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DETECTING AN OBSTACLE IMMERSED IN A FLUID: THE STOKES CASE

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Abstract. This paper presents a theoretical study of a detection of an object immersed in a fluid. The fluid motion is governed by the Stokes equations. We detail the Dirichlet case for which the results are just stated in [3]. We make a shape sensitivity analysis of order two in order to prove the existence of the first and the second orders shape derivatives. The strategy adopted to detect the object is to minimize a least-squares functional. We characterize the gradient of the functional using an adjoint problem. Finally, we study the stability of this setting. We give the expression of the shape Hessian at a critical point and the compactness of the Riesz operator corresponding to this shape Hessian is shown. The ill-posedness of the identification problem follows which explains the need of regularization to numerically solve this problem.

Keywords: Stationary Stokes problem, sensitivity with respect to the domain of order two, geometric inverse problem.

AMS classification: 35R30, 35Q30, 49Q10, 49Q12, (76D07).

§1. Introduction

Notations and references on the Stokes equations. For a domain Ω, $\langle \cdot\,,\,\cdot \rangle_{\Omega}$ and $\langle \cdot\,,\,\cdot \rangle_{\partial\Omega}$ will denote respectively the duality products $\langle \cdot\,,\,\cdot \rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}^1_0(\Omega)}$ and $\langle \cdot\,,\,\cdot \rangle_{\mathbf{H}^{-1/2}(\partial\Omega),\mathbf{H}^{1/2}(\partial\Omega)}$. Moreover, $\mathbf{n}$ represents the external unit normal to $\partial\Omega$.

In this paper, we use some existence, uniqueness and regularity results concerning the Stokes equations: we refer for example to [9, Chapter 1]. Moreover, we also use some local regularity arguments: see [5, Theorem IV.5.1] for details.

Setting of the problem. Let Ω a bounded, connected open subset of $\mathbb{R}^N$ (with $N = 2$ or $N = 3$) with a $C^{1,1}$ boundary. Let $\delta > 0$ fixed (small). We define $\mathcal{O}_\delta$ the set of all open subsets ω of Ω with a $C^{2,1}$ boundary such that $\mathrm{d}(x, \partial\Omega) > \delta$ for all $x \in \omega$ and such that $\Omega \setminus \overline{\omega}$ is connected. We also define Ω_δ an open set with a C^∞ boundary such that

$$\{x \in \Omega\,;\ \mathrm{d}(x, \partial\Omega) > \delta/2\} \subset \Omega_\delta \subset \{x \in \Omega\,;\ \mathrm{d}(x, \partial\Omega) > \delta/3\}\,.$$

Let $\boldsymbol{f_b}$ be an admissible boundary measurement. Let $\boldsymbol{g} \in \mathbf{H}^{3/2}(\partial\Omega)$ such that $\boldsymbol{g} \neq \mathbf{0}$ and satisfying the following condition:

$$\int_{\partial\Omega} \boldsymbol{g} \cdot \mathbf{n} = 0. \tag{1}$$

Let us consider, for $\omega \in \mathcal{O}_\delta$, the following overdetermined Stokes boundary values problem:

$$\begin{cases} -\operatorname{div}(\sigma(\boldsymbol{u}, p)) = \mathbf{0} & \text{in } \Omega \setminus \overline{\omega}, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega \setminus \overline{\omega}, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \partial\Omega, \\ \boldsymbol{u} = \mathbf{0} & \text{on } \partial\omega, \\ \sigma(\boldsymbol{u}, p)\mathbf{n} = \boldsymbol{f_b} & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\sigma(\boldsymbol{u}, p) = \nu(\nabla\boldsymbol{u} + {}^t\nabla\boldsymbol{u}) - p\,\mathrm{I}$ is the stress tensor and $\nu > 0$ is a given constant representing the kinematic viscosity of the liquid.

We assume there exists $\omega \in \mathcal{O}_\delta$ such that (2) has a solution. This means that the measurement $\boldsymbol{f_b}$ is perfect, *i.e.* without error. Thus, we consider the following geometric inverse problem:

$$\textit{find } \omega \in \mathcal{O}_\delta \textit{ and a pair } (\boldsymbol{u}, p) \textit{ which satisfies the overdetermined system (2).} \tag{3}$$

To solve this inverse problem, we consider, for $\omega \in \mathcal{O}_\delta$, the least-squares functional

$$J(\omega) = \frac{1}{2}\int_{\partial\Omega} |\sigma(\boldsymbol{u}(\omega), p(\omega))\,\mathbf{n} - \boldsymbol{f_b}|^2,$$

where $(\boldsymbol{u}(\omega), p(\omega)) \in \mathbf{H}^2(\Omega \setminus \overline{\omega}) \times \mathrm{H}^1(\Omega \setminus \overline{\omega})$ is a solution of the Stokes problem

$$\begin{cases} -\operatorname{div}(\sigma(\boldsymbol{u}, p)) = \mathbf{0} & \text{in } \Omega \setminus \overline{\omega}, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega \setminus \overline{\omega}, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \partial\Omega, \\ \boldsymbol{u} = \mathbf{0} & \text{on } \partial\omega. \end{cases} \tag{4}$$

Since we imposed the compatibility condition (1), problem (4) has a unique solution once a normalization condition on the pressure p is imposed (see for example [9, Chapter 1]). Such a solution $(\boldsymbol{u}, p)$ is called the state of the system. Here, we choose the normalization

$$\int_{\partial\Omega} (\sigma(\boldsymbol{u}, p)\mathbf{n}) \cdot \mathbf{n} = \int_{\partial\Omega} \boldsymbol{f_b} \cdot \mathbf{n}. \tag{5}$$

Then, we try to minimize the least-squares criterion J:

$$\omega^* = \operatorname*{arg\,min}_{\omega \in \mathcal{O}_\delta} J(\omega). \tag{6}$$

Indeed, if ω^* is solution of the inverse problem (3), then $J(\omega^*) = 0$ and (6) holds. Conversely, if ω^* solves (6) with $J(\omega^*) = 0$, then this domain ω^* is a solution of the inverse problem.

Introduction of the needed functional tools. Let $\boldsymbol{U} = \{\boldsymbol{\theta} \in \mathbf{W}^{3,\infty}(\mathbb{R}^N);\ \operatorname{supp}\boldsymbol{\theta} \subset \overline{\Omega_\delta}\}$ and $\boldsymbol{\mathcal{U}} = \{\boldsymbol{\theta} \in \boldsymbol{U};\ \|\boldsymbol{\theta}\|_{3,\infty} < 1\}$ be the space of admissible deformations. Notice that if $\boldsymbol{\theta} \in \boldsymbol{\mathcal{U}}$ then $(\mathbf{I} + \boldsymbol{\theta})$ is a diffeomorphism. For such a $\boldsymbol{\theta} \in \boldsymbol{U}$ and $\omega \in \mathcal{O}_\delta$, we check $\Omega = (\mathbf{I} + \boldsymbol{\theta})(\Omega)$ and we define the perturbed domain $\omega_\theta = (\mathbf{I} + \boldsymbol{\theta})(\omega)$ which is so that $\Omega \setminus \overline{\omega_\theta} \in \mathcal{O}_\delta$.

Let $T > 0$, that we will have to fix small. We will use the shape calculus introduced in [7] by F. Murat and J. Simon. Thus, we consider the function

$$\phi : t \in [0, T) \mapsto \mathbf{I} + t\,\boldsymbol{V} \in \mathbf{W}^{3,\infty}(\mathbb{R}^N),$$

where $\boldsymbol{V} \in \boldsymbol{U}$. Note that for small t, $\phi(t)$ is a diffeomorphism of $\mathbb{R}^N$ and that $\phi'(0) = \boldsymbol{V}$ vanishes on $\partial\Omega$ and even on the tubular neighborhood $\Omega \setminus \overline{\Omega_\delta}$ of $\partial\Omega$. For $t \in [0, T)$, we define $\omega_t = \phi(t)(\omega)$ and $\mathbf{n}_t$ the external unit normal of $\Omega \setminus \overline{\omega_t}$.

Outlines of the paper. This paper is organized as follows. In Section 2, we state the main results of this work. We first mention an identifiability result proved by C. Alvarez *et al.* in [1]. We claim the existence of the first order shape derivative of the state and we characterize this derivative. We then give the expression of the gradient of the least-squares functional introducing an adjoint problem. Furthermore, we discuss higher order shape derivatives and we characterize the shape Hessian at a possible solution of the original inverse problem. Finally, we justify the instability of the problem: the Riesz operator corresponding to the shape Hessian at a critical shape is compact, which means that the functional is degenerate for the high frequencies. In Section 3, we present some preliminary results: we recall an extension of the usual implicit functions Theorem proved by J. Simon in [8] and we prove some results used in section 4 where the main results of this work are proved. In Section 5, we compare the Neumann case exposed in [3] and the Dirichlet case treated in this paper: we point out the difficulties and the mistakes made in the statement of the Dirichlet case in [3].

§2. Statement of the main results

Identifiability result. According to [1, Theorem 1.2] proved by C. Alvarez *et al.*, the inverse problem (3) is well posed, in the sense that the solution (which exists by assumption) is unique. Indeed, this identifiability result claims that given a fixed $\boldsymbol{g}$, two different geometries ω_0 and ω_1 in $\mathcal{O}_\delta$ yield two different measures $\boldsymbol{f_{b1}}$ and $\boldsymbol{f_{b2}}$.

Sensitivity with respect to the domain. Secondly, we aim to make a sensitivity (with respect to the shape) analysis. The Stokes problem on $\Omega \setminus \overline{\omega_t}$

$$\begin{cases} -\operatorname{div}(\sigma(\boldsymbol{u_t}, p_t)) = \mathbf{0} & \text{in } \Omega \setminus \overline{\omega_t}, \\ \operatorname{div} \boldsymbol{u_t} = 0 & \text{in } \Omega \setminus \overline{\omega_t}, \\ \boldsymbol{u_t} = \boldsymbol{g} & \text{on } \partial\Omega, \\ \boldsymbol{u_t} = \mathbf{0} & \text{on } \partial\omega_t, \end{cases} \tag{7}$$

admits a unique solution $(\boldsymbol{u_t}, p_t) \in \mathbf{H}^2(\Omega \setminus \overline{\omega_t}) \times \mathrm{H}^1(\Omega \setminus \overline{\omega_t})$ satisfying the normalization condition $\int_{\partial\Omega} (\sigma(\boldsymbol{u_t}, p_t)\mathbf{n}) \cdot \mathbf{n} = \int_{\partial\Omega} \boldsymbol{f_b} \cdot \mathbf{n}$.

Proposition 1 (First order shape derivatives of the state)**.** *The solution $(\boldsymbol{u}, p)$ is differentiable with respect to the domain and the derivatives $(\boldsymbol{u}', p') \in \mathbf{H}^2(\Omega \setminus \overline{\omega}) \times \mathrm{H}^1(\Omega \setminus \overline{\omega})$ is the only* *solution of the following boundary values problem*

$$\begin{cases} -\operatorname{div}(\sigma(\boldsymbol{u}', p')) = \mathbf{0} & \text{in } \Omega \setminus \overline{\omega}, \\ \operatorname{div} \boldsymbol{u}' = 0 & \text{in } \Omega \setminus \overline{\omega}, \\ \boldsymbol{u}' = \mathbf{0} & \text{on } \partial\Omega, \\ \boldsymbol{u}' = -\partial_{\mathbf{n}} \boldsymbol{u} \, (\boldsymbol{V} \cdot \mathbf{n}) & \text{on } \partial\omega, \end{cases} \tag{8}$$

with the normalization condition $\int_{\partial\Omega} (\sigma(\boldsymbol{u}', p')\mathbf{n}) \cdot \mathbf{n} = 0$.

Proposition 2 (First order shape derivatives of the functional)**.** *For* $\boldsymbol{V}$ *in* $\boldsymbol{U}$*, the least-squares functional* J *is differentiable at* ω *in the direction* $\boldsymbol{V}$ *with*

$$\mathrm{D}\, J(\omega) \cdot \boldsymbol{V} = -\int_{\partial\omega} [(\sigma(\boldsymbol{w}, q)\mathbf{n}) \cdot \partial_{\mathbf{n}} \boldsymbol{u}] \, (\boldsymbol{V} \cdot \mathbf{n}),$$

where $(\boldsymbol{w}, q) \in \mathbf{H}^1(\Omega \setminus \overline{\omega}) \times \mathrm{L}^2(\Omega \setminus \overline{\omega})$ *is the solution of the Stokes boundary values problem:*

$$\begin{cases} 2 - \operatorname{div}(\sigma(\boldsymbol{w}, q)) = \mathbf{0} & \text{in } \Omega \setminus \overline{\omega}, \\ \operatorname{div} \boldsymbol{w} = 0 & \text{in } \Omega \setminus \overline{\omega}, \\ \boldsymbol{w} = \sigma(\boldsymbol{u}, p)\mathbf{n} - \boldsymbol{f_b} & \text{on } \partial\Omega, \\ \boldsymbol{w} = \mathbf{0} & \text{on } \partial\omega, \end{cases} \tag{9}$$

with the normalization condition $\langle \sigma(\boldsymbol{w}, q)\mathbf{n} \,,\, \mathbf{n} \rangle_{\partial\Omega} = 0$.

Remark 1. Propositions 1 and 2 remain true under weaker assumptions. Indeed, the proofs are still valid if ω has a $C^{1,1}$ boundary and $\boldsymbol{V} \in \mathbf{W}^{2,\infty}(\mathbb{R}^N)$. However, in this case, the expression of $\mathrm{D}\, J(\omega) \cdot \boldsymbol{V}$ has to be understood as a duality product $\mathrm{H}^{-1/2} \times \mathrm{H}^{1/2}$ and $(\boldsymbol{u}', p')$ only belongs to $\mathbf{H}^1(\Omega \setminus \overline{\omega}) \times \mathrm{L}^2(\Omega \setminus \overline{\omega})$. Moreover, we will prove Proposition 1 only assuming Ω is Lipschitz.

Second order analysis: justification of the instability. Finally, we want to study the stability of the optimization problem (6) at ω^*.

Proposition 3 (Characterization of the shape Hessian at a critical shape)**.** *The solution* $(\boldsymbol{u}, p)$ *is twice differentiable with respect to the domain. Moreover, for* $\boldsymbol{V} \in \boldsymbol{U}$*, we have*

$$\mathrm{D}^2 J(\omega^*) \cdot \boldsymbol{V} \cdot \boldsymbol{V} = -\int_{\partial\omega^*} [(\sigma(\boldsymbol{w}', q')\mathbf{n}) \cdot \partial_{\mathbf{n}} \boldsymbol{u}] \; (\boldsymbol{V} \cdot \mathbf{n}),$$

where $(\boldsymbol{w}', q') \in \mathbf{H}^1(\Omega \setminus \overline{\omega^*}) \times \mathrm{L}^2(\Omega \setminus \overline{\omega^*})$ *is the solution of the following problem:*

$$\begin{cases} -\operatorname{div}(\sigma(\boldsymbol{w}', q')) = \mathbf{0} & \text{in } \Omega \setminus \overline{\omega^*}, \\ \operatorname{div} \boldsymbol{w}' = 0 & \text{in } \Omega \setminus \overline{\omega^*}, \\ \boldsymbol{w}' = \sigma(\boldsymbol{u}', p')\mathbf{n} & \text{on } \partial\Omega, \\ \boldsymbol{w}' = \mathbf{0} & \text{on } \partial\omega^*, \end{cases}$$

with the normalization condition $\langle \sigma(\boldsymbol{w}', q')\mathbf{n} \,,\, \mathbf{n} \rangle_{\partial\Omega} = 0$.

Proposition 4 (Compactness at a critical point)**.** *The Riesz operator corresponding to* $\mathrm{D}^2 J(\omega^*)$ *defined from* $\mathbf{H}^{1/2}(\partial\omega^*)$ *to* $\mathbf{H}^{-1/2}(\partial\omega^*)$ *is compact.*

This last statement points out the lack of stability of the optimization problem (6). This compactness result means, roughly speaking, that in a neighborhood of ω^* (*i.e.* for t small), J behaves as its second order approximation and one cannot expect an estimate of the kind $C\,t \leq \sqrt{J(\omega_t)}$ with a constant C uniform in $\boldsymbol{V}$. This proposition emphasizes that the gradient has not a uniform sensitivity with respect to the deformation directions: J is degenerate for the high frequencies. This explains the numerical difficulties encountered to solve numerically this problem. For more details, we refer to [3, §2.3].

§3. Differentiability results

To prove the existence of the shape derivatives of the state, we have to prove the existence of the total first variations. In order to prove it, we use a generalized implicit function theorem proved by J. Simon (see [8, Theorem 6]) that we recall the statement for the reader's convenience.

Theorem 5 (J. Simon [8])**.** *We give us*

- *an open set* $\mathcal{U}$ *in a Banach space* U, $u_0 \in \mathcal{U}$, *two reflexive Banach spaces* E_1 *and* E_2,
- *a map* $F : \mathcal{U} \times E_1 \to E_2$, *such that* $F(u,\cdot) \in \mathcal{L}(E_1, E_2)$ *for all* $u \in \mathcal{U}$,
- *a function* $m : \mathcal{U} \to E_1$ *and a function* $f : \mathcal{U} \to E_2$ *such that*

$$F(u, m(u)) = f(u) \quad \forall u \in \mathcal{U}.$$

(i) *Assume that*

- $u \mapsto F(u,\cdot)$ *is differentiable at* u_0 *into* $\mathcal{L}(E_1, E_2)$,
- f *is differentiable at* u_0,
- $\|F(u_0, x)\|_{E_2} \geq \alpha \|x\|_{E_1} \quad \forall x \in E_1$, *for some* $\alpha > 0$.

Then, the map $u \mapsto m(u)$ *is differentiable at* u_0. *Its derivative* $m'(u_0,\cdot)$ *is the unique solution of*

$$F(u_0, m'(u_0, v)) = f'(u_0, v) - \partial_u F(u_0, m(u_0), v) \quad \forall v \in U.$$

(ii) *In addition, assume that for some integer* $k \geq 1$, $u \mapsto F(u,\cdot)$ *and* f *are* k *times differentiable at* u_0. *Then, the map* $u \mapsto m(u)$ *is* k *times differentiable at* u_0.

Let $\boldsymbol{\theta} \in \mathcal{U}$. We set $(\boldsymbol{u_\theta}, p_\theta)$ the unique solution in $\mathbf{H}^1(\Omega \setminus \overline{\omega_\theta}) \times \mathrm{L}^2(\Omega \setminus \overline{\omega_\theta})$ of

$$\begin{cases} -\operatorname{div}(\sigma(\boldsymbol{u_\theta}, p_\theta)) = \mathbf{0} & \text{in } \Omega \setminus \overline{\omega_\theta}, \\ \operatorname{div} \boldsymbol{u_\theta} = 0 & \text{in } \Omega \setminus \overline{\omega_\theta}, \\ \boldsymbol{u_\theta} = \boldsymbol{g} & \text{on } \partial\Omega, \\ \boldsymbol{u_\theta} = \mathbf{0} & \text{on } \partial\omega_\theta, \end{cases}$$

with $\langle (\sigma(\boldsymbol{u_\theta}, p_\theta)\mathbf{n}) \cdot \mathbf{n}\,,\ 1\rangle_{\partial\Omega} = \langle \boldsymbol{f_b} \cdot \mathbf{n}\,,\ 1\rangle_{\partial\Omega}$. Let us consider $\boldsymbol{G} \in \mathbf{H}^1(\Omega)$ such that

$$\boldsymbol{G} = \boldsymbol{g} \ \text{ on } \partial\Omega, \quad \operatorname{div} \boldsymbol{G} = 0 \ \text{ in } \Omega \quad \text{and} \quad \boldsymbol{G} = \mathbf{0} \ \text{ in } \Omega_\delta.$$

Thus $(\boldsymbol{z_\theta} = \boldsymbol{u_\theta} - \boldsymbol{G}, p_\theta) \in \mathbf{H}^1_0(\Omega \setminus \overline{\omega_\theta}) \times \mathrm{L}^2(\Omega \setminus \overline{\omega_\theta})$ is such that

$$\begin{cases} \displaystyle\int_{\Omega\setminus\overline{\omega_\theta}} \sigma(\boldsymbol{z_\theta}, p_\theta) : \nabla \boldsymbol{\varphi}_\theta = -\int_{\Omega\setminus\overline{\omega_\theta}} \nu \nabla \boldsymbol{G} : \nabla \boldsymbol{\varphi}_\theta, & \forall \boldsymbol{\varphi_\theta} \in \mathbf{H}^1_0(\Omega \setminus \overline{\omega_\theta}), \\ \displaystyle\int_{\Omega\setminus\overline{\omega_\theta}} \xi_\theta \operatorname{div} \boldsymbol{z_\theta} = 0, & \forall \xi_\theta \in \mathrm{L}^2(\Omega \setminus \overline{\omega_\theta}), \\ \langle (\sigma(\boldsymbol{z_\theta}, p_\theta)\mathbf{n}) \cdot \mathbf{n}\,,\, 1 \rangle_{\partial\Omega} = \langle (\boldsymbol{f_b} - \sigma(\boldsymbol{G}, 0)\mathbf{n}) \cdot \mathbf{n}\,,\, 1 \rangle_{\partial\Omega}\,. \end{cases} \tag{10}$$

Let us define the key objects of our differentiability proof:

$$\boldsymbol{v_\theta} = \boldsymbol{z_\theta} \circ (\mathbf{I} + \boldsymbol{\theta}) \in \mathbf{H}^1_0(\Omega \setminus \overline{\omega}) \quad \text{and} \quad q_\theta = p_\theta \circ (\mathbf{I} + \boldsymbol{\theta}) \in \mathrm{L}^2(\Omega \setminus \overline{\omega}).$$

For $k \geq -1$ and $m \geq 0$ integers with $k < m$, we note $\mathrm{X}^{k,m}(\Omega \setminus \overline{\omega}, \Omega_\delta \setminus \overline{\omega})$ the space of functions in $\mathrm{H}^k(\Omega \setminus \overline{\omega})$ such that their restriction to $\Omega_\delta \setminus \overline{\omega}$ belongs to $\mathrm{H}^m(\Omega_\delta \setminus \overline{\omega})$. This space endowed with the norm $\|u\|_{\mathrm{X}^{k,m}(\Omega\setminus\overline{\omega},\Omega_\delta\setminus\overline{\omega})} = \left(\|u\|^2_{\mathrm{H}^k(\Omega\setminus\overline{\omega})} + \|u\|^2_{\mathrm{H}^m(\Omega_\delta\setminus\overline{\omega})}\right)^{1/2}$ is hilbertian.

First order differentiability. To prove the existence of the first order shape derivative, we first have to prove the following three lemmas:

Lemma 6 (Characterization of $(\boldsymbol{v_\theta}, q_\theta)$)**.** *For $\boldsymbol{\theta} \in \boldsymbol{\mathcal{U}}$, the pair $(\boldsymbol{v_\theta}, q_\theta)$ satisfies for all test functions $\boldsymbol{\varphi} \in \mathbf{H}^1_0(\Omega \setminus \overline{\omega})$ and $\xi \in \mathrm{L}^2(\Omega \setminus \overline{\omega})$*

$$\begin{cases} \displaystyle\int_{\Omega\setminus\overline{\omega}} [(\nu \nabla \boldsymbol{v_\theta} A(\boldsymbol{\theta})) : \nabla \boldsymbol{\varphi} - q_\theta B(\boldsymbol{\theta}) : \nabla \boldsymbol{\varphi}] = \int_{\Omega\setminus\overline{\omega}} -\nu \nabla \boldsymbol{G} : \nabla \boldsymbol{\varphi}, \\ \displaystyle\int_{\Omega\setminus\overline{\omega}} (\nabla \boldsymbol{v_\theta} : B(\boldsymbol{\theta}))\, \xi = 0, \\ \langle (\sigma(\boldsymbol{v}, q)\mathbf{n}) \cdot \mathbf{n}\,,\, 1 \rangle_{\partial\Omega} = \langle (\boldsymbol{f_b} - \sigma(\boldsymbol{G}, 0)\mathbf{n}) \cdot \mathbf{n}\,,\, 1 \rangle_{\partial\Omega}\,, \end{cases}$$

with

$$\begin{aligned} J_{\boldsymbol{\theta}} &= \det(\mathrm{I} + \nabla \boldsymbol{\theta}) \in \mathrm{W}^{2,\infty}\left(\overline{\Omega_\delta}\right), \\ A(\boldsymbol{\theta}) &= J_{\boldsymbol{\theta}}\, (\mathrm{I} + \nabla \boldsymbol{\theta})^{-1} (\mathrm{I} + {}^t\nabla \boldsymbol{\theta})^{-1} \in \mathrm{W}^{2,\infty}\left(\overline{\Omega_\delta}, \mathcal{M}_{N,N}\right), \\ B(\boldsymbol{\theta}) &= J_{\boldsymbol{\theta}} (\mathrm{I} + {}^t\nabla \boldsymbol{\theta})^{-1} \in \mathrm{W}^{2,\infty}\left(\overline{\Omega_\delta}, \mathcal{M}_{N,N}\right). \end{aligned}$$

Lemma 7 (Differentiability of $\boldsymbol{\theta} \mapsto (\boldsymbol{v_\theta}, q_\theta)$)**.** *The function*

$$\boldsymbol{\theta} \in \boldsymbol{\mathcal{U}} \mapsto (\boldsymbol{v_\theta}, q_\theta) \in \mathbf{X}^{1,2}(\Omega \setminus \overline{\omega}, \Omega_\delta \setminus \overline{\omega}) \times \mathrm{X}^{0,1}(\Omega \setminus \overline{\omega}, \Omega_\delta \setminus \overline{\omega})$$

is differentiable in a neighborhood of $\mathbf{0}$.

Lemma 8 (Differentiability of $\boldsymbol{\theta} \mapsto (\boldsymbol{u_\theta}, p_\theta)$)**.** *There exists $\widetilde{\boldsymbol{u}}_{\boldsymbol{\theta}}$, $\widetilde{p}_\theta$ some respective extensions of $\boldsymbol{u_\theta} \in \mathbf{H}^1(\Omega \setminus \overline{\omega})$, $p_\theta \in \mathrm{L}^2(\Omega \setminus \overline{\omega})$ such that the functions*

$$\boldsymbol{\theta} \in \boldsymbol{\mathcal{U}} \mapsto \widetilde{\boldsymbol{u}}_{\boldsymbol{\theta}} \in \mathbf{H}^1(\Omega) \quad \textit{and} \quad \boldsymbol{\theta} \in \boldsymbol{\mathcal{U}} \mapsto \widetilde{p}_\theta \in \mathrm{L}^2(\Omega)$$

are differentiable at $\mathbf{0}$.

Remark 2. We will prove this three lemmas under weaker assumptions: ω with a $C^{1,1}$ boundary, Ω with a Lipschitz boundary and $\boldsymbol{\theta} \in \mathbf{W}^{2,\infty}(\mathbb{R}^N)$.

Proof of Lemma 6: characterization of $(\boldsymbol{v_\theta}, q_\theta)$. We make a change of variables in (10). First, notice that, since $\operatorname{div} \boldsymbol{z_\theta} = 0$ in $\Omega \setminus \overline{\omega_\theta}$,

$$\int_{\Omega\setminus\overline{\omega_\theta}} \sigma(\boldsymbol{z_\theta}, p_\theta) : \nabla \boldsymbol{\varphi}_\theta = \int_{\Omega\setminus\overline{\omega_\theta}} (\nu \nabla \boldsymbol{z_\theta} : \nabla \boldsymbol{\varphi}_\theta - p_\theta \operatorname{div} \boldsymbol{\varphi}_\theta), \quad \forall \boldsymbol{\varphi_\theta} \in \mathbf{H}^1_0(\Omega \setminus \overline{\omega_\theta}).$$

Let $\boldsymbol{\varphi} \in \mathbf{H}^1_0(\Omega \setminus \overline{\omega})$, $\xi \in \mathrm{L}^2(\Omega \setminus \overline{\omega})$ and $\boldsymbol{\theta} \in \boldsymbol{\mathcal{U}}$. Then we proceed in the same manner than the proof of Lemma 3.1 in [3]: we use the test functions $\boldsymbol{\varphi_\theta} = \boldsymbol{\varphi} \circ (\mathbf{I} + \boldsymbol{\theta})^{-1} \in \mathbf{H}^1_0(\Omega \setminus \overline{\omega_\theta})$ and $\xi_\theta = \xi \circ (\mathbf{I} + \boldsymbol{\theta})^{-1} \in \mathrm{L}^2(\Omega \setminus \overline{\omega_\theta})$ in the variational formulation (10) and we make the change of variables $x = (\mathbf{I} + \boldsymbol{\theta})y$. Noticing that $\boldsymbol{\theta} \equiv \mathbf{0}$ in $\Omega \setminus \overline{\Omega_\delta}$ (and therefore on $\partial\Omega$) and that $\boldsymbol{G} \equiv 0$ in Ω_δ, we obtain the result. □

The proof of Lemma 7 is based on Simon's Theorem: we adapt the method used in the proof of Lemma 3.2 in [3].

Proof of Lemma 7: differentiability of $\theta \mapsto (\boldsymbol{v_\theta}, q_\theta)$. Let us check the assumptions of Simon's Theorem.

First step: notations. We need some additional tools: a third domain $\widetilde{\Omega}_\delta$ which is an open set with a C^∞ boundary such that $\Omega_\delta \subset\subset \widetilde{\Omega}_\delta \subset\subset \Omega$ and a truncation function $\Phi \in C^\infty_c(\widetilde{\Omega}_\delta)$ such that $\Phi \equiv 1$ in Ω_δ. Then, we define the spaces

$$\begin{aligned} \boldsymbol{E_1} &= \left\{(\boldsymbol{v}, q) \in \mathbf{H}^1_0(\Omega \setminus \overline{\omega}) \times \mathrm{L}^2(\Omega \setminus \overline{\omega})\,;\ (\Phi\boldsymbol{v}, \Phi q) \in \mathbf{H}^2(\Omega \setminus \overline{\omega}) \times \mathrm{H}^1(\Omega \setminus \overline{\omega})\right\}, \\ \boldsymbol{E_2} &= \left\{(\boldsymbol{f}, g) \in \mathbf{H}^{-1}(\Omega \setminus \overline{\omega}) \times \mathrm{L}^2(\Omega \setminus \overline{\omega});\ (\Phi\boldsymbol{f}, \Phi g) \in \mathbf{L}^2(\Omega \setminus \overline{\omega}) \times \mathrm{H}^1(\Omega \setminus \overline{\omega})\right\} \times \mathbb{R}. \end{aligned}$$

Note that $\boldsymbol{E_1}$ and $\boldsymbol{E_2}$ are Hilbert spaces with respective norms

$$\begin{aligned} \|(\boldsymbol{v}, q)\|^2_{\boldsymbol{E_1}} &= \|\boldsymbol{v}\|^2_{\mathbf{H}^1(\Omega\setminus\overline{\omega})} + \|q\|^2_{\mathrm{L}^2(\Omega\setminus\overline{\omega})} + \|\Phi\boldsymbol{v}\|^2_{\mathbf{H}^2(\Omega\setminus\overline{\omega})} + \|\Phi q\|^2_{\mathrm{H}^1(\Omega\setminus\overline{\omega})}, \\ \|((\boldsymbol{f}, g), r)\|^2_{\boldsymbol{E_2}} &= \|\boldsymbol{f}\|^2_{\mathbf{H}^{-1}(\Omega\setminus\overline{\omega})} + \|g\|^2_{\mathrm{L}^2(\Omega\setminus\overline{\omega})} + \|\Phi\boldsymbol{f}\|^2_{\mathbf{L}^2(\Omega\setminus\overline{\omega})} + \|\Phi g\|^2_{\mathrm{H}^1(\Omega\setminus\overline{\omega})} + |r|^2. \end{aligned}$$

Moreover, we can also notice that $\boldsymbol{E_1} \hookrightarrow \mathbf{X}^{1,2}(\Omega \setminus \overline{\omega}, \Omega_\delta \setminus \overline{\omega}) \times \mathrm{X}^{0,1}(\Omega \setminus \overline{\omega}, \Omega_\delta \setminus \overline{\omega})$ and that $\boldsymbol{E_2} \hookrightarrow \mathbf{X}^{-1,0}(\Omega \setminus \overline{\omega}, \Omega_\delta \setminus \overline{\omega}) \times \mathrm{X}^{0,1}(\Omega \setminus \overline{\omega}, \Omega_\delta \setminus \overline{\omega}) \times \mathbb{R}$. Using the notations introduced in Lemma 6, we also define, for $\boldsymbol{\theta} \in \boldsymbol{\mathcal{U}}$ and $(\boldsymbol{v}, q) \in \boldsymbol{E_1}$, the following functions:

- $\boldsymbol{f_1}(\boldsymbol{\theta}) \in \mathbf{H}^{-1}(\Omega \setminus \overline{\omega})$ by $\forall \boldsymbol{\varphi} \in \mathbf{H}^1_0(\Omega \setminus \overline{\omega})$,

$$\langle \boldsymbol{f_1}(\boldsymbol{\theta})\,,\, \boldsymbol{\varphi}\rangle_{\Omega\setminus\overline{\omega}} = -\int_{\Omega\setminus\overline{\omega}} \nu J_{\boldsymbol{\theta}} \nabla \boldsymbol{G} : \nabla \boldsymbol{\varphi} = -\int_{\Omega\setminus\overline{\Omega_\delta}} \nu \nabla \boldsymbol{G} : \nabla \boldsymbol{\varphi},$$

- $\boldsymbol{F_1}(\boldsymbol{\theta}, (\boldsymbol{v}, q)) \in \mathbf{H}^{-1}(\Omega \setminus \overline{\omega})$ by $\forall \boldsymbol{\varphi} \in \mathbf{H}^1_0(\Omega \setminus \overline{\omega})$,

$$\langle \boldsymbol{F_1}(\boldsymbol{\theta}, (\boldsymbol{v}, q))\,,\, \boldsymbol{\varphi}\rangle_{\Omega\setminus\overline{\omega}} = \int_{\Omega\setminus\overline{\omega}} \{[\nu \nabla \boldsymbol{v}\, A(\boldsymbol{\theta})] : \nabla \boldsymbol{\varphi} - q\, B(\boldsymbol{\theta}) : \nabla \boldsymbol{\varphi}\},$$

- $\boldsymbol{m}(\boldsymbol{\theta}) = (\boldsymbol{v_\theta}\,,\, q_\theta)$ and $\boldsymbol{f}(\boldsymbol{\theta}) = (\boldsymbol{f_1}(\boldsymbol{\theta})\,,\, 0\,,\, \langle(\boldsymbol{f_b} - \sigma(\boldsymbol{G}, 0)\mathbf{n}) \cdot \mathbf{n}\,,\, 1\rangle_{\partial\Omega})$,

- $\boldsymbol{F}(\boldsymbol{\theta},(\boldsymbol{v},q)) = \big(\boldsymbol{F_1}(\boldsymbol{\theta},(\boldsymbol{v},q))\,,\ \nabla\boldsymbol{v} : B(\boldsymbol{\theta})\,,\ \langle(\sigma(\boldsymbol{v},q)\mathbf{n})\cdot\mathbf{n}\,,\ 1\rangle_{\partial\Omega}\big)$.

By the characterization of $(\boldsymbol{v_\theta}, q_\theta)$ obtained in Lemma 6,

$$\boldsymbol{F}(\boldsymbol{\theta}, \boldsymbol{m}(\boldsymbol{\theta})) = \boldsymbol{f}(\boldsymbol{\theta}) \quad \forall\boldsymbol{\theta}\in\boldsymbol{\mathcal{U}}.$$

Second step: differentiability of $\boldsymbol{F}$ *and* $\boldsymbol{f}$ *at* $\mathbf{0}$. In the same way as what is done in the proof of Lemma 3.2 in [3], we prove that $\boldsymbol{F}$ and $\boldsymbol{f}$ are C^∞ in a neighborhood of $\mathbf{0}$.

Third step: existence of $\alpha > 0$ *such that* $\|\boldsymbol{F}(\mathbf{0},(\boldsymbol{v},q))\|_{\boldsymbol{E_2}} \geq \alpha\|(\boldsymbol{v},q)\|_{\boldsymbol{E_1}}$. We consider a pair $(\boldsymbol{v},q)\in\boldsymbol{E_1}$ and we define $(\boldsymbol{\xi},\eta,\boldsymbol{r})\in\boldsymbol{E_2}$ by $\boldsymbol{F}(\mathbf{0},(\boldsymbol{v},q)) = (\boldsymbol{\xi},\eta,\boldsymbol{r})$. Then,

$$\begin{cases} \displaystyle\int_{\Omega\setminus\overline{\omega}} \{\nu\nabla\boldsymbol{v} : \nabla\boldsymbol{\varphi} - q\ \mathrm{div}\,\boldsymbol{\varphi}\} = \langle\boldsymbol{\xi}\,,\ \boldsymbol{\varphi}\rangle_{\Omega\setminus\overline{\omega}} & \forall\boldsymbol{\varphi}\in\mathbf{H}^1_0(\Omega\setminus\overline{\omega}),\\ \displaystyle\int_{\Omega\setminus\overline{\omega}} \phi\ \mathrm{div}\,\boldsymbol{v} = \int_{\Omega\setminus\overline{\omega}}\phi\,\eta & \forall\phi\in\mathrm{L}^2(\Omega\setminus\overline{\omega}),\\ \langle(\sigma(\boldsymbol{v},q)\mathbf{n})\cdot\mathbf{n}\,,\ 1\rangle_{\partial\Omega} = r. \end{cases}$$

The compatibility condition of the previous problem is automatically satisfied because of $\int_{\Omega\setminus\overline{\omega}}\eta = \int_{\partial(\Omega\setminus\overline{\omega})}\boldsymbol{v}\cdot\mathbf{n} = 0$ since $\boldsymbol{v}\in\mathbf{H}^1_0(\Omega\setminus\overline{\omega})$. Thus, proceeding in the same manner than in the proof of Lemma 3.2 in [3], we check using a local regularity argument that there exists a constant $\alpha > 0$ such that

$$\|\boldsymbol{F}(\mathbf{0},(\boldsymbol{v},q))\|_{\boldsymbol{E_2}} \geq \alpha\|(\boldsymbol{v},q)\|_{\boldsymbol{E_1}}.$$

Fourth step: conclusion. By Simon's Theorem, the function $\boldsymbol{\theta}\in\boldsymbol{\mathcal{U}}\mapsto(\boldsymbol{v_\theta},q_\theta)\in\boldsymbol{E_1}$ is differentiable (and even C^∞) in a neighborhood of $\mathbf{0}$. We conclude using the fact that $\boldsymbol{E_1}$ is continuously embedded in $\mathbf{X}^{1,2}(\Omega\setminus\overline{\omega},\Omega_\delta\setminus\overline{\omega})\times\mathrm{X}^{0,1}(\Omega\setminus\overline{\omega},\Omega_\delta\setminus\overline{\omega})$. □

Proof of Lemma 8: differentiability of $\boldsymbol{\theta}\mapsto(\boldsymbol{u_\theta},p_\theta)$. This proof is exactly the same than the proof of Lemma 3.3 in [3]. We refer to this one for details. The idea is to use the differentiability result by composition by $(\mathbf{I}+\boldsymbol{\theta})^{-1}$ (see [6, Lemma 5.3.9]). □

Higher order differentiability. To prove the existence of the second total variations, we will proceed in the same way that what is done previously. We mimic the proof of Lemma 7, only increasing the local regularity in the used spaces to prove that the function

$$\boldsymbol{\theta}\in\boldsymbol{\mathcal{U}}\mapsto(\boldsymbol{v_\theta},q_\theta)\in\mathbf{X}^{1,3}(\Omega\setminus\overline{\omega},\Omega_\delta\setminus\overline{\omega})\times\mathrm{X}^{0,2}(\Omega\setminus\overline{\omega},\Omega_\delta\setminus\overline{\omega})$$

is twice differentiable in a neighborhood of $\mathbf{0}$. Then, proceeding in exactly the same way than in the proof of Lemma 3.5 in [3], we prove the following lemma:

Lemma 9 (Second order shape differentiability)**.** *The solution* $(\boldsymbol{u},p)$ *is twice differentiable with respect to the domain.*

§4. Proof of the main results

First order shape derivatives of the state. *Proof of Proposition 1.* The existence of the shape derivative $(\boldsymbol{u}',p')$ is proved using the Fréchet differentiability Lemma 8. Using the variational formulation of problem (7), we use classical shape derivatives calculus to characterize

$(\boldsymbol{u}', p')$ (see [6, proof of Theorem 5.3.1] concerning the Laplacian case for example). We just precise that, since $\boldsymbol{u} = \mathbf{0}$ on $\partial\omega$, $\nabla\boldsymbol{u} = \partial_{\mathbf{n}}\boldsymbol{u} \otimes \mathbf{n}$, where $\otimes$ is the tensorial product. Hence the classical boundary condition $\boldsymbol{u}' = -\nabla\boldsymbol{u}\,\boldsymbol{V}$ on $\partial\omega$ can be written $\boldsymbol{u}' = -\partial_{\mathbf{n}}\boldsymbol{u}\,(\boldsymbol{V}\cdot\mathbf{n})$. □

First order shape derivatives of the functional. For all $t \in [0, T)$, consider $(\boldsymbol{u_t}, p_t)$ solution of (7) and define,

$$J(\omega_t) = j(t) = \frac{1}{2}\int_{\partial\Omega} |\sigma(\boldsymbol{u_t}, p_t)\,\mathbf{n} - \boldsymbol{f_b}|^2.$$

Proof of Proposition 2. First step: derivative of j and adjoint problem. Noting $(\boldsymbol{u}', p')$ the shape derivative of $(\boldsymbol{u}, p)$, we differentiate j with respect to t at 0 to obtain

$$j'(0) = \nabla J(\omega)\cdot\boldsymbol{V} = \int_{\partial\Omega} (\sigma(\boldsymbol{u}', p')\,\mathbf{n})\cdot(\sigma(\boldsymbol{u}, p)\,\mathbf{n} - \boldsymbol{f_b})\,. \tag{11}$$

Then, we consider the adjoint problem (9). Since we choose the normalization condition (5), the compatibility condition of the adjoint problem is satisfied. Therefore it admits a unique solution $(\boldsymbol{w}, q) \in \mathbf{H}^1(\Omega\setminus\overline{\omega})\times \mathrm{L}^2(\Omega\setminus\overline{\omega})$ with $\langle\sigma(\boldsymbol{w}, q)\,\mathbf{n}\,,\,\mathbf{n}\rangle_{\Omega\setminus\overline{\omega}} = 0$.

Second step: writing of $j'(0)$ as an integral on $\partial\omega$. We proceed by successive integrations by parts. We multiply the first equation of the adjoint problem (9) by $\boldsymbol{u}'$ to get

$$\int_{\Omega\setminus\overline{\omega}} \nu\,\nabla\boldsymbol{w} : \nabla\boldsymbol{u}' = -\left\langle -\sigma(\boldsymbol{w}, q)\,\mathbf{n}\,,\,\boldsymbol{u}'\right\rangle_{\partial(\Omega\setminus\overline{\omega})}\,, \tag{12}$$

since $\operatorname{div}\boldsymbol{u}' = 0$ in $\Omega\setminus\overline{\omega}$ (see Proposition 1). Then, we multiply the first equation of the problem (8) by $\boldsymbol{w}$ to obtain

$$\int_{\Omega\setminus\overline{\omega}} \nu\,\nabla\boldsymbol{u}' : \nabla\boldsymbol{w} = -\left\langle -\sigma(\boldsymbol{u}', p')\,\mathbf{n}\,,\,\boldsymbol{w}\right\rangle_{\partial(\Omega\setminus\overline{\omega})}\,, \tag{13}$$

since $\operatorname{div}\boldsymbol{w} = 0$ in $\Omega\setminus\overline{\omega}$. Gathering (11), (12) and (13) and using the boundary conditions of $(\boldsymbol{u}', p')$ and $(\boldsymbol{w}, q)$ (see problems (8) and (9)), we obtain the announced result. □

Characterization of the shape Hessian at a critical point. We consider $\omega^* \in \mathcal{O}_\delta$ a critical shape of the functional J.

Proof of Proposition 3. First step: second order shape differentiability. By Lemma 9, the second order shape derivative exists which is noted $(\boldsymbol{u}'', p'')$.

Second step: second derivative of j and derivative of the adjoint problem. Let $\boldsymbol{V} \in \boldsymbol{U}$. We differentiate the function j twice with respect to t. At $t = 0$, it holds

$$j''(0) = \mathrm{D}^2 J(\omega)\cdot\boldsymbol{V}\cdot\boldsymbol{V} = \int_{\partial\Omega}\left[(\sigma(\boldsymbol{u}'', p'')\,\mathbf{n})\cdot((\sigma(\boldsymbol{u}, p)\,\mathbf{n}) - \boldsymbol{f_b}) + |\sigma(\boldsymbol{u}', p')\,\mathbf{n}|^2\right].$$

Since ω^* solves the inverse problem, $\sigma(\boldsymbol{u}, p)\,\mathbf{n} = \boldsymbol{f_b}$ on $\partial\Omega$. Therefore

$$\mathrm{D}^2 J(\omega^*)\cdot\boldsymbol{V}\cdot\boldsymbol{V} = 2\int_{\partial\Omega} |\sigma(\boldsymbol{u}', p')\,\mathbf{n}|^2. \tag{14}$$

We introduce $(\boldsymbol{w}, q) \in \mathbf{H}^1(\Omega \setminus \overline{\omega}) \times \mathrm{L}^2(\Omega \setminus \overline{\omega})$ with $\langle \sigma(\boldsymbol{w}, q)\,\mathbf{n}\,,\,\mathbf{n}\rangle_{\partial\Omega} = 0$ the solution of the adjoint system (9). Notice that, for $\omega = \omega^*$, $\sigma(\boldsymbol{u}, p)\,\mathbf{n} = \boldsymbol{f_b}$ on $\partial\Omega$. Hence, the uniqueness of the solution of the Stokes problem enforces that $\boldsymbol{w} = \mathbf{0}$ in $\Omega \setminus \overline{\omega^*}$. Therefore, characterizing $\boldsymbol{w'}$ and q', the shape derivatives of $\boldsymbol{w}$ and q, in the same manner that we characterized $\boldsymbol{u'}$ and p' (see Proposition 1), we obtain the system (3).

Third step: writing of $j''(0)$ as an integral on $\partial\omega$. We multiply the first equation of problem (3) by $\boldsymbol{u'}$ to get

$$\int_{\Omega\setminus\overline{\omega^*}} \nu\nabla\boldsymbol{w'} : \nabla\boldsymbol{u'} = -\left\langle -\sigma(\boldsymbol{w'}, q')\,\mathbf{n}\,,\,\boldsymbol{u'}\right\rangle_{\partial\omega^*}. \tag{15}$$

We multiply the first equation of problem (8) by $\boldsymbol{w'}$ to get

$$\int_{\Omega\setminus\overline{\omega^*}} \nu\nabla\boldsymbol{u'} : \nabla\boldsymbol{w'} = -\left\langle -\sigma(\boldsymbol{u'}, p')\,\mathbf{n}\,,\,\boldsymbol{w'}\right\rangle_{\partial\Omega}. \tag{16}$$

Therefore, gathering (14), (15) and (16), we obtain the announced result. □

Justifying the ill-posedness of the problem. *Proof of Proposition 4.* The proof is an adaptation of the proof of Proposition 2.8 in [3]. The idea is to decompose the shape Hessian as a composition of linear continuous operators and a compact operator. The compactness is proved using a local regularity argument. □

§5. Conclusion

The formal calculus of the shape derivative for the Stokes equations is easier in the Dirichlet case than in the Neumann case which is presented by M. Badra *et al.* in [3], particularly the characterization of $(\boldsymbol{u'}, p')$. However, an other difficulty arises here, due to the introduction of the adjoint problem (9). Indeed, the boundary condition $\sigma(\boldsymbol{u}, p)\,\mathbf{n} - \boldsymbol{f_b}$ on $\partial\Omega$ imposed in (9) has to belong in $\mathbf{H}^{1/2}(\partial\Omega)$. Thus, we have to assume that $\partial\Omega$ is $C^{1,1}$ while we can work with a Lipschitz domain in the Neumann case. Moreover, if we want to make the measurement on a part O of $\partial\Omega$ like what is done in [3], we are confronted to the same difficulty. Indeed, the boundary condition on $\partial\Omega$ of the adjoint problem (9) would be then $(\sigma(\boldsymbol{u}, p)\,\mathbf{n} - \boldsymbol{f_b})\mathbb{1}_O$ which does not belong to $\mathbf{H}^{1/2}(\partial\Omega)$, even if Ω is smooth. A solution could be to use the *very weak solutions* (see *e.g.* [2, §4.2, Definition 1]), even if this method need again that $\partial\Omega$ is $C^{1,1}$. Then, it would be necessary to prove the differentiability with respect to the domain of the *very weak solution* $(\boldsymbol{w}, q) \in \mathbf{L}^2(\Omega \setminus \overline{\omega}) \times \mathrm{H}^{-1}(\Omega \setminus \overline{\omega})/\mathbb{R}$ of the adjoint problem, which is not classical. An other solution is to use a smooth cut-off function as what is done in [4].

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