Monografías Matemáticas *García de Galdeano* **37**, 149–159 (2012)

REDUCTION OF GIBBS PHENOMENON FOR 1D RBF INTERPOLATION

Diego Izquierdo, María Cruz López de Silanes
and María Cruz Parra

Abstract. The Gibbs phenomenon can be observed in different interpolation methods. Radial basis functions (RBF) is a modern meshfree interpolation technique in any number of dimensions. Here we investigate the Gibbs phenomenon for 1D RBF interpolation numerically, and propose a procedure to reduce Gibbs oscillations using nonsmooth basis functions locally. The accuracy in the smooth region is enhanced by applying piecewise linear basis functions in the proximity of discontinuity.

Keywords: Radial basis functions, RBF, Gibbs phenomenon, interpolation.

AMS classification: 65D05, 41A05.

§1. Introduction

Radial basis functions interpolation is a modern meshfree technique in any number of dimensions collected in [7] and introduced by Hardy using multiquadrics [5].

Gibbs phenomenon is the peculiar manner in which the Fourier series of a function f behaves at a jump discontinuity. The overshoot does not die out as the frequency increases, but approaches a finite limit. Gibbs phenomenon can also be observed in different interpolation methods. Fornberg and Flyer [3] perform cardinal interpolation for discontinuous functions with centers $x_j = j \in \mathbb{Z}$ and study expansion coefficients for some RBFs. Guessab, Moncayo and Schmeisser [4] define a class of nonlinear four point subdivision schemes. These schemes include as a particular case the PPH scheme (or power-2 scheme) previously studied by Amat, Donat, Liandrat and Trillo [1]. The general schemes, by using generalized harmonic means, reduce the Gibbs phenomenon around jump discontinuities, as occurs with power-2 scheme. Their properties (e.g. stability, convexity preservation, approximation order) are more balanced than those of the power-p schemes.

Jung [6] makes a complete study of RBF interpolation on $\mathbb{R}$ of step function with uniformly distributed centers in $[-1, 1]$ and uses multiquadric with shape parameter, γ, $\Phi(x) = \sqrt{|x|^2 + \gamma^2}$. Jung proposes a method to reduce Gibbs phenomenon adapting shape parameter, i.e. to define $\gamma = 0$ at centers next to discontinuity. Actually, multiquadric is changed by linear RBF at these centers. Here, our aim is to describe a similar interpolation technique that eliminates oscillations next to discontinuity, using different RBFs.

This paper is divided into the following sections. In Section 2, we establish the necessary notations and preliminaries for RBF interpolation on $\mathbb{R}^d$, a technique described in [7]. In Section 3, we consider an interpolation example of the discontinuous function studied in [6]. First, we study local performance of interpolation with two centers and then interpolation with N centers uniformly distributed in $[-1, 1]$. We use RBFs of [2, Appendix D] to obtain interpolant features for different step functions and two other functions. Finally, in Section 4, we develop a local piecewise linear interpolation for discontinuous functions using different RBFs to reduce the Gibbs oscillations in the vicinity of the discontinuity. This technique adapts and expands the method described in [6] to most RBF of [2, Appendix D]. Then, this technique is applied to some examples presented in the previous section. We finish the section obtaining some errors for an example in [4] and compare these results with those given there. The numerical and graphical examples presented in this paper have been executed using Mathematica 8.0.

§2. RBF interpolation

Definition 1. A function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is said to be radial if there exists a continuous function $\phi : [0, +\infty) \to \mathbb{R}$ such that $\Phi(x) = \phi(\|x\|_2)$ for all $x \in \mathbb{R}^d$.

Let $N \in \mathbb{N}$. We interpolate an unknown function $f : \Omega \subseteq \mathbb{R}^d \to \mathbb{R}$, with data values $F = (f_1, \dots, f_N)^\top$ at given data sites $X = \{x_1, \dots, x_N\} \subseteq \Omega$, the set of centers, so that we look for an interpolant as

$$s_{f,X}(x) = \sum_{j=1}^{N} \alpha_j \Phi(x - x_j), \qquad x \in \mathbb{R}^d,$$

with expansion coefficients vector, $\alpha = (\alpha_1, \dots, \alpha_N)^\top$, so that the interpolation conditions are verified

$$s_{f,X}(x_j) = f_j, \quad 1 \le j \le N. \tag{1}$$

Let $A_{\Phi,X} = (\Phi(x_j - x_k))_{1 \le j,k \le N}$ be the interpolation matrix. If there exists a unique solution of the system

$$A_{\Phi,X}\alpha = F,$$

then $s_{f,X}$ will be defined.

Definition 2. A function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is positive definite on $\mathbb{R}^d$ if, for all $N \in \mathbb{N}$, all pairwise distinct $x_1, \dots, x_N \in \mathbb{R}^d$ and all $\alpha \in \mathbb{R}^N \setminus \{0\}$, the quadratic form

$$\sum_{j=1}^{N} \sum_{k=1}^{N} \alpha_j \alpha_k \Phi(x_j - x_k)$$

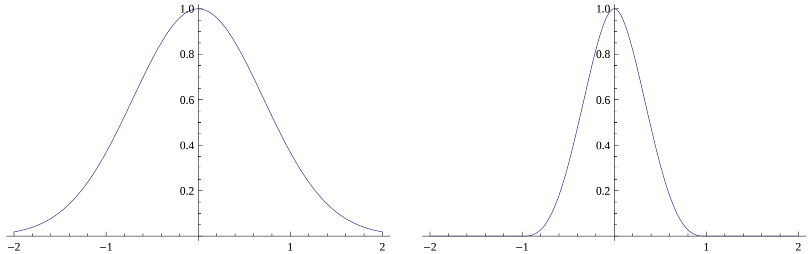
is positive.

By definition $A_{\Phi,X}$ is symmetric. If it is positive definite, then the interpolant will be defined. In this way, we can also say that Φ is positive definite when the interpolation matrix $A_{\Phi,X}$ is positive definite.

Not every RBF used for interpolation is a positive definite function, although the corresponding quadratic form is positive for some expansion coefficients. In general, RBF interpolation uses a conditionally positive definite function of some order.

Definition 3. Let $m \in \mathbb{N}$. A function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is conditionally positive definite of order m on $\mathbb{R}^d$ if, for all $N \in \mathbb{N}$, all pairwise distinct $x_1, \dots, x_N \in \mathbb{R}^d$ and all $\alpha \in \mathbb{R}^N \backslash \{0\}$ satisfying

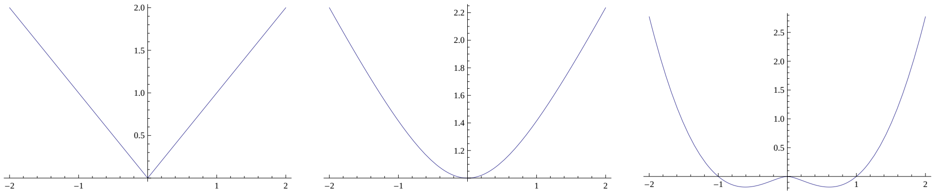
$$\sum_{j=1}^{N} \alpha_j p(x_j) = 0$$



$\phi_1(|x|) = e^{-|x|^2}$
Gaussian RBF

$\phi_2(|x|) = (1 - |x|)_+^3(3|x| + 1)$
C^2-Wendland RBF

Figure 1: Two positive definite functions on $\mathbb{R}$.



$\phi_3(|x|) = |x|$
Linear RBF
$(m = 1)$

$\phi_4(|x|) = \sqrt{1 + |x|^2}$
Multiquadric RBF
$(m = 1)$

$\phi_5(|x|) = |x|^2 \log(|x|)$
Thin Plate Spline RBF
$(m = 2)$

Figure 2: Three conditionally positive definite functions on $\mathbb{R}$.

for all real-valued polynomials of degree less than m, the quadratic form

$$\sum_{j=1}^{N} \sum_{k=1}^{N} \alpha_j \alpha_k \Phi(x_j - x_k)$$

is positive.

For any $m \in \mathbb{N}$, we denote by $\pi_{m-1}(\mathbb{R}^d)$ the space of polynomial functions defined over $\mathbb{R}^d$ of degree $\leq m - 1$ with respect to the set of variables. If we want to interpolate f using a conditionally positive definite function of order m, we will look for an interpolant of the form

$$s_{f,X}(x) = \sum_{j=1}^{N} \alpha_j \Phi(x - x_j) + \sum_{k=1}^{Q} \beta_k p_k, \qquad x \in \mathbb{R}^d, \tag{2}$$

where $\{p_1, \ldots, p_Q\}$ is a basis of the polynomial space $\pi_{m-1}(\mathbb{R}^d)$.

The coefficients $\alpha = (\alpha_1, \ldots, \alpha_N)^\top$ and $\beta = (\beta_1, \ldots, \beta_Q)^\top$ in (2) are uniquely determined by (1) and the additional conditions

$$\sum_{j=1}^{N} \alpha_j p_k(x_j) = 0, \qquad 1 \leq k \leq Q.$$

If we define the matrix $P = (p_k(x_j)) \in \mathbb{R}^{N\times Q}$, α and β will be the solution of the system

$$\begin{pmatrix} A_{\Phi,X} & P \\ P^\top & 0 \end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix} F \\ 0\end{pmatrix}.$$

In this paper, we consider the RBF interpolation on $\mathbb{R}$, i.e. for the case $d = 1$. For more details and proofs, revise [7].

§3. Interpolation of a discontinuous function

In this section, we study features of an interpolant $s_{f,X}$ for a piecewise function

$$f(x) = \begin{cases} f_1(x), & -1 \le x < 0, \\ f_2(x), & 0 < x \le 1, \end{cases} \tag{3}$$

with f_1 and f_2 continuous, and such that it has a finite jump discontinuity at $x_c = 0$, i.e. $|f^+ - f^-| \neq 0$, where $f^- = \lim_{x\to 0^-} f(x)$ and $f^+ = \lim_{x\to 0^+} f(x)$.

First we present a study of RBF interpolation with two centers near discontinuity and then we make a general study with N centers in $[-1, 1]$.

3.1. Local performance of interpolation

We now select two centers in a small neighbourhood of the discontinuity. Let $X = \{-\delta/2, \delta/2\}$ for $\delta > 0$. Most RBFs produce a strictly monotone interpolant $s_\delta(x)$ defined in $[-\delta/2, \delta/2]$. By definition, $s_\delta(x)$ is continuous, so we can then evaluate it at $x_c = 0$:

- If Φ is positive definite, we will get as interpolant

$$s_\delta(x) = \alpha_1\Phi(x + \delta/2) + \alpha_2\Phi(x - \delta/2),$$

where

$$\alpha_1 = \frac{f(\delta/2)\Phi(\delta) - f(-\delta/2)\Phi(0)}{\Phi^2(\delta) - \Phi^2(0)} \quad \text{and} \quad \alpha_2 = \frac{f(-\delta/2)\Phi(\delta) - f(\delta/2)\Phi(0)}{\Phi^2(\delta) - \Phi^2(0)}.$$

Then

$$s_\delta(0) = (f(\delta/2) + f(-\delta/2))\frac{\Phi(\delta/2)}{\Phi(\delta) + \Phi(0)}.$$

- If Φ is conditionally positive definite of order one, we will get as interpolant

$$s_\delta(x) = \alpha_1\Phi(x + \delta/2) + \alpha_2\Phi(x - \delta/2) + \beta_1,$$

where

$$\alpha_1 = \frac{f(\delta/2) - f(-\delta/2)}{2(\Phi(\delta) - \Phi(0))} = -\alpha_2 \quad \text{and} \quad \beta_1 = \frac{f(-\delta/2) + f(\delta/2)}{2}.$$

Then

$$s_\delta(0) = \frac{f(\delta/2) + f(-\delta/2)}{2}.$$

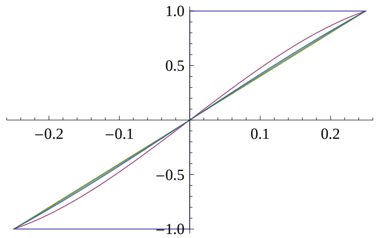


Figure 3: $s_\delta(x)$, with $\delta = 1/2$, for $f_1(x) = -1$ and $f_2(x) = 1$.

- If Φ is any conditionally positive definite of order two, we will get as interpolant
$$s_\delta(x) = \alpha_1\Phi(x + \delta/2) + \alpha_2\Phi(x - \delta/2) + \beta_2 x + \beta_1,$$
where
$$\alpha_1 = 0 = \alpha_2, \quad \beta_1 = \frac{f(\delta/2) + f(-\delta/2)}{2} \quad \text{and} \quad \beta_2 = \frac{f(\delta/2) - f(-\delta/2)}{\delta}.$$
Then
$$s_\delta(0) = \frac{f(\delta/2) + f(-\delta/2)}{2}.$$

Let us observe that, if Φ is any conditionally positive definite function of a higher order, we will not get a unique interpolant. In Figure 3, we show interpolants $s_\delta(x)$, with $\delta = 1/2$, for the fuctions ϕ_1, ϕ_2, ϕ_3 and ϕ_4 defined in Figures 1 and 2, with $f_1(x) = -1$ and $f_2(x) = 1$. The graphic shows that interpolants are strictly increasing and $s_\delta(0) = 0$.

3.2. Interpolation with N centers

We reduce interpolation study to an even number N of centers $X = \{x_1, \ldots, x_N\}$, but the same results are obtained for an odd N.

We consider that centers are uniformly distributed in $[-1, 1]$, that is, for $j = 1, \ldots, N$, $x_j = -1 + 2(j-1)/(N-1)$. Discontinuity exists at $x_c = (x_{N/2} + x_{N/2+1})/2 = 0$. Any RBF used to interpolate produces a continuous interpolant $s_{f,X}$, defined in Section 2. For most RBFs of [2, Appendix D], $s_{f,X}$ has the same features. We have obtained lots of examples, using the mentioned RBFs, for different step functions and functions in Example 2. The next two examples show the interpolant features.

Example 1. Let f be given by (3) with $f_1(x) = -1$ and $f_2(x) = 1$. We interpolate it with $N = 4, 16, 32, 64$ and 128, using the RBFs ϕ_2, ϕ_3, $\widetilde{\phi}(r) = \phi_4(\sqrt{50}r)$ and ϕ_5 (see Figure 4). We have modified ϕ_4 to get good interpolation matrices in the sense that Mathematica is able to solve the associated systems. We observe that $s_{f,X}$ is strictly increasing in $(x_{N/2}, x_{N/2+1})$. In addtion, ϕ_3-interpolants do not present oscillations near the discontinuity. In fact, by definition of f in this example, Jung [6] shows that any ϕ_3-interpolant is
$$s_{f,X}(x) = \begin{cases} -1, & x < x_{N/2}, \\ (N-1)x, & x_{N/2} \le x \le x_{N/2+1}, \\ 1, & x > x_{N/2+1}. \end{cases}$$

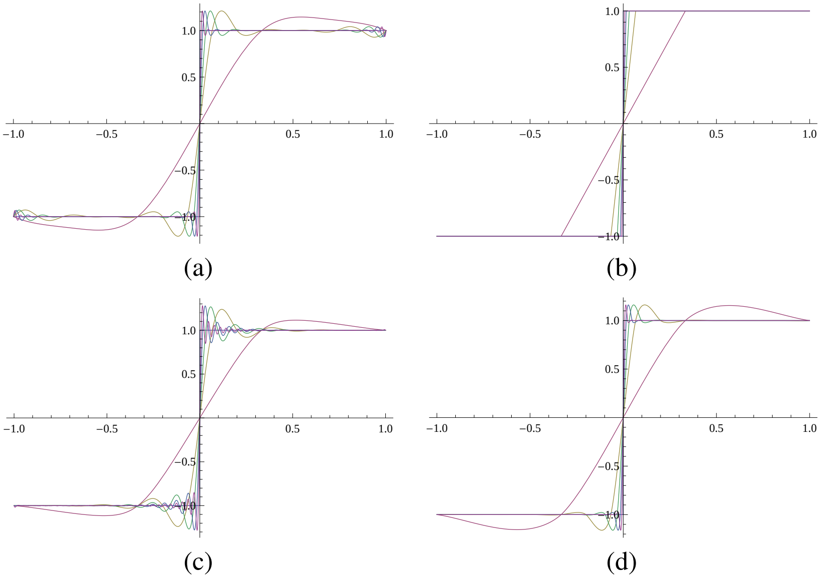


Figure 4: Interpolants of the function f given in Example 1. The used RBFs are ϕ_2 in (a), ϕ_3 in (b), $\widetilde{\phi}$ in (c) and ϕ_5 in (d).

Example 2. We consider the two non-step functions

$$g_1(x) = \begin{cases} \sin x, & x < 0, \\ \cos x, & x > 0, \end{cases} \quad \text{and} \quad g_2(x) = \begin{cases} \log(1 - x), & x < 0, \\ 0.5(x - 0.5)^3, & x > 0, \end{cases}$$

and we interpolate them with $N = 4, 16, 32, 64$, and 128 centers. The function g_1 has also been considered in [6].

Figure 5 shows several interpolants of g_1 and g_2, using ϕ_3 as RBF. These interpolants do not present oscillations. They are polygonal functions with vertices at $(x_i, f(x_i))$ for $i = 1, \ldots, N$, and so they are not differential functions at vertices. Therefore, ϕ_3-interpolants are not good approximations of functions. In Figure 6, we show interpolants of g_1 on top and of g_2 on the bottom. We use $\widetilde{\phi}$ at (a) and (d), ϕ_2 at (b) and (e), and ϕ_5 at (c) and (f).

Numerical experiments for not oscillatory differentiable RBFs of [2, Appendix D] yield interpolants with the same features:

- The interpolant of f has oscillations near x_c. Oscillations do not disappear even for high values of N, Gibbs phenomenon, but increase up to a limit. Maximum oscillations are located in $(x_{N/2-1}, x_{N/2})$ and $(x_{N/2+1}, x_{N/2+2})$.
- $s_{f,X}$ is a strictly increasing monotone function in $(x_{N/2}, x_{N/2+1})$ if $f(x_{N/2}) < f(x_{N/2+1})$ and strictly decreasing if $f(x_{N/2}) > f(x_{N/2+1})$.
- The expansion coefficient α_i is related to the center x_i, for $i = 1, \ldots, N$. Taking centers each time close to x_c the absolute values of associated expansion coefficients become much bigger than at the boundary.

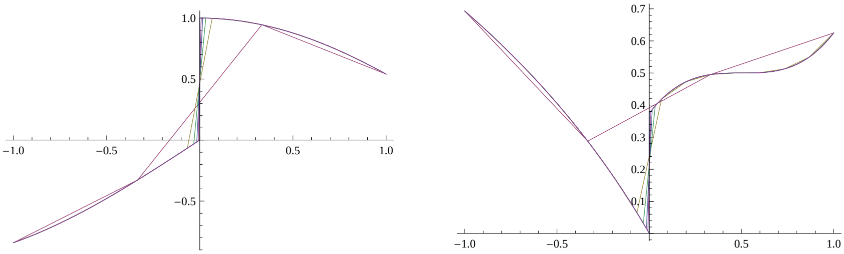


Figure 5: ϕ_3-interpolants of the functions g_1 (left) and g_2 (right) given in Example 2.

Now we define a rate R to measure maximum oscillation on the right of discontinuity. Let $s^\star$ be the value of the interpolant at maximum oscillation located in $(x_{N/2+1}, x_{N/2+2})$. For step functions f with $f_1(x) = f^-$ and $f_2(x) = f^+$, we define the ratio R between the maximum absolute value of over/under-shoots and the jump discontinuity by

$$R = \frac{|s^\star - f^+|}{|f^+ - f^-|}. \tag{4}$$

We consider different step functions and compute R, i.e. oscillations performance, with different RBFs, number of centers and jump discontinuities. Table 1 collects this information and shows that maximum oscillation limit depends on the discontinuity jump and the RBF used, for a given N. Values of Table 1 point out that R is a relative measure of the maximum oscillation since R is invariant for fixed N and RBF. This means that R does not depend on the jump discontinuity for fixed N and RBF. Looking through Table 1, we can affirm that the interpolation using Φ_5 produces a maximum oscillation limit about 8% of jump.

Remark 1. All results in this section could also be obtained for any interval and with a discontinuity at another point.

§4. Local piecewise linear interpolation

In the previous section, we have described the behaviour of the interpolant $s_{f,X}$ of a function with a discontinuity for N centers uniformly distributed. The interpolant does not reproduce the discontinuity of function and the Gibbs phenomenon appears.

Anyway, we observe a special performance of interpolant using ϕ_3 as RBF: $s_{f,X}$ has no oscillation because it is a piecewise linear function.

Looking through Fornberg's paper [3], we confirm that RBF expansion coefficients are bigger near discontinuity. Moreover, Jung [6] gives a method to eliminate oscillations of interpolant using multiquadrics. Jung's paper adapts the interpolation by changing the shape parameter of multiquadrics, $\gamma = 0$, at centers with expansion coefficients in absolute value bigger than that at the boundary. This is changing multiquadric by linear RBF, ϕ_3. We realize that it is enough to change RBF at centers next to the discontinuity: $x_{N/2}$ and $x_{N/2+1}$. We can eliminate oscillations using ϕ_3 only at those centers and most RBFs of [2, Appendix D] at the other centers.

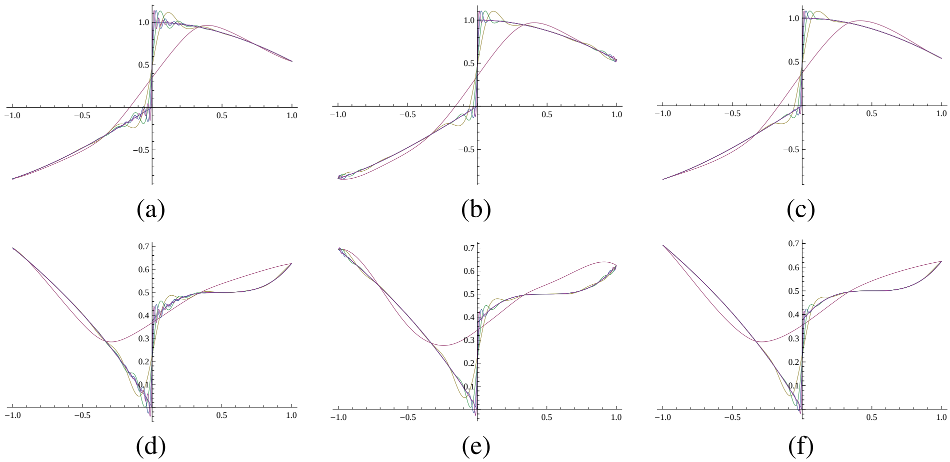


Figure 6: Interpolants $s_{g_1,X}$ (top row) and $s_{g_2,X}$ (bottom row). The used RBFs are $\widetilde{\phi}$ in (a) and (d), ϕ_2 in (b) and (e), and ϕ_5 in (c) and (f).

RBF	(f^-, f^+)	$N = 4$	$N = 16$	$N = 32$	$N = 64$	$N = 128$
Φ_2	$(-1, 1)$	0.07269	0.10546	0.10538	0.10540	0.10545
	$(0, 1)$	0.07269	0.10546	0.10538	0.10540	0.10545
	$(-1.5, 1.5)$	0.07269	0.10546	0.10538	0.10540	0.10545
	$(-0.4, 0.4)$	0.07269	0.10546	0.10538	0.10540	0.10545
$\widetilde{\Phi}$	$(-1, 1)$	0.05727	0.11899	0.13324	0.13877	0.14041
	$(0, 1)$	0.05727	0.11899	0.13324	0.13877	0.14036
	$(-1.5, 1.5)$	0.05727	0.11899	0.13324	0.13877	0.14055
	$(-0.4, 0.4)$	0.05727	0.11899	0.13324	0.13877	0.14029
Φ_5	$(-1, 1)$	0.07740	0.08046	0.08046	0.08046	0.08046
	$(0, 1)$	0.07741	0.08046	0.08046	0.08046	0.08046
	$(-1.5, 1.5)$	0.07740	0.08046	0.08046	0.08046	0.08046
	$(-0.4, 0.4)$	0.07740	0.08046	0.08046	0.08046	0.08046

Table 1: Values of R for different RBF, (f^-, f^+) and N

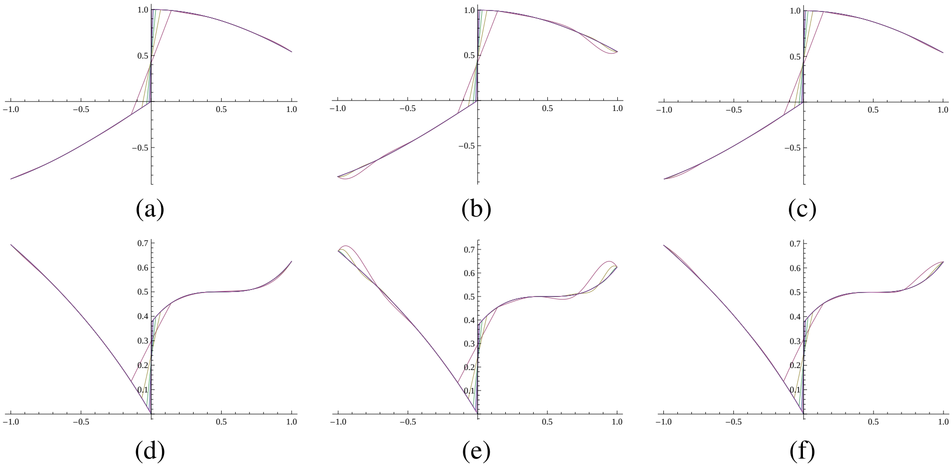


Figure 7: Interpolants $\widetilde{s}_{g_1,X}$ (top row) and $\widetilde{s}_{g_2,X}$ (bottom row). The used RBFs are $\widetilde{\phi}$ in (a) and (d), ϕ_2 in (b) and (e), and ϕ_5 in (c) and (f).

In the conditions described in Section 3, we seek an interpolant $\widetilde{s}_{f,X}$, using ϕ_3 at centers next to discontinuity, of the form

$$\widetilde{s}_{f,X}(x) = \sum_{\substack{j=1 \\ j\neq N/2,N/2+1}}^{N} \widetilde{\alpha}_j\, \phi(|x-x_j|) + \sum_{j=1}^{2} \widetilde{\alpha}_{N/2-1+j}\, |x-x_{N/2-1+j}| + \sum_{k=1}^{\widetilde{m}} \lambda_k p_k, \quad x \in \mathbb{R},$$

where $\{p_1, \ldots, p_{\widetilde{m}}\}$ is a basis of the polynomial space $\pi_{\widetilde{m}-1}(\mathbb{R})$. The coefficients $\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_N$ and $\lambda_1, \ldots, \lambda_{\widetilde{m}}$ are determined by (1) and the additional conditions

$$\sum_{j=1}^{N} \widetilde{\alpha}_j p_k(x_j) = 0, \qquad 1 \leq k \leq \widetilde{m}.$$

We use $\widetilde{m} = 1$ for Φ positive definite and $\widetilde{m} = m$ for Φ conditionally positive definite of order m. Finally we add the constant needed by the linear RBF ϕ_3.

Next, we present two examples. Example 3 shows graphical behaviour of this method for two functions studied in the previous section. Example 4 provides some errors at some distance from discontinuity to show the fitting of the new interpolant.

Example 3. We apply this technique to Example 2 to eliminate oscillations of the interpolants in Figure 6. In Figure 7, we observe that the oscillations are eliminated and interpolants fit better to the function at $[-1, x_{N/2}] \cup [x_{N/2+1}, 1]$. This technique eliminates oscillations because we get an interpolant that is a straight line by $(x_{N/2}, f(x_{N/2}))$ and $(x_{N/2+1}, f(x_{N/2+1}))$ in $[x_{N/2}, x_{N/2+1}]$.

RBF	$x = \frac{-41}{46}$	$x = \frac{-24}{46}$	$x = \frac{-8}{46}$	$x = \frac{-7}{46}$	$x = \frac{-5}{46}$	$x = \frac{-4}{46}$	$x = \frac{-3}{46}$	$x = 0$
$\widetilde{\phi}_1(r) = \phi_1(4.1r)$	1.95e−4	2.00e−5	7.21e−6	3.16e−6	5.07e−6	1.97e−5	3.02e−5	8.16e−5
$\widetilde{\phi}_2(r) = \phi_2(1.3r)$	1.12e−3	3.41e−6	2.30e−4	1.79e−4	1.16e−4	1.49e−4	1.12e−4	2.20e−4
$\widetilde{\phi}_4(r) = \phi_4(2r)$	7.31e−8	1.46e−6	1.41e−5	1.31e−5	2.47e−5	5.12e−5	5.60e−5	3.60e−5

Table 2: Values of E for different points and RBFs.

Example 4. Let

$$g_3(x) = \begin{cases} \exp(x), & x \in [-1, 0), \\ 3, & x = 0, \\ 5 + \sin x, & x \in (0, 1], \end{cases}$$

be a function given in [4]. We apply the described technique with $N = 24$ centers for different RBFs. Let $E(x) = \left|f(x) - \widetilde{s}_{g_3,X}(x)\right|$ be the error function. It is obvious that $E(x_i) = 0$ for $i = 1, \ldots, N$. Errors close to 1 occur at next to discontinuity due to the approximation of the technique near to discontinuity. Table 2 shows the values of E for different points and RBFs. We observe that these errors are similar to the ones obtained in [4] for the same example.

Finally, as conclusions, we have investigated the Gibbs phenomenon for 1D RBF interpolation numerically, and proposed a procedure to reduce oscillations using nonsmooth basis functions locally. This technique is the first step of an approximation method of discontinuous functions which we plan to develop in the future.

Acknowledgements

This work has been partially supported by the Ministry of Science and Innovation (Spain), through the Research Project MTM2009-07315 and the Government of Aragón through two Research groups, E65 and T68, and Research Network CTPR04/10. The authors want to thank Juan José Torrens for his invaluable help. They also thank María Moncayo for suggestions and support and appreciate the comments of the anonymous referee.

References

[1] Amat, S., Donat, R., Liandrat, J., and Trillo, C. Analysis of a new nonlinear subdivision scheme. Applications in image processing. *Found. Comput. Math. 6* (2006), 193–225.

[2] Fasshauer, G. E. *Meshfree Approximation Methods with Matlab*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.

[3] Fornberg, B., and Flyer, N. The Gibbs phenomenon for radial basis functions. In *The Gibbs Phenomenon in Various Representations and Applications*. Sampling Publishing, Potsdam (New York), 2005, pp. 201–224.

[4] Guessab, A., Moncayo, M., and Schmeisser, G. A class of nonlinear four-point subdivision schemes properties in terms of conditions. *Adv. Comput. Math.* (In press).

[5] Hardy, R. L. Multiquadric equations of topography and other irregular surfaces. *J. Geophys. Res. 76* (1971), 1905–1915.

[6] Jung, J.-H. A note on the gibbs phenomenon with multiquadric radial basis functions. *Appl. Numer. Math. 57* (2007), 213–229.

[7] Wendland, H. *Scattered Data Approximation*. Cambridge Univ. Press, Cambridge, 2005.

Diego Izquierdo, María Cruz López de Silanes and María Cruz Parra
Departamento de Matemática Aplicada
Universidad de Zaragoza
C/ María de Luna 3. 50018 Zaragoza. Spain
dizquier@unizar.es, mcruz@unizar.es , cparra@unizar.es