $\ell = 0, 1, 2, \ldots, \ell_{\max}$, we compute $\Delta U_h^{k+1,\ell}$, solution to the linear system

$$F'\left(U_h^{k+1,\ell}\right) \Delta U_h^{k+1,\ell} = -F\left(U_h^{k+1,\ell}\right),$$

   where $F'\left(U_h^{k+1,\ell}\right)$ is the Jacobian of F at point $U_h^{k+1,\ell}$,
5. we finally let $U_h^{k+1,\ell+1} = U_h^{k+1,\ell} + \Delta U_h^{k+1,\ell}$.

By definition of the Jacobian,

$$F'_{ij}\left(U_h^{k+1,\ell}\right) = \frac{\partial F_i}{\partial u_j^{k+1,\ell}}\left(U_h^{k+1,\ell}\right)$$

and we get that

$$\begin{aligned} F'_{ij}\left(U_h^{k+1,\ell}\right) = {} & \frac{1}{\Delta t}\int_{x_l}^{x_r} f'\left(\frac{u_h^{k+1,\ell}-u_h^{k,\ell}}{\Delta t}\right)\phi_i\phi_j\,dx - \frac{1}{\Delta t}\int_{x_l}^{x_r} b'\left(u^{k+1,\ell}\right)\left(u_h^{k,\ell}\right)'\phi'_i\phi_j\,dx \\ & + \int_{x_l}^{x_r}\left[\phi_j a'\left(u_h^{k+1,\ell}\right)\left(u_h^{k+1,\ell}\right)' + a\left(u_h^{k+1}\right)\phi'_j\right]\phi'_i\,dx \\ & + \frac{1}{\Delta t}\int_{x_l}^{x_r}\left[\phi_j b'\left(u_h^{k+1,\ell}\right)\left(u_h^{k+1,\ell}\right)' + b\left(u_h^{k+1,\ell}\right)\phi'_j\right]\phi'_i\,dx. \end{aligned}$$

Thus, we can compute the coefficient matrix and the right-hand side matrix.

§3. Numerical simulations

In this section, we illustrate the solution to the problem (1) with different given data. In the following examples, $]x_l, x_r[\, = \,]-\pi, \pi[$ and

- either $u_0 = 0$ and $g(t,x) = 1$ if $x \in [\pi/4, \pi/2]$, $g(t,x) = -1$ if $x \in [-\pi/2, -\pi/4]$, $g(t,x) = 0$ otherwise (configuration 1),
- or $u_0(x) = 4x/\pi$ if $x \in [-\pi/4, \pi/4]$, $u_0(x) = 2-4x/\pi$ if $x \in \,]\pi/4, \pi/2]$, $u_0(x) = -2-4x/\pi$ if $x \in [-\pi/2, -\pi/4[$, $u_0(x) = 0$ otherwise, and $g(t,x) = 0$ (configuration 2).

3.1. Linear pseudoparabolic equation or Sobolev' equation

Here, $f(r) = r$, $a(r) = 1$ and $b(r) = \tau$ with $\tau = 0, 1/2, 1$ and 5. We present the simulation of configuration 1 (*i.e.* $u_0 = 0$) in Figure 1 and that of Configuration 2 (*i.e.* $u_0 \neq 0$) in Figure 2.

Remark first that the pseudoparabolic perturbation slows down the evolution of the system. The second remark concerns the space regularity of the solution for $t > 0$: in the pseudoparabolic case, the initial condition fixes the regularity of the solution. Indeed, the first step in the time-iteration solves the elliptic problem: $u - (\Delta t + \tau)\,\Delta u = \Delta_t g + u_0 - \tau\Delta u_0$. Consequently, if $\tau > 0$ and if u_0 is in $H_0^1(\Omega)$, it will be the same for the solution u.

In Figures 3 and 4, we illustrate the same problem unless b where $b(r) = 0.1$ if $r < 0$, $b(r) = k$ else. We can see the dissymmetry of the solution.

3.2. Nonlinear pseudoparalolic equation

In Figures 5 and 6, $f(r) = r$, $a(r) = \arctan(r)$ and $b(r) = \tau$ where $\tau = 0.1, 0.2, 0.5, 1$. Since the sign of a changes, we observe diffusive and anti-diffusive effects illustrated by a convergence to a Dirac mass, especially for small ϵ.

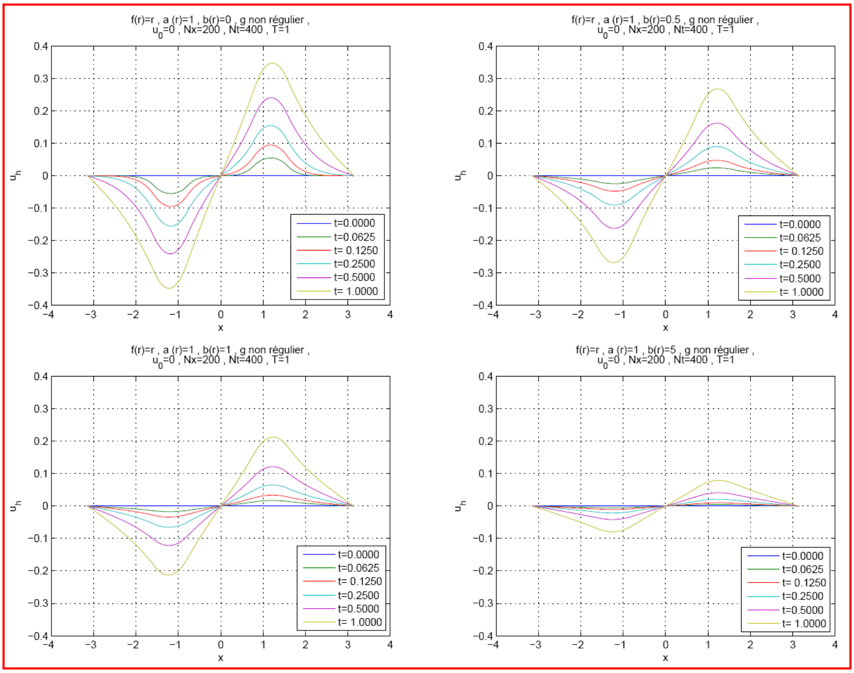

Figure 1: $\partial_t u - \partial^2_{xx} u - \tau \partial^3_{xxt} u = g$ with $u_0(x) = 0$, $\tau = 0, 1/2, 1, 5$.

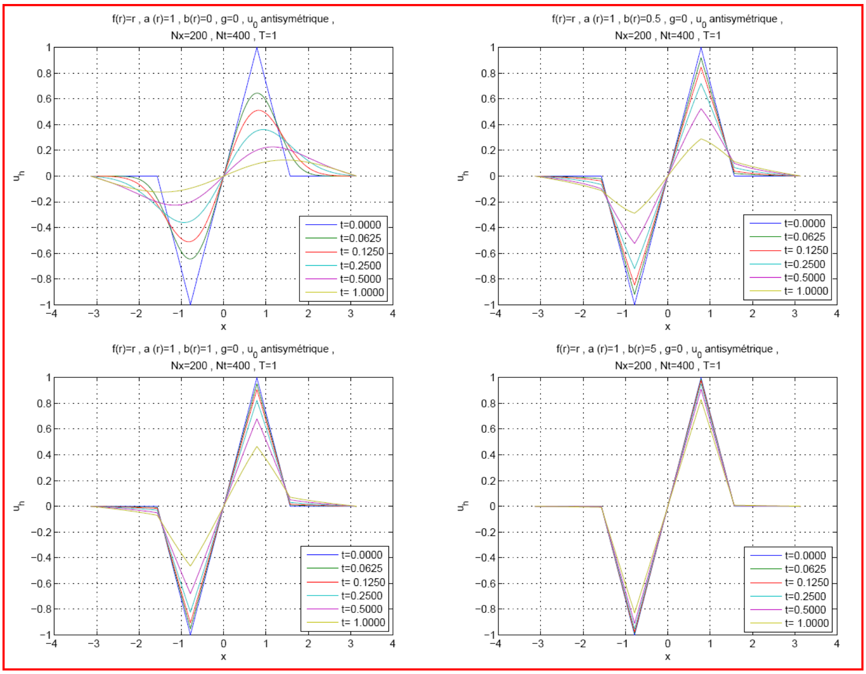

Figure 2: $\partial_t u - \partial^2_{xx} u - \tau \partial^3_{xxt} u = 0$ with $u_0(x) \neq 0$, $\tau = 0, 1/2, 1, 5$.

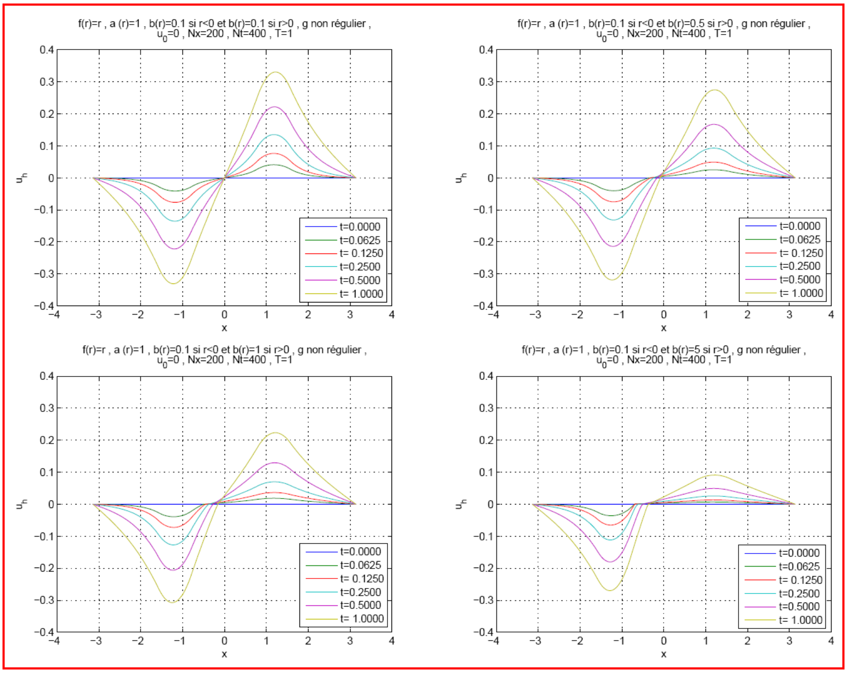

Figure 3: $\partial_t u - \partial^2_{xx} u - \tau \partial^3_{xxt} u = g$ with $u_0(x) = 0$, $b(r) = \tau r^+ - 0.1 r^-$, $\tau = 0, 1/2, 1, 5$.

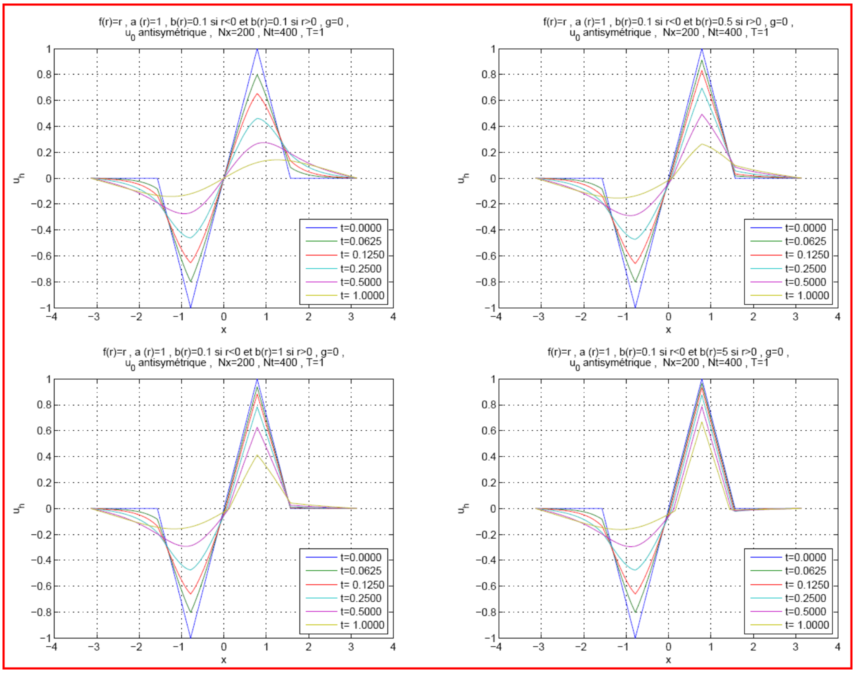

Figure 4: $\partial_t u - \partial^2_{xx} u - \tau \partial^3_{xxt} u = 0$ with $u_0(x) \neq 0$, $b(r) = \tau r^+ - 0.1 r^-$, $\tau = 0, 1/2, 1, 5$.

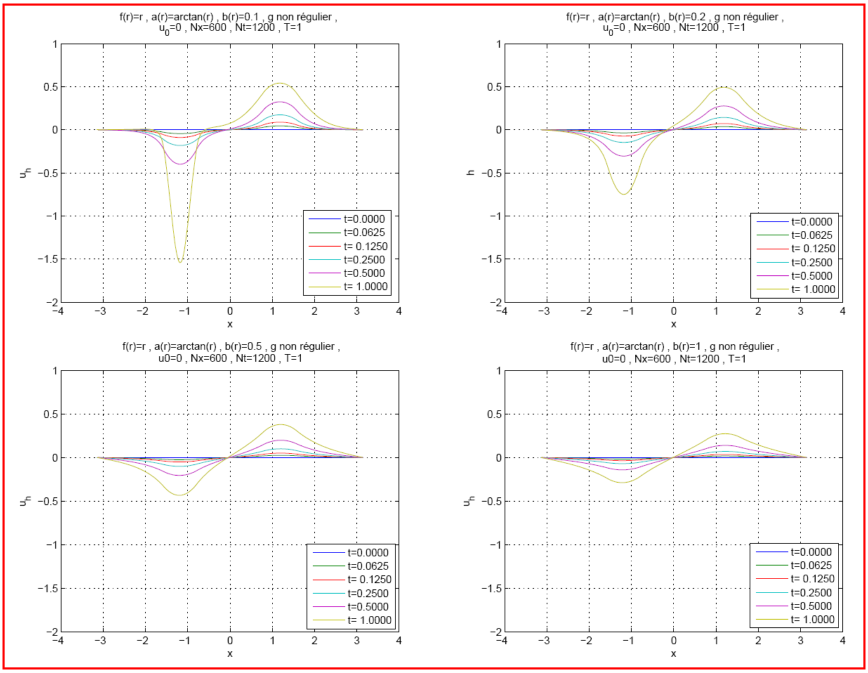

Figure 5: $\partial_t u - \partial_x[\arctan(u)\partial_x u] - \tau\partial^3_{xxt}u = g$ with $u_0 = 0$, $\tau = 0.1, 0.2, 0.5, 1$.

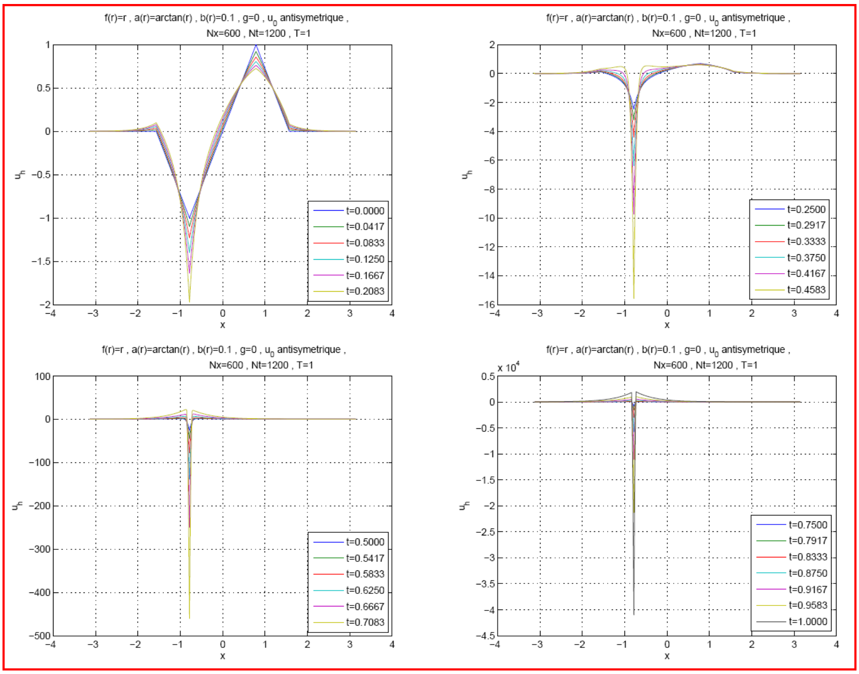

Figure 6: $\partial_t u - \partial_x[\arctan(u)\partial_x u] - \tau\partial^3_{xxt}u = 0$ with $u_0 \neq 0$, $\tau = 0.1$ and small times.

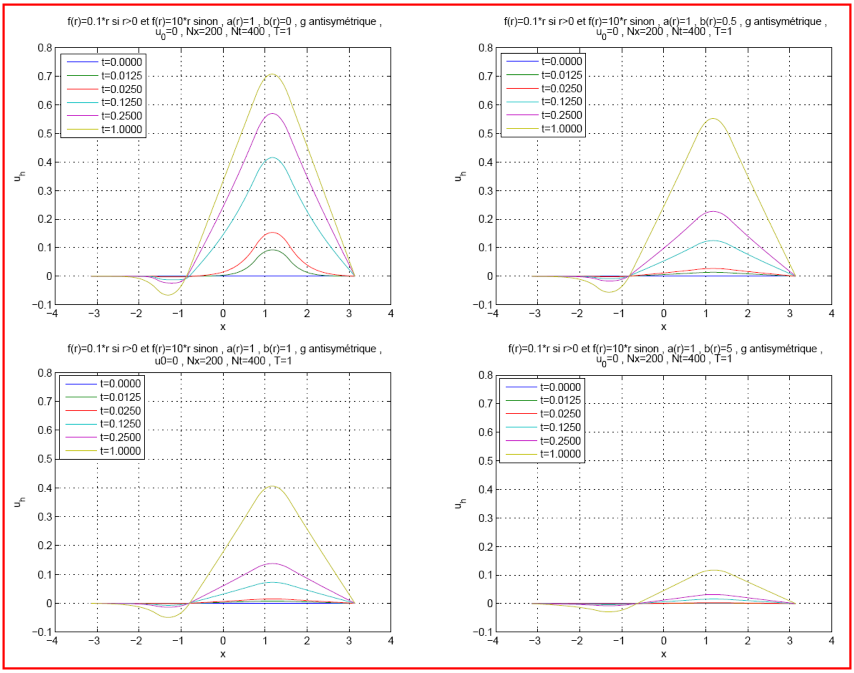

Figure 7: $f(\partial_t u) - \partial^2_{xx} u - \tau \partial^3_{xxt} u = g$ with $u_0(x) = 0$, $\tau = 0, 1/2, 1, 5$.

3.3. Barenblatt's Equation

In Figures 7 to 10, $f(r) = r/10$ if $r > 0$ and $f(r) = 10r$ otherwise, $a(r) = 1$ and $b(r) = \tau$ with different values of $\tau = 0.1, 0.2, 0.5, 1$. The two configurations are illustrated, as well as the asymptotic behaviour.

Note that, in spite of odd data, $x \mapsto u(t, x)$ is not a odd function any more if $t > 0$. Indeed, for negative x, $t \mapsto u(t, x)$ is an increasing function. Thus, the equation is formally $\partial_t u - 10\,\Delta u - 10\epsilon\Delta\partial_t u = 10g$. Else, for positive x, $t \mapsto u(t, x)$ is a decreasing function. Thus, the equation is formally $\partial_t u - \frac{1}{10}\Delta u - \frac{\epsilon}{10}\Delta\partial_t u = \frac{g}{10}$.

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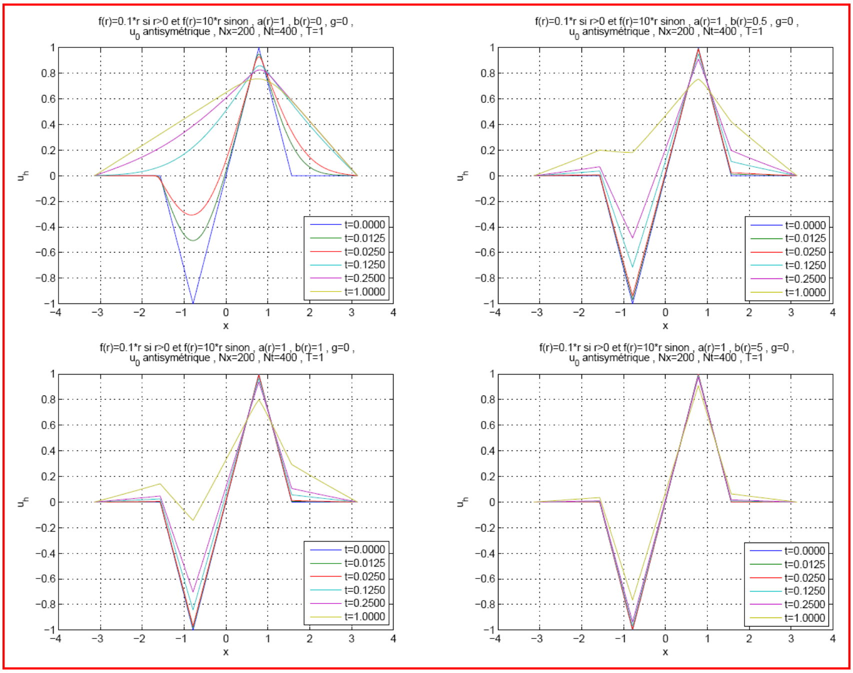

Figure 8: $f(\partial_t u) - \partial^2_{xx}u - \tau\partial^3_{xxt}u = 0$ with $u_0(x) \neq 0$, $\tau = 0, 1/2, 1, 5$.

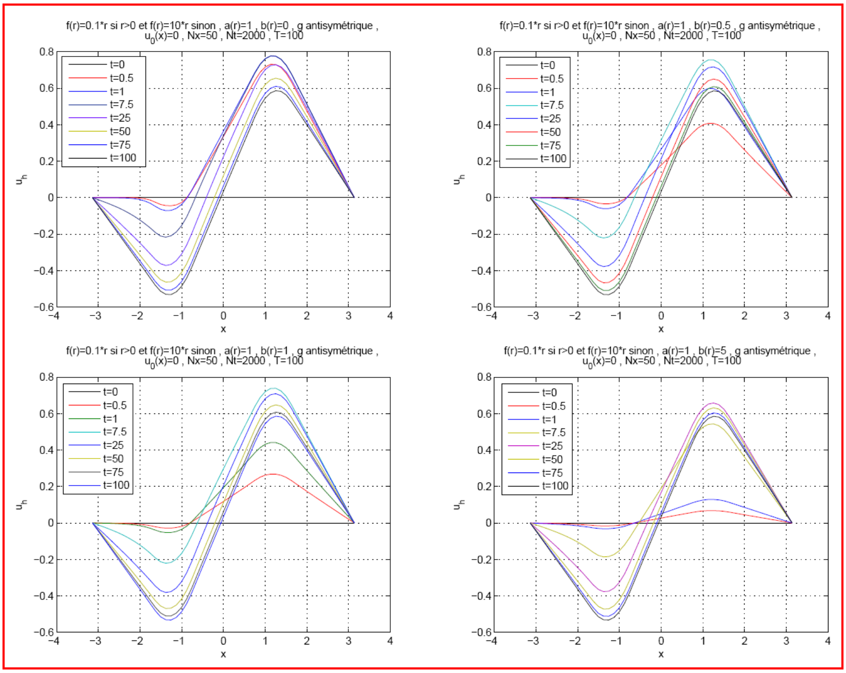

Figure 9: $f(\partial_t u) - \partial^2_{xx}u - \tau\partial^3_{xxt}u = g$ with $u_0(x) = 0$, $\tau = 0, 1/2, 1, 5$, $t \to \infty$.

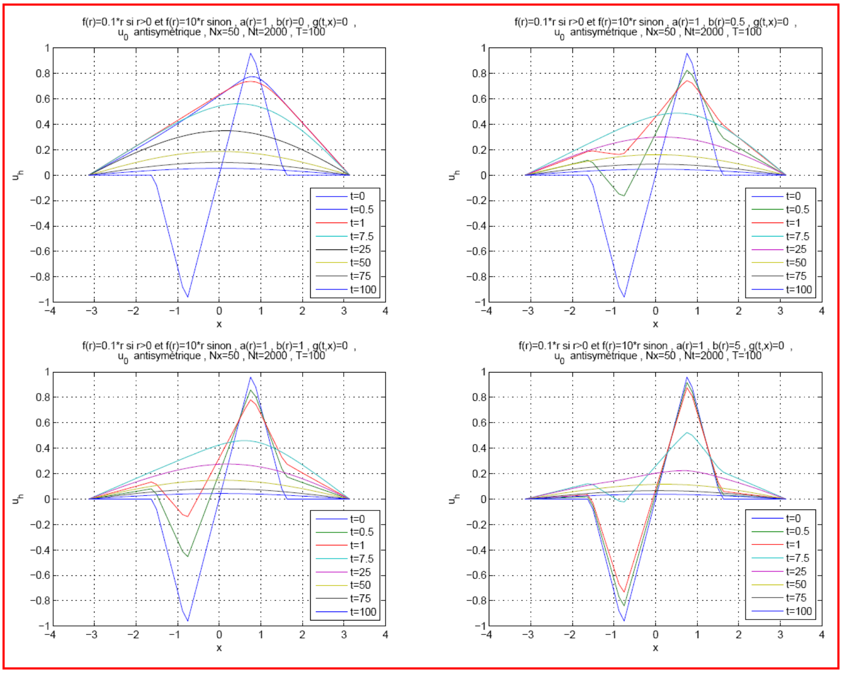

Figure 10: $f(\partial_t u) - \partial^2_{xx} u - \tau \partial^3_{xxt} u = 0$ with $u_0(x) \neq 0$, $\tau = 0, 1/2, 1, 5$, $t \to \infty$.

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LMA, University of Pau - Postal address IPRA BP 1155 Pau Cedex (France)
robert.luce@univ-pau.fr - guy.vallet@univ-pau.fr

Departement of Mathematics, Royal University of Phnom Penh
Russian Federation Boulevard, Toul Kork, Phnom Penh (Cambodia).
seamngonn@yahoo.fr