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CONSTRUCTION OF MAJORIZING SEQUENCES FOR OPERATORS WITH UNBOUNDED SECOND DERIVATIVE

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Abstract. The aim of this paper is to construct majorizing sequences for Newton's method in Banach spaces, when the second Fréchet derivative of the operator involved is unbounded, and prove then the semilocal convergence of the method. The new results are illustrated with a nonlinear integral equation of mixed Hammerstein type.

Keywords: Newton's method, semilocal convergence, majorizing sequence, Hammerstein's integral equation.

AMS classification: 45G10, 47H99, 65J15.

§1. Introduction

We present a study for approximating a solution x^* of the equation

$$F(x) = 0, \tag{1}$$

where F is a nonlinear operator defined on a non-empty open convex subset Ω of a Banach space X with values in a Banach space Y, by the most famous iterative method, Newton's method, whose algorithm is:

$$x_{n+1} = x_n - [F'(x_n)]^{-1}F(x_n), \quad n = 0, 1, 2, \ldots, \tag{2}$$

where the starting point x_0 is given.

The generalization of Newton's method to Banach spaces is due to the Russian mathematician L. V. Kantorovich, who publishes several papers at the mid-twentieth century. Initially, see [3] , Kantorovich proves the semilocal convergence of Newton's method under the conditions: $\|\Gamma_0\| \leq \beta$, $\|\Gamma_0 F(x_0)\| \leq \eta$ and

$$\|F''(x)\| \leq K, \quad x \in \Omega, \tag{3}$$

where it is supposed that the operator $\Gamma_0 = [F'(x_0)]^{-1} \in \mathcal{L}(Y, X)$ exists at some $x_0 \in \Omega$, where $\mathcal{L}(Y, X)$ is the set of bounded linear operators from Y into X. The great majority of the results appearing in the literature are concerning with the need for the operator F'' to be bounded in the domain Ω, where the solution x^* must exist. According to this, the number of equations that can be solved by Newton's method is limited. For instance, we cannot analyse the convergence of Newton's method to a solution of an equation where the second derivative of the operator involved is not bounded in a domain, what usually happens in some nonlinear integral equations of mixed Hammerstein type [2]; i.e.:

$$x(s) = u(s) + \sum_{i=1}^{m} \int_a^b G_i(s,t) H_i(x(t))\, dt, \quad s \in [a,b], \tag{4}$$

where $-\infty < a < b < \infty$, G_i, H_i $(i = 1, 2, \ldots, m)$ and u are known functions and x is a continuous function (solution) to be determined. In particular, for nonlinear integral equations of the form

$$x(s) = u(s) + \int_a^b G(s,t)[x(t)^{2+p} + \frac{1}{2}x(t)^2]\, dt, \quad s \in [a,b], \tag{5}$$

with $p \in [0,1]$, where u is a continuous function and the kernel G is the Green function

$$G(s,t) = \begin{cases} \dfrac{(b-s)(t-a)}{b-a}, & t \leq s, \\ \dfrac{(s-a)(b-t)}{b-a}, & s \leq t. \end{cases}$$

Integral equations of this type can be found in the dynamic model of a chemical reactor, which is governed by a control equation and justify the analysis and computation of mixed Hammerstein equations [1].

Solving nonlinear integral equation (5) is equivalent to solve (1), where

$$F : \Omega \subseteq C[a,b] \longrightarrow C[a,b], \quad \Omega = \{x \in C[a,b] : \ x(s) > 0, \ s \in [a,b]\},$$

$$[F(x)](s) = x(s) - u(s) - \int_a^b G(s,t)[x(t)^{2+p} + \frac{1}{2}x(t)^2]\, dt, \quad p \in (0,1].$$

Taking into account the expression of F, it follows

$$[F'(x)y](s) = y(s) - \int_a^b G(s,t)[(2+p)x(t)^{1+p} + x(t)]y(t)\, dt,$$

$$[F''(x)(yz)](s) = -\int_a^b G(s,t)[(2+p)(1+p)x(t)^p + 1]z(t)y(t)\, dt. \tag{6}$$

Notice that condition (3) is not satisfied since $\|F''(x)\|$ is not bounded in all Ω. To see this, we use *reductio ad absurdum*. We suppose $\|F''(x)\| \leq K$ in Ω for the max-norm and denote $M = \max_{[a,b]} \int_a^b |G(s,t)|\, dt$. Then, if $x(t) = \big((K - M + \epsilon)/(M(2+p)(1+p))\big)^{1/p}$, with $\epsilon \in (M-K, +\infty)$ if $M > K$ or $\epsilon \in (0, +\infty)$ if $M \leq K$, and $y(t) = z(t) = 1$, it follows that

$$\|[F''(x)(yz)](s)\| = \left\| \int_a^b G(s,t)[(2+p)(1+p)x(t)^p + 1]\, dt \right\|$$
$$= \left\| \frac{K+\epsilon}{M} \int_a^b G(s,t)\, dt \right\| = K + \epsilon > K.$$

Thus, the last is contradictory to the given statement, since there does not exist a constant K such that $\|F''(x)\| \leq K$ in all Ω. To solve the last, we can use an elegant alternative which consists of relaxing condition (3) by the following one:

$$\|F''(x)\| \leq \omega(\|x\|), \quad x \in \Omega, \tag{7}$$

where $\omega : \mathbb{R}_+ \cup \{0\} \longrightarrow \mathbb{R}$ is a continuous non-decreasing real function.

In this paper, we prove the semilocal convergence of Newton's method under condition (7) instead of condition (3) and illustrate the new result with a nonlinear integral equation of mixed Hammerstein type. The results and their proofs are given in Banach spaces and based on the concept of majorizing sequence:

> Let $\{x_n\}$ be a sequence in a Banach space X and $\{t_n\}$ a scalar sequence. The sequence $\{t_n\}$ majorizes to the sequence $\{x_n\}$ if
>
> $$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n, \quad n = 0, 1, 2, \ldots$$

Emphasize that the interest of majorizing sequences is that the convergence of the sequence in Banach spaces is deduced from the convergence of the scalar sequence, as we can see in the following result [3]:

> Let $\{x_n\}$ be a sequence in a Banach space X and $\{t_n\}$ a scalar majorizing sequence of $\{x_n\}$. If $\{t_n\}$ converges to $t^* < \infty$, there exists $x^* \in X$ such that $x^* = \lim_n x_n$ and $\|x^* - x_n\| \leq t^* - t_n$, for $n \geq 0$.

Throughout the paper we denote $\overline{B(x,\rho)} = \{y \in X : \|y - x\| \leq \rho\}$ and $B(x,\rho) = \{y \in X : \|y - x\| < \rho\}$.

§2. Semilocal convergence

Once the definition of majorizing sequence is introduced, Kantorovich establishes the semilocal convergence of Newton's method under the conditions $\|\Gamma_0\| \leq \beta$, $\|\Gamma_0 F(x_0)\| \leq \eta$ and (3), so that the semilocal convergence of Newton's method is then guaranteed from the quadratic polynomial (see [3])

$$f(t) = \frac{K}{2}(t - t_0)^2 - \frac{t - t_0}{\beta} + \frac{\eta}{\beta}$$

and the scalar sequence $\{t_n\}$,

$$t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}, \quad n = 0, 1, 2, \ldots, \tag{8}$$

which majorizes sequence (2).

The main aim of this paper is to present a new version of the Kantorovich study, where condition (3) is relaxed by condition (7). Specifically, we suppose

(C_1) There exists $x_0 \in \Omega$ such that the operator $\Gamma_0 = [F'(x_0)]^{-1}$ is well-defined and $\|\Gamma_0\| \leq \beta$,

(C_2) $\|\Gamma_0 F(x_0)\| \leq \eta$,

(C_3) $\|F''(x)\| \leq \omega(\|x\|)$, $x \in \Omega$, where $\omega : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}$ is a continuous real non-decreasing function.

If we follow a similar way to Kantorovich, we cannot consider a quadratic polynomial to define the scalar majorizing sequence, since condition (C_3) does not permit it. So, from (C_1)–(C_3), we can construct the function

$$f(t) = \int_{t_0}^{t} \int_{t_0}^{\theta} \omega(\xi) d\xi d\theta - \frac{t - t_0}{\beta} + \frac{\eta}{\beta}, \quad t_0 \geq 0, \tag{9}$$

where ω is the function defined in (7).

Before establishing the new semilocal convergence of Newton's method, we give some previous results that are needed. Lemmas 1 and 2 are technical and the proofs follow immediately.

Lemma 1. *Let ω and f be the real functions defined in (7) and (9), respectively. Then:*

a) If there exists a solution $\alpha > 0$ of the equation

$$W(t) - W(t_0) - \frac{1}{\beta} = 0, \tag{10}$$

where W is a primitive for ω in $\mathbb{R}_+$, then α is the unique minimum of f in $\mathbb{R}_+$.

b) The function f is non-increasing in (t_0, α),

c) If $f(\alpha) \leq 0$, then equation $f(t) = 0$ has at least one solution in $\mathbb{R}_+$. Moreover, if we denote the smallest positive root of $f(t) = 0$ by t^*, we have $t^* \in (t_0, \alpha]$.

Lemma 2. *Let (8) with $f(t)$ defined in (9). Suppose that there exists a positive root α of (10) such that $f(\alpha) \leq 0$. Then, $\{t_n\}$ is a non-decreasing sequence that converges to t^*.*

Next, we prove that sequence $\{x_n\}$ is well-defined. To do this, firstly, we see that $x_1 \in B(x_0, t^* - t_0)$; and secondly, if we assume that $B(x_0, t^* - t_0) \subseteq \Omega$, it follows that $x_n \in B(x_0, t^* - t_0)$, for all $n = 2, 3, 4, \ldots$

To see that x_1 is well-defined, we take into account that $\Gamma_0 = [F'(x_0)]^{-1}$ and $\|\Gamma_0\| \leq -1/f'(t_0) = \beta$ and $\|x_1 - x_0\| \leq \eta = t_1 - t_0$, so that $x_1 \in B(x_0, t^* - t_0)$. In the following result we see that $x_n \in B(x_0, t^* - t_0)$ and $\{t_n\}$ is a majorizing sequence.

Lemma 3. *Let F be a nonlinear twice continuously differentiable operator defined on a non-empty open convex domain Ω of a Banach space X with values in a Banach space Y. We suppose that conditions (C_1)–(C_3) hold and $f(\alpha) \leq 0$, where $f(t)$ is defined in (9) and α is a solution of (10), $\|x_0\| \leq t_0$ and $B(x_0, t^* - t_0) \subseteq \Omega$. Then, $x_n \in B(x_0, t^* - t_0)$, for all $n \in \mathbb{N}$. Moreover, the sequence $\{t_n\}$ defined in (8) majorizes to the sequence $\{x_n\}$ defined in (2); i.e: $\|x_{n+1} - x_n\| \leq t_{n+1} - t_n$ with $n = 0, 1, 2, \ldots$*

Proof. Firstly, by the Banach lemma, observe that there exists $\Gamma_1 = [F'(x_1)]^{-1}$ and $\|\Gamma_1\| \leq$

$-1/f'(t_1)$, since $\|I - \Gamma_0 F'(x_1)\| < 1$. Indeed,

$$\|I - \Gamma_0 F'(x_1)\| = \left\| \int_{x_0}^{x_1} \Gamma_0 F''(x)\, dx \right\| = \left\| \int_0^1 \Gamma_0 F''(x_0 + t(x_1 - x_0))(x_1 - x_0)\, dt \right\|$$

$$\leq \|\Gamma_0\| \int_0^1 \|F''(x_0 + t(x_1 - x_0))\| \, \|x_1 - x_0\| \, dt \leq \beta(t_1 - t_0) \int_0^1 \omega(\|x_0 + t(x_1 - x_0)\|)\, dt$$

$$\leq \beta(t_1 - t_0) \int_0^1 \omega(t_0 + t(t_1 - t_0))\, dt = 1 - \frac{f'(t_1)}{f'(t_0)} < 1,$$

since $\omega(t) = f''(t)$ and ω is a non-decreasing function. Therefore,

$$\| [\Gamma_0 F'(x_1)]^{-1} \| \leq \frac{f'(t_0)}{f'(t_1)} \qquad \text{and} \qquad \|\Gamma_1\| \leq \| [\Gamma_0 F'(x_1)]^{-1} \| \, \|\Gamma_0\| \leq -\frac{1}{f'(t_1)}.$$

Secondly, since $\|x_0\| \leq t_0$, then $\|x_1\| \leq \|x_1 - x_0\| + \|x_0\| \leq t_1$, then $\|x_1\| \leq t_1$.

Thirdly, the Taylor series expansion of $F(x)$ about x_0 is

$$F(x_1) = F(x_0) + F'(x_0)(x_1 - x_0) + \int_{x_0}^{x_1} F''(x)(x_1 - x)\, dx$$

$$= \int_0^1 F''(x_0 + \tau(x_1 - x_0))(1 - \tau)(x_1 - x_0)^2\, d\tau,$$

so that

$$\|F(x_1)\| \leq \int_0^1 \omega(\|x_0 + \tau(x_1 - x_0)\|)(1 - \tau)\|x_1 - x_0\|^2\, d\tau$$

$$\leq \int_0^1 \omega(\|x_0\| + \tau\|x_1 - x_0\|)(1 - \tau)\|x_1 - x_0\|^2\, d\tau$$

$$\leq \int_0^1 \omega(t_0 + \tau(t_1 - t_0))(1 - \tau)(t_1 - t_0)^2\, d\tau = f(t_1),$$

since

$$f(t_1) = \int_0^1 f''(t_0 + \tau(t_1 - t_0))(1 - \tau)(t_1 - t_0)^2\, d\tau = \int_0^1 \omega(t_0 + \tau(t_1 - t_0))(1 - \tau)(t_1 - t_0)^2\, d\tau.$$

Fourthly, from $\|\Gamma_1\| \leq -1/f'(t_1)$ and $\|F(x_1)\| \leq f(t_1)$, it follows that

$$\|x_2 - x_1\| \leq \|\Gamma_1 F(x_1)\| \leq \|\Gamma_1\| \, \|F(x_1)\| \leq -\frac{f(t_1)}{f'(t_1)} = t_2 - t_1.$$

Fifthly, we see that $\|x_2 - x_0\| \leq \|x_2 - x_1\| + \|x_1 - x_0\| \leq t_2 - t_0$, so that $x_2 \in B(x_0, t^* - t_0)$.

Finally, if we assume, for $n \in \mathbb{N}$, that

[I_n] there exists $\Gamma_n = [F'(x_n)]^{-1}$ and $\|\Gamma_n\| \leq -\frac{1}{f'(t_n)}$,

[II_n] $\|x_n\| \leq t_n$,

$[III_n]$ $\|F(x_n)\| \leq f(t_n)$,

$[IV_n]$ $\|x_{n+1} - x_n\| \leq t_{n+1} - t_n$,

$[V_n]$ $\|x_{n+1} - x_0\| \leq t^* - t_0$,

it follows in the same way that $[I_{n+1}]$–$[V_{n+1}]$ hold, so that $[I_n]$–$[V_n]$ are true for all positive integers n by mathematical induction. Consequently, (8) is a majorizing sequence of (2). □

We are now ready to prove in the next theorem the semilocal convergence of Newton's method when the operator F satisfies (C_1)–(C_3). The proof of the theorem follows from the previous lemmas.

Theorem 4. *Let F be a nonlinear twice continuously differentiable operator defined on a non-empty open convex domain Ω of a Banach space X with values in a Banach space Y. Suppose that conditions (C_1)–(C_3) are satisfied. If $f(\alpha) \leq 0$, where $f(t)$ is defined in (9), $\|x_0\| \leq t_0$ and $B(x_0, R) \subseteq \Omega$ with $R = t^* - t_0$, then Newton's method (2) converges to a solution x^* of (1). Moreover, $x_n, x^* \in \overline{B(x_0, R)}$, for all $n \in \mathbb{N}$ and $\|x^* - x_n\| \leq t^* - t_n$, $n \geq 0$. If r is the biggest positive root of the equation*

$$\int_R^t \int_{t_0}^{t_0+u} \omega(z)\, dz\, du = \frac{t - R}{\beta}, \tag{11}$$

the solution x^* is unique in $B(x_0, r) \cap \Omega$ if $r > R$ or in $\overline{B(x_0, R)}$ if $r = R$.

Proof. On the one hand, from Lemma 3 and the fact that the scalar sequence $\{t_n\}$ is convergent, it follows that there exists x^* such that $x^* = \lim_n x_n$, since $\{t_n\}$ is a majorizing sequence of $\{x_n\}$, and x_n, $x^* \in \overline{B(x_0, R)}$, for all $n \in \mathbb{N}$.

On the other hand, as

$$\|F(x_n)\| = \|F'(x_n)(x_{n+1} - x_n)\| \leq \|F'(x_n)\|\, \|x_{n+1} - x_n\|$$

and

$$\left\|F'(x_n) - F'(x_0)\right\| = \left\|\int_{x_0}^{x_n} F''(x)dx\right\|$$
$$= \left\|\int_0^1 F''(x_0 + t(x_n - x_0))(x_n - x_0)\, dt\right\| \leq \int_0^1 \|F''(x_0 + t(x_n - x_0))\|\, \|x_n - x_0\|\, dt$$
$$\leq \int_0^1 \omega(\|x_0 + t(x_n - x_0)\|)\, \|x_n - x_0\|\, dt \leq \omega(t_0 + R)R,$$

we have,

$$\|F'(x_n)\| \leq \|F'(x_n) - F'(x_0)\| + \|F'(x_0)\| \leq \omega(t_0 + R)R + \|F'(x_0)\|,$$

and consequently $\{\|F'(x_n)\|\}$ is bounded and $\lim_n \|F(x_n)\| = 0$. Now, by the continuity of F, it is clear that x^* is a solution of $F(x) = 0$.

To see the unicity of x^*, when $r > R$, we suppose that y^* is another solution of $F(x) = 0$ in $B(x_0, r) \cap \Omega$. Since

$$0 = F(y^*) - F(x^*) = \int_{x^*}^{y^*} F'(x)dx = \int_0^1 F'(x^* + t(y^* - x^*))(y^* - x^*)dt,$$

it suffices to see that there exists the operator

$$\left[\Gamma_0 \int_0^1 F'(x^* + t(y^* - x^*))dt\right]^{-1}. \tag{12}$$

Indeed, from

$$\begin{aligned} I - \Gamma_0 \int_0^1 F'(x^* + t(y^* - x^*))\,dt &= \Gamma_0 \left[\int_0^1 F'(x_0)dt - \int_0^1 F'(x^* + t(y^* - x^*))\,dt\right] \\ &= -\Gamma_0 \int_0^1 \left(\int_{x_0}^{x^*+t(y^*-x^*)} F''(z)\,dz\right) dt, \end{aligned}$$

if we take norms, we have

$$\begin{aligned} &\left\| I - \Gamma_0 \int_0^1 F'(x^* + t(y^* - x^*))dt \right\| \le \|\Gamma_0\| \left\| \int_0^1 \int_{x_0}^{x^*+t(y^*-x^*)} F''(z)\,dz\,dt \right\| \\ &\le \beta \int_0^1 \int_0^1 \left\| F''(x_0 + v((x^* - x_0) + t(y^* - x^*)))((x^* - x_0) + t(y^* - x^*)) \right\| \,dv\,dt \\ &\le \beta \int_0^1 \int_0^1 \left\| F''(x_0 + v((x^* - x_0) + t(y^* - x^*))) \right\| \left\| (x^* - x_0) + t(y^* - x^*) \right\| dv\,dt \\ &\le \beta \int_0^1 \|(x^* - x_0) + t(y^* - x^*)\| \left(\int_0^1 \left\| F''(x_0 + v((x^* - x_0) + t(y^* - x^*))) \right\| \,dv\right) dt \\ &\le \beta \int_0^1 ((1-t)\|x^* - x_0\| + t\|y^* - x_0\|) \left(\int_0^1 \omega(\|x_0 + v((x^* - x_0) + t(y^* - x^*))\|)\,dv\right) dt \\ &< \beta \int_0^1 ((1-t)R + tr) \left(\int_0^1 \omega(\|x_0\| + \|v((x^* - x_0) + t(y^* + x_0 - x_0 - x^*))\|)\,dv\right) dt \\ &\le \beta \int_0^1 ((1-t)R + tr) \left(\int_0^1 \omega(t_0 + v(R + t(r - R)))\,dv\right) dt \end{aligned}$$

and, since

$$\beta \int_0^1 ((1-t)R + tr) \left(\int_0^1 \omega(t_0 + v(R + t(r - R)))\,dv\right) dt = \frac{\beta}{r - R} \int_R^r \int_{t_0}^{t_0+u} \omega(z)dzdu = 1,$$

by the Banach lemma, operator (12) exists.

If $r = R$, we suppose that y^* is another solution of $F(x) = 0$ in $\overline{B(x_0, R)}$. Since $\|y^* - x_0\| \le t^* - t_0$, by mathematical induction we suppose that $\|y^* - x_k\| \le t^* - t_k$ for $k = 0, 1, \ldots, n$. Then, having into account that $F(y^*) = 0$ and $x_{n+1} = x_n - \Gamma_n F(x_n)$ we can write

$$y^* - x_{n+1} = -\Gamma_n \int_0^1 F''(x_n + t(y^* - x_n))(1 - t)(y^* - x_n)^2\,dt,$$

as $\|x_n + t(y^* - x_n)\| \le t_n + t(y^* - t_n)$, we obtain

$$\|y^* - x_{n+1}\| \le -\frac{M}{f'(t_n)}\|y^* - x_n\|^2, \tag{13}$$

being $M = \int_0^1 \omega(t_n + t(y^* - t_n))(1-t)\,dt$.

In the same way for f function, we have

$$t^* - t_{n+1} = -\frac{1}{f'(t_n)} \int_0^1 f''(t_n + t(t^* - t_n))(1-t)(t^* - t_n)^2\,dt,$$

and therefore we obtain

$$t^* - t_{n+1} = -\frac{M}{f'(t_n)}(t^* - t_n)^2. \tag{14}$$

Then, from (13) and (14) we prove that $\|y^* - x_{n+1}\| \leq t^* - t_{n+1}$. So $\|y^* - x_n\| \leq t^* - t_n$ for all n, therefore as $\lim_n t_n = t^*$ and $\lim_n x_n = x^*$, it follows that $y^* = x^*$. □

§3. Application to a particular equation (5)

We have seen in the introduction that second derivative (6) is not bounded in all $\Omega = \{x \in C[a,b] : \ x(s) > 0,\ s \in [a,b]\}$. On the contrary, we see in the following that the alternative condition given by (C_3) in Theorem 4 holds, and consequently the convergence of Newton's method to a solution of (5) is then guaranteed from Theorem 4. From (C_3) we deduce

$$\omega(z) = M\left(1 + (2+p)(1+p)z^p\right). \tag{15}$$

Moreover, for a fixed $x_0(s)$, we have

$$\|I - F'(x_0)\| \leq M\left((2+p)\|x_0^{1+p}\| + \|x_0\|\right),$$

and by the Banach lemma, we obtain

$$\|\Gamma_0\| \leq \frac{1}{1 - M\left((2+p)\|x_0^{1+p}\| + \|x_0\|\right)} = \beta,$$

provided that $M\left((2+p)\|x_0^{1+p}\| + \|x_0\|\right) < 1$. Furthermore, since $\|F(x_0)\| \leq \|x_0 - u\| + M\left(\|x_0^{2+p}\| + \frac{1}{2}\|x_0^2\|\right)$, it follows that

$$\|\Gamma_0 F(x_0)\| \leq \|\Gamma_0\|\,\|F(x_0)\| \leq \frac{\|x_0 - u\| + M\left(\|x_0^{2+p}\| + \frac{1}{2}\|x_0^2\|\right)}{1 - M\left((2+p)\|x_0^{1+p}\| + \|x_0\|\right)} = \eta.$$

Once the parameters β and η are calculated and function (15) is known, we use Theorem 4 to prove the existence of solution of equation (5) and guarantee the convergence of Newton's method.

To determine the domain of existence of solution, we consider the following particular equation (5):

$$x(s) = 1 + \int_0^1 G(s,t)\left(x(t)^{5/2} + \frac{1}{2}x(t)^2\right)dt, \quad s \in [0,1], \tag{16}$$

where the kernel G is the Green function.

If we repeat what is done for (5) with $u(s) = 1$, $p = 1/2$, $[a,b] = [0,1]$ and choose $x_0(s) = 1/2$, we can guarantee by the Banach lemma that the operator Γ_0 exists and $\|\Gamma_0\| \leq 32(12 + \sqrt{2})/355$, since

$$\|[(I - F'(x_0))y](s)\| \leq \frac{1}{64}\left(4 + 5\sqrt{2}\right)\|y\| \qquad \text{and} \qquad \|I - F'(x_0)\| < 1.$$

Moreover, $\|F(x_0)\| \leq (33 + \sqrt{2})/64$ and

$$\beta = 1.2091\ldots, \qquad \eta = 0.6501\ldots, \qquad \omega(z) = \frac{1}{8} + \frac{\sqrt{z}}{32}.$$

Since $t_0 \geq \|x_0\| = 1/2$ in Theorem 4, we take $t_0 = 1/2$, so that the equation

$$W(t) - W(t_0) - \frac{1}{\beta} = \frac{1}{96}(2t\sqrt{t} + 12t - 7\sqrt{2} - 96) = 0,$$

has only one root: $\alpha = 5.1992\ldots$

If we now construct the function $f(t)$ of theorem 4, we obtain

$$f(t) = (0.0083\ldots)t^2\sqrt{t} + (0.0625\ldots)t^2 - (0.8968\ldots)t + (0.9690\ldots),$$

so that $f(\alpha) = -1.4908\ldots < 0$. The smallest positive root of $f(t) = 0$ is $t^* = 1.1943\ldots$ and $t^* - \|x_0\| = 0.6943\ldots = R$, so that the domain of existence of solution is

$$\{\varphi \in C[0,1];\ \|\varphi - \frac{1}{2}\| \leq 0.6943\ldots\}.$$

Moreover, as the biggest positive root of the corresponding equation (11) is $r = 8.5193\ldots$, then the domain of uniqueness of solution is

$$\{\varphi \in C[0,1];\ \|\varphi - \frac{1}{2}\| < 8.5193\ldots\} \cap \Omega.$$

Note that in practice we can observe that the domain of existence of solution is optimum when $t_0 = \|x_0\|$.

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