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# AN APPLICATION OF CARLEMAN INEQUALITIES FOR A CURVED QUANTUM GUIDE

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**Abstract.** We consider in this paper the Schrödinger operator $-i\partial_t - \Delta$ on a curved quantum guide in $\mathbb{R}^2$ for which the reference curve is asymptotically straight. Using an adapted Carleman estimate, we establish a local estimation result for the curvature with a single observation.

*Keywords:* Schrödinger Operators, quantum guide, curvature, Carleman estimate, inverse problem.
*AMS classification:* 35J10.

## §1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a curved quantum guide with a fixed width $d > 0$ and let $T > 0$. We consider the Schrödinger operator

$$H := -i\partial_t - \Delta \text{ in } \Omega \times (0, T).$$

We proceed as in [8] and [4]. We denote by $\Gamma = (\Gamma_1, \Gamma_2)$ the function which characterizes the reference curve and by $N = (N_1, N_2)$ the outgoing normal. We denote by

$$\Omega_1 := \mathbb{R} \times (d, 2d).$$

Each point $(x, y)$ of $\Omega$ is described by the curvilinear coordinates $(s, u)$ as follows:

$$f : \Omega_1 \longrightarrow \Omega \quad \text{with} \quad (x, y) = f(s, u) = \Gamma(s) + uN(s). \tag{1}$$

We assume $\Gamma_1'(s)^2 + \Gamma_2'(s)^2 = 1$ and we recall that the signed curvature $\gamma$ of $\Gamma$ is defined by $\gamma(s) = -\Gamma_1''(s)\Gamma_2'(s) + \Gamma_2''(s)\Gamma_1'(s)$, named so because $|\gamma(s)|$ represents the curvature of the reference curve at $s$. We assume throughout this paper that:

**Assumption 1.**

- $\gamma \in C^3(\mathbb{R})$, $\gamma^{(k)} \in L^\infty(\mathbb{R})$ *for each* $k = 0, 1, 2, 3$, *where* $\gamma^{(k)}$ *denotes the k-th derivatives of* $\gamma$.
- $\gamma(s) \to 0$ *as* $|s| \to \infty$ *and* $1 - 2d\|\gamma\|_\infty > 0$, *where* $\|\gamma\|_\infty := \sup_{s\in\mathbb{R}} |\gamma(s)| = \|\gamma\|_{L^\infty(\mathbb{R})}$.

Note that, by the inverse function theorem, the map $f$ defined by (1) is a diffeomorphism provided $1 - u\gamma(s) \neq 0$, for all $u, s$, which is guaranteed by Assumption 1. The curvilinear coordinates $(s, u)$ are locally orthogonal so the metric in $\Omega$ is expressed with respect to them through a diagonal metric tensor $\begin{pmatrix} (1-u\gamma(s))^2 & 0 \\ 0 & 1 \end{pmatrix}$. The transition to the curvilinear coordinates

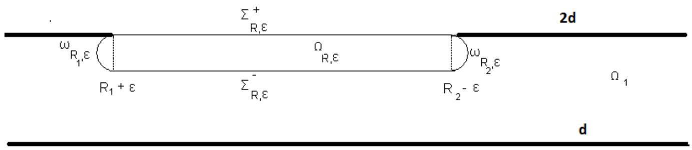


Figure 1: Geometry of the problem

represents an isometric map of $L^2(\Omega)$ to $L^2(\Omega_1, g^{1/2}\ dsdu)$, where $g(s,u) := (1 - u\gamma(s))^2$ is the Jacobian $\partial(x,y)/\partial(s,u)$. Therefore we can replace the operator $H$ (with the metric $dx\,dy$ on $\Omega$) by the operator $H_g$ (with the metric $g^{1/2}ds\,du$ on $\Omega_1$), where

$$H_g := -i\partial_t - g^{-1/2}\partial_s(g^{-1/2}\partial_s) - g^{-1/2}\partial_u(g^{1/2}\partial_u).$$

Then we can rewrite the operator $H_g$ into a Schrödinger-type operator (with the metric $ds\,du$ on $\Omega_1$). Indeed, using the unitary transformation $U_g(\psi) = g^{1/4}\psi$, setting $H_\gamma := U_g H_g U_g^{-1}$, we get

$$H_\gamma = -i\partial_t - \partial_s(c_\gamma(s,u)\partial_s) - \partial_u^2 + V_\gamma(s,u)$$

with

$$c_\gamma(s,u) = \frac{1}{(1 - u\gamma(s))^2} \tag{2}$$

and

$$V_\gamma(s,u) = -\frac{\gamma^2(s)}{4(1-u\gamma(s))^2} - \frac{u\gamma''(s)}{2(1-u\gamma(s))^3} - \frac{5u^2\gamma'^2(s)}{4(1-u\gamma(s))^4}.$$

Let $R := (R_1, R_2) \in \mathbb{R}^2$ and $\epsilon > 0$. We denote by

$$\Omega_{R,\epsilon} := \omega_{R,\epsilon} \cup (]R_1 + \epsilon, R_2 - \epsilon[\times]2d - 2\epsilon, 2d[)$$

a regular bounded domain in $\Omega_1$, with

$$\omega_{R,\epsilon} := \omega_{R_1,\epsilon} \cup \omega_{R_2,\epsilon},$$
$$\omega_{R_1,\epsilon} := \{(s,u) \in \mathbb{R}^2,\ R_1 < s < R_1 + \epsilon,\ 2d - 2\epsilon < u < 2d,\ (s - R_1 - \epsilon)^2 + (u - 2d + \epsilon)^2 < \epsilon\},$$
$$\omega_{R_2,\epsilon} := \{(s,u) \in \mathbb{R}^2,\ R_2 - \epsilon < s < R_2,\ 2d - 2\epsilon < u < 2d,\ (s - R_2 + \epsilon)^2 + (u - 2d + \epsilon)^2 < \epsilon\}.$$

Note that $\omega_{R_1,\epsilon}$ and $\omega_{R_2,\epsilon}$ are half-balls and let (see Figure 1)

$$\Sigma^+_{R,\epsilon} := [R_1 + \epsilon, R_2 - \epsilon] \times \{2d\}, \qquad \Gamma_{R,\epsilon} := \partial\Omega_{R,\epsilon} - \Sigma^-_{R,\epsilon},$$
$$\Sigma^-_{R,\epsilon} := [R_1 + \epsilon, R_2 - \epsilon] \times \{2d - 2\epsilon\}, \qquad \Gamma_\epsilon := (\partial\omega_{R_1,\epsilon} \cup \partial\omega_{R_2,\epsilon}) \cap \partial\Omega_{R,\epsilon}.$$

We now consider the following Schrödinger equation

$$\begin{cases} H_\gamma z := -i\partial_t z(s,u,t) - \partial_s(c_\gamma(s,u)\partial_s z(s,u,t)) - \partial_u^2 z(s,u,t) + V_\gamma(s,u)z(s,u,t) = 0, \\ (s,u,t) \in \Omega_{R,\epsilon} \times (0,T), \\ z(s,u,t) = l(x,y,t),\ (s,u) \in \partial\Omega_{R,\epsilon},\ t \in (0,T), \\ z(s,u,0) = z_0(s,u),\ (s,u) \in \Omega_{R,\epsilon}. \end{cases} \tag{3}$$

Our problem can be stated as follows: Is it possible to determine the curvature $\gamma$ from the measurement of $\partial_\nu(\partial_t z)$ on $\Sigma^+_{R,\epsilon}$?

Let $z$, depending on $\epsilon$ (resp. $\widetilde{z}$, depending on $\epsilon$ too) be a solution of (3) associated with $(\gamma, l, z_0)$ (resp. $(\widetilde{\gamma}, l, z_0)$). We assume that $z_0$ is a real-valued function and that $(\gamma - \widetilde{\gamma})(s) \neq 0$ and $(\gamma' - \widetilde{\gamma}')(s) \neq 0$ for all $s \in [R_1, R_2]$. Our main result is

$$\|\gamma - \widetilde{\gamma}\|^2_{L^2(\Omega_{R,\epsilon})} \leq C\|\partial_\nu(\partial_t z - \partial_t \widetilde{z})\|^2_{L^2(\Sigma^+_{R,\epsilon}\times(-T,T))} + C\epsilon,$$

where $C$ is a positive constant which depends on $d, T$ and where the above norms are weighted Sobolev norms.

This paper gives a quantum mechanics application of an inverse problem and we use for that the important tool of Carleman estimates. Indeed, the method of Carleman inequalities has been introduced in the field of inverse problems by Bukhgeim and Klibanov [2, 3, 11, 12, 13, 14] and constitutes a very efficient tool to derive observability estimates. Note also that even if the spectral properties of curved quantum guides have been intensively studied for several years (see [7, 8, 9] e.g.), up to our knowledge there are few results for inverse problems associated with curved quantum guide (see [4]). The main difficulty here is to recover the curvature $\gamma$ via two coefficients $c_\gamma$ and $V_\gamma$. Few results have already been obtained for the simultaneous identification of two coefficients with one observation and these two coefficients were not linked up (see [6]). This is not the case here where the coefficients $c_\gamma$ and $V_\gamma$ both depend on $\gamma$. Another difficulty when we work with Carleman estimates is the existence of the weight function $\widetilde{\beta}$ (see Assumption 2). And usually this imposes restrictive conditions for the diffusion coefficient i.e. in our case for $c_\gamma$ and therefore for $\gamma$. This is why, due to these two difficulties which come from our model (a curved guide with an asymptotically straight curvature $\gamma$), we work in the subdomain $\Omega_{R,\epsilon}$ instead of the whole strip $\Omega_1$ and we get an additional term $C\epsilon$ in the right hand side of our main result (which was not the case in [5,6]). This paper is organized as follows: Section 2 is devoted to the Carleman inequality adapted to our problem. In Section 3 we state and prove our main result.

## §2. Carleman inequality

In this section we obtain a Carleman estimate for a function $q$ equal to zero on $\partial\Omega_{R,\epsilon}\times(-T, T)$ and solution of the Schrödinger equation $H_\gamma q \in L^2(\Omega_{R,\epsilon} \times (-T, T))$. We prove a Carleman estimate for $q$ with a single observation acting on $\Gamma_{R,\epsilon} \times (-T, T)$ in the right-hand side of the estimate. Note that this estimate is quite similar to the one obtained in [1] or [5] but the computations are different. Indeed the weight function $\widetilde{\beta}$ does not satisfy the same pseudo-convexity assumptions (see Assumption 2(iii)). This is the main difference compared to [5] and this is due to the particular form of the operator $H_\gamma$ where the diffusion coefficient $c_\gamma$ only appears in the derivatives respect to $s$.

We use the following notations

$$c := c_\gamma, \quad \nabla_c\beta := \begin{pmatrix} \sqrt{c}\partial_s\beta \\ \partial_u\beta \end{pmatrix} \quad \text{and} \quad \nu_c := \begin{pmatrix} \sqrt{c}\partial_s\nu \\ \partial_u\nu \end{pmatrix},$$

where $\nu$ denotes the unit outward normal to $\partial\Omega_{R,\epsilon}$ and we proceed as in [1] or [5]. Let $\widetilde{\beta} := \widetilde{\beta}(s, u)$ be a positive function such that there exists positive constants $\beta_0$ and $C_{pc}$ which satisfy:

**Assumption 2.**

*(i)* $\widetilde{\beta} \in C^4(\overline{\Omega_{R,\epsilon}})$, *and* $\widetilde{\beta}(s,u) \geq 0$ *for all* $(s,u) \in \Omega_{R,\epsilon}$.

*(ii)* $|\nabla_c \widetilde{\beta}| \geq \beta_0 > 0$ *in* $\Omega_{R,\epsilon}$, *and* $\nabla_c \widetilde{\beta} \cdot \nu_c \leq 0$ *in* $\Sigma^-_{R,\epsilon}$.

*(iii)* $2\,\mathrm{Re}\, D_c^2 \widetilde{\beta}(\xi, \overline{\xi}) - \frac{1}{c}\nabla_c c \cdot \nabla_c \widetilde{\beta} |\xi_1|^2 + 2|\nabla_c \widetilde{\beta} \cdot \xi|^2 \geq C_{pc}|\xi|^2$ *for all* $\xi = (\xi_1, \xi_2) \in \mathbb{C}$, *where*

$$D_c^2 \widetilde{\beta} = \begin{pmatrix} \partial_s(c\partial_s \widetilde{\beta}) & \sqrt{c}\,\partial^2_{su}\widetilde{\beta} \\ \frac{1}{\sqrt{c}}\,\partial_u(c\partial_s \widetilde{\beta}) & \partial^2_u \widetilde{\beta} \end{pmatrix}. \tag{4}$$

This assumption imposes restrictive conditions for the choice of the coefficient $c := c_\gamma$ and thus for the curvature $\gamma$ in connection with the function $\widetilde{\beta}$ as in [5, 6]. Note that there exists functions satisfying such conditions. Indeed if we assume that $\widetilde{\beta}(s,u) := \beta_1(s) + \beta_2(u)$, these conditions can be written in the following form:

$$A := 2\partial_s(c\partial_s\beta_1) - c\partial_s c\partial_s\beta_1 - \partial_u c\partial_u\beta_2 + 2c(\partial_s\beta_1)^2 \geq cst > 0 \text{ and } 2AC - B^2 \geq cst > 0,$$

with $B := (1/\sqrt{c})\partial_u c\partial_s\beta_1 + 2\sqrt{c}\partial_s\beta_1\partial_u\beta_2$ and $C := \partial^2_u\beta_2 + (\partial_u\beta_2)^2$. For example if $\widetilde{\beta}(s,u) = e^s + e^u$, these two last conditions become

$$A = (1 - u\gamma(s))^{-3}[(2 - c(s,u))2u\gamma'(s)e^s - 2\gamma(s)e^u] + 2c(s,u)(e^s + e^{2s})$$

and

$$\begin{aligned} 2AC - B^2 = 4c(s,u)\Big[&(2 - c(s,u))u\gamma'(s)(1 - u\gamma(s))^{-1}e^s(e^u + e^{2u}) \\ &- \gamma(s)(1 - u\gamma(s))^{-1}e^u(e^u + e^{2u}) + e^s e^u(1 + e^s + e^u) \\ &- \gamma(s)(1 - u\gamma(s))^{-1}e^{2s}(\gamma(s)(1 - u\gamma(s))^{-1} + 2e^u)\Big]. \end{aligned}$$

We have $A \geq cst > 0$ and $2AC - B^2 \geq cst > 0$ for any curvature $\gamma$ in

$$\{\gamma \in C^1(\mathbb{R}),\ \gamma' \geq 0,\ \gamma \leq 0,\ (1 - 2d\|\gamma\|_\infty)^{-2} < 2,\ \gamma(s) > -2e^{2d-2\epsilon}(1 - 2d\|\gamma\|_\infty)\ \forall s \in [R_1, R_2]\}.$$

Similar restrictive conditions upon the function $c$ in connection with the function $\widetilde{\beta}$ have also been highlighted for the hyperbolic case in [13, 14].

Then we define $\beta = \widetilde{\beta} + K$ with $K = m\|\widetilde{\beta}\|_{L^\infty(\Omega_{R,\epsilon})}$ and $m > 1$. For $\lambda > 0$ we define on $\Omega_{R,\epsilon} \times (-T, T)$ the functions $\phi$ and $\eta$ by

$$\phi(s,u,t) = \frac{e^{\lambda\beta(s,u)}}{(T-t)(T+t)} \quad \text{and} \quad \eta(s,u,t) = \frac{e^{2\lambda K} - e^{\lambda\beta(s,u)}}{(T-t)(T+t)}. \tag{5}$$

For $S > 0$ we set $\psi = e^{-S\eta}q$ and $M\psi := e^{-S\eta}H_\gamma q$. Following [1], we write $M\psi - V_\gamma\psi = M_1\psi + M_2\psi$, with

$$M_1\psi := -i\partial_t\psi - \Delta_c\psi - S^2\lambda^2\phi^2\psi|\nabla_c\beta|^2, \tag{6}$$

$$M_2\psi := -iS\,\partial_t\eta\psi + 2S\lambda\phi\nabla_c\beta \cdot \nabla_c\overline{\psi} + S\lambda^2\phi\psi|\nabla_c\beta|^2 + S\lambda\phi\psi\Delta_c\beta, \tag{7}$$

where

$$\nabla_c\beta := \begin{pmatrix} \sqrt{c}\partial_s\beta \\ \partial_u\beta \end{pmatrix}, \quad \Delta_c\beta := \partial_s(c\partial_s\beta) + \partial^2_u\beta, \quad \nabla_c\beta \cdot \nabla_c\psi = c\partial_s\beta\partial_s\overline{\psi} + \partial_u\beta\partial_u\overline{\psi}.$$

Then the following result holds:

**Theorem 3.** *Let $H_\gamma$, $M$, $M_1$, $M_2$ be the operators defined as above. Assume that Assumptions 1 and 2 are satisfied. Then, there exist $\Lambda_0 > 0$, $S_0 > 0$ and a positive constant $C$ depending on $T$ such that, for any $\lambda > \Lambda_0$ and any $S > S_0$,*

$$S\lambda \int_{\Omega_{R,\epsilon}\times(-T,T)} \phi|\nabla q|^2 e^{-2S\eta} + S^3\lambda^4 \int_{\Omega_{R,\epsilon}\times(-T,T)} \phi^3|q|^2 e^{-2S\eta} + \int_{\Omega_{R,\epsilon}\times(-T,T)} |M_1(e^{-S\eta}q)|^2$$
$$+ \int_{\Omega_{R,\epsilon}\times(-T,T)} |M_2(e^{-S\eta}q)|^2 \leq C \int_{\Omega_{R,\epsilon}\times(-T,T)} |H_\gamma q|^2 e^{-2S\eta} + CS\lambda \int_{\Gamma_{R,\epsilon}\times(-T,T)} \phi|\partial_\nu q|^2 e^{-2S\eta}$$

*for all $q$ satisfying $H_\gamma q \in L^2(\Omega_{R,\epsilon} \times (-T,T))$, $q \in L^2(-T,T;H_0^1(\Omega_{R,\epsilon}))$, $\partial_\nu q = \nabla q \cdot \nu$, and $\partial_\nu q \in L^2(-T,T;L^2(\Gamma_{R,\epsilon}))$.*

*Proof.* We proceed as in [1], [5] or [6]. We have:

$$\int_{\Omega_{R,\epsilon}\times(-T,T)} |M\psi - V_\gamma\psi|^2 = \int_{\Omega_{R,\epsilon}\times(-T,T)} (|M_1\psi|^2 + |M_2\psi|^2) + 2\,\mathrm{Re} \int_{\Omega_{R,\epsilon}\times(-T,T)} M_1\psi\overline{M_2\psi}. \quad (8)$$

Multiplying each term of $M_1\psi$ by each term of $\overline{M_2\psi}$ (see (6) and (7)), we will calculate under the following form:

$$\mathrm{Re} \int_{\Omega_{R,\epsilon}\times(-T,T)} M_1\psi\overline{M_2\psi} = I_{11} + I_{12} + I_{13} + I_{21} + I_{22} + I_{23} + I_{31} + I_{32} + I_{33}. \quad (9)$$

We denote by $Q := \Omega_{R,\epsilon} \times (-T,T)$. We obtain by integrating by parts:

$$I_{11} = \mathrm{Re} \int_Q (-i\partial_t\psi)\overline{(-iS\,\partial_t\eta\psi)} = -\frac{S}{2} \int_Q \partial_t^2\eta\,|\psi|^2. \quad (10)$$

Since $I_{12} = \mathrm{Re} \int_Q (-i\partial_t\psi)2S\lambda\phi\nabla_c\beta\cdot\nabla_c\psi = S\lambda\,\mathrm{Im} \int_Q \phi\partial_t\psi\nabla_c\beta\cdot\nabla_c\psi - S\lambda\,\mathrm{Im} \int_Q \phi\partial_t\overline{\psi}\nabla_c\beta\cdot\nabla_c\overline{\psi}$, integrating by parts in time for the first term and in space for the second term, we get

$$I_{12} = S\lambda^2\,\mathrm{Im} \int_Q \phi\partial_t\overline{\psi}\psi|\nabla_c\beta|^2 + S\lambda\,\mathrm{Im} \int_Q \phi\partial_t\overline{\psi}\psi\Delta_c\beta - S\lambda\,\mathrm{Im} \int_Q \partial_t\phi\psi\nabla_c\beta\cdot\nabla_c\psi. \quad (11)$$

Moreover, $I_{13} = \mathrm{Re} \int_Q (-i\partial_t\psi)[S\lambda^2\phi\overline{\psi}|\nabla_c\beta|^2 + S\lambda\phi\overline{\psi}\Delta_c\beta]$ becomes

$$I_{13} = -S\lambda\,\mathrm{Im} \int_Q \phi\partial_t\overline{\psi}\psi\Delta_c\beta - S\lambda^2\,\mathrm{Im} \int_Q \phi\partial_t\overline{\psi}\psi|\nabla_c\beta|^2 \quad (12)$$

and integrating by parts in space we have

$$I_{21} = \mathrm{Re} \int_Q (-\Delta_c\psi)\overline{(-iS\,\partial_t\eta\psi)} = -S\lambda\,\mathrm{Im} \int_Q \partial_t\phi\psi\nabla_c\beta\cdot\nabla_c\psi. \quad (13)$$

So from (11)–(13) note that $I_{12} + I_{13} + I_{21} = -2S\lambda\,\mathrm{Im} \int_Q \partial_t\phi\psi\nabla_c\beta\cdot\nabla_c\psi$. Furthermore, $I_{22} = \mathrm{Re} \int_Q (-\Delta_c\psi)2S\lambda\phi\nabla_c\beta\cdot\nabla_c\psi$. By integrating by parts twice in space we obtain that

$$I_{22} = 2S\lambda^2 \int_Q \phi|\nabla_c\beta\cdot\nabla_c\overline{\psi}|^2 + 2S\lambda\,\mathrm{Re} \int_Q \phi D_c^2\beta(\nabla_c\psi, \nabla_c\overline{\psi})$$

$$-S\lambda\int_Q \phi\frac{1}{c}\nabla_c c\cdot\nabla_c\beta|\sqrt{c}\partial_s\psi|^2 - S\lambda\int_{\partial\Omega_R\times(-T,T)}\phi\nabla_c\beta\cdot\nu_c|\nabla_c\psi|^2 \qquad (14)$$
$$-S\lambda^2\int_Q\phi|\nabla_c\beta|^2|\nabla_c\psi|^2 - S\lambda\int_Q\phi\Delta_c\beta|\nabla_c\psi|^2,$$

with $D_c^2\beta$ defined by (4). We have also $I_{23} = \mathrm{Re}\int_Q(-\Delta_c\psi)[S\lambda^2\phi\overline{\psi}|\nabla_c\beta|^2 + S\lambda\phi\overline{\psi}\Delta_c\beta]$ and, by integrations by parts twice in space, we obtain:

$$\begin{aligned}I_{23} &= S\lambda\int_Q\phi|\nabla_c\psi|^2\Delta_c\beta - S\lambda^3\int_Q|\psi|^2\phi|\nabla_c\beta|^2\Delta_c\beta - S\lambda^2\int_Q|\psi|^2\phi\nabla_c\beta\cdot\nabla_c(\Delta_c\beta)\\ &- \frac{S\lambda^2}{2}\int_Q|\psi|^2\phi\Delta_c(|\nabla_c\beta|^2) - \frac{S\lambda^2}{2}\int_Q|\psi|^2\phi(\Delta_c\beta)^2 - \frac{S\lambda}{2}\int_Q|\psi|^2\phi\Delta_c(\Delta_c\beta) \qquad (15)\\ &- \frac{S\lambda^4}{2}\int_Q|\psi|^2\phi|\nabla_c\beta|^4 - S\lambda^3\int_Q|\psi|^2\phi\nabla_c\beta\cdot\nabla_c(|\nabla_c\beta|^2) + S\lambda^2\int_Q\phi|\nabla_c\beta|^2|\nabla_c\psi|^2.\end{aligned}$$

And we obviously have

$$I_{31} = \mathrm{Re}\int_Q(-S^2\lambda^2\phi^2\psi|\nabla_c\beta|^2)\overline{(-iS\partial_t\eta\psi)} = 0. \qquad (16)$$

Moreover

$$\begin{aligned}I_{32} &= \mathrm{Re}\int_Q(-S^2\lambda^2\phi^2\psi|\nabla_c\beta|^2)2S\lambda\phi\nabla_c\beta\cdot\nabla_c\psi\\ &= S^3\lambda^3\int_Q\phi^3|\psi|^2(\nabla_c\beta\cdot\nabla_c(|\nabla_c\beta|^2) + |\nabla_c\beta|^2\Delta_c\beta) + 3S^3\lambda^4\int_Q\phi^3|\nabla_c\beta|^4|\psi|^2,\end{aligned} \qquad (17)$$

$$\begin{aligned}I_{33} &= \mathrm{Re}\int_Q(-S^2\lambda^2\phi^2\psi|\nabla_c\beta|^2)[S\lambda^2\phi\overline{\psi}|\nabla_c\beta|^2 + S\lambda\phi\overline{\psi}\Delta_c\beta]\\ &= -S^3\lambda^3\int_Q\phi^3|\psi|^2|\nabla_c\beta|^2\Delta_c\beta - S^3\lambda^4\int_Q\phi^3|\nabla_c\beta|^4|\psi|^2.\end{aligned} \qquad (18)$$

Therefore, from (10) to (18), (9) becomes

$$\begin{aligned}\mathrm{Re}\int_Q M_1\psi\overline{M_2\psi} = &-\frac{S}{2}\int_Q\partial_t^2\eta|\psi|^2 - S\lambda\int_Q\phi\frac{1}{c}\nabla_c c\cdot\nabla_c\beta|\sqrt{c}\partial_s\psi|^2\\ &- 2S\lambda\,\mathrm{Im}\int_Q\partial_t\phi\psi\nabla_c\beta\cdot\nabla_c\psi + 2S\lambda^2\int_Q\phi|\nabla_c\beta\cdot\nabla_c\psi|^2\\ &+ 2S\lambda\,\mathrm{Re}\int_Q\phi D_c^2\beta(\nabla_c\overline{\psi},\nabla_c\psi) - S\lambda\int_{\partial\Omega_{R,\epsilon}\times(-T,T)}\phi|\nabla_c\psi|^2\nabla_c\beta\cdot\nu_c\\ &- S\lambda^3\int_Q\phi|\psi|^2|\nabla_c\beta|^2\Delta_c\beta - S\lambda^2\int_Q\phi|\psi|^2\nabla_c\beta\cdot\nabla_c(\Delta_c\beta) \qquad (19)\\ &- \frac{S\lambda^2}{2}\int_Q\phi|\psi|^2(\Delta_c\beta)^2 - S\lambda^3\int_Q\phi|\psi|^2\nabla_c\beta\cdot\nabla_c(|\nabla_c\beta|^2)\\ &- \frac{S\lambda^2}{2}\int_Q\phi|\psi|^2\Delta_c(|\nabla_c\beta|^2) - \frac{S\lambda}{2}\int_Q\phi|\psi|^2\Delta_c(\Delta_c\beta)\end{aligned}$$

$$- \frac{S\lambda^4}{2}\int_Q \phi|\psi|^2|\nabla_c\beta|^4 + 2S^3\lambda^4\int_Q \phi^3|\psi|^2|\nabla_c\beta|^4$$
$$+ S^3\lambda^3\int_Q \phi^3|\psi|^2\nabla_c\beta\cdot\nabla_c(|\nabla_c\beta|^2). \tag{20}$$

Now, if we call by $X$ the terms in (19) which are neglectable with respect to the quatities $S\lambda^2\int_Q \phi|\nabla_c\beta\cdot\nabla_c\psi|^2$ or $S^3\lambda^4\int_Q \phi^3|\psi|^2|\nabla_c\beta|^4$, we get:

$$\begin{aligned} X = & -\frac{S}{2}\int_Q \partial_t^2\eta|\psi|^2 - 2S\lambda\,\mathrm{Im}\int_Q \partial_t\phi\psi\nabla_c\beta\cdot\nabla_c\psi - S\lambda^3\int_Q \phi|\psi|^2|\nabla_c\beta|^2\Delta_c\beta \\ & - S\lambda^2\int_Q \phi|\psi|^2\nabla_c\beta\cdot\nabla_c(\Delta_c\beta) - \frac{S\lambda^2}{2}\int_Q \phi|\psi|^2(\Delta_c\beta)^2 - \frac{S\lambda}{2}\int_Q \phi|\psi|^2\Delta_c(\Delta_c\beta) \\ & - \frac{S\lambda^4}{2}\int_Q \phi|\psi|^2|\nabla_c\beta|^4 - S\lambda^3\int_Q \phi|\psi|^2\nabla_c\beta\cdot\nabla_c(|\nabla_c\beta|^2) - \frac{S\lambda^2}{2}\int_Q \phi|\psi|^2\Delta_c(|\nabla_c\beta|^2) \\ & + S^3\lambda^3\int_Q \phi^3|\psi|^2\nabla_c\beta\cdot\nabla_c(|\nabla_c\beta|^2). \end{aligned}$$

So (19) becomes

$$\begin{aligned} \mathrm{Re}\int_Q M_1\psi\overline{M_2\psi} = & X + 2S\lambda^2\int_Q \phi|\nabla_c\beta\cdot\nabla_c\psi|^2 + 2S\lambda\,\mathrm{Re}\int_Q \phi D_c^2\beta(\nabla_c\overline{\psi},\nabla_c\psi) \\ & - S\lambda\int_Q \phi\frac{1}{c}\nabla_c c\cdot\nabla_c\beta|\sqrt{c}\partial_s\psi|^2 - S\lambda\int_{\partial\Omega_{R,\epsilon}\times(-T,T)} \phi|\nabla_c\psi|^2\nabla_c\beta\cdot\nu_c \\ & + 2S^3\lambda^4\int_Q \phi^3|\psi|^2|\nabla_c\beta|^4 \end{aligned}$$

and there exists a positive constant $k$ such that

$$|X| \le kS\lambda^4\int_Q \phi|\psi|^2 + kS^3\lambda^3\int_Q \phi^3|\psi|^2 + kS\lambda\int_Q \phi|\nabla_c\beta\cdot\nabla_c\psi|^2. \tag{21}$$

Moreover, from (8), (21) and Assumption 2, we get

$$\begin{aligned} & \int_Q [|M_1\psi|^2 + |M_2\psi|^2] + 4S\lambda^2\int_Q \phi|\nabla_c\beta\cdot\nabla_c\psi|^2 - 2S\lambda\int_Q \phi\frac{1}{c}\nabla_c c\cdot\nabla_c\beta|\sqrt{c}\partial_s\psi|^2 \\ & \quad + 4S\lambda\,\mathrm{Re}\int_Q \phi D_c^2\beta(\nabla_c\overline{\psi},\nabla_c\psi) + 4S^3\lambda^4\beta_0^4\int_Q \phi^3|\psi|^2 \\ & \le Cs\lambda\int_{\partial\Omega_{R,\epsilon}\times(-T,T)} \phi|\nabla_c\psi|^2\nabla_c\beta\cdot\nu_c + CS\lambda^4\int_Q \phi|\psi|^2 + CS^3\lambda^3\int_Q \phi^3|\psi|^2 \\ & \quad + CS\lambda\int_Q \phi|\nabla_c\beta\cdot\nabla_c\psi|^2 + C\int_Q |M\psi|^2 + C\int_Q V_\gamma^2|\psi|^2. \end{aligned} \tag{22}$$

Since $V_\gamma$ is bounded on $\Omega_{R,\epsilon}$ and since $\phi$ is a positive continuous function there exists a positive constant depending upon $T$ such that $V_\gamma \le cst\,\phi^3$. Choosing such $S$ and $\lambda$ sufficiently large, we deduce that there exists a positive constant $C_1$ such that (22) becomes

$$\int_Q [|M_1\psi|^2 + |M_2\psi|^2] + S\lambda^2 \int_Q \phi|\nabla_c\beta\cdot\nabla_c\psi|^2 + S\lambda\,\mathrm{Re}\int_Q \phi D_c^2\beta(\nabla_c\overline{\psi},\nabla_c\psi) + S^3\lambda^4\int_Q \phi^3|\psi|^2 - S\lambda\int_Q \phi\frac{1}{c}\nabla_c c\cdot\nabla_c\beta|\sqrt{c}\partial_s\psi|^2 \leq C_1 S\lambda\int_{\Gamma_{R,\epsilon}\times(-T,T)} \phi|\nabla_c\psi|^2\nabla_c\beta\cdot\nu_c + C_1\int_Q |M\psi|^2.$$

Finally, we come back to $q = e^{S\eta}\psi$. And this concludes the proof. □

## §3. Inverse problem

First, using an idea developed in [10], we prove the following lemma:

**Lemma 4.** *Let $z_0$ be a real function in $C^2(\overline{\Omega_{R,\epsilon}})$ and define the following first order differential operator $P_0 g := \partial_s z_0 \partial_s g$. Let $\eta_0$ be a real function in $C^2(\overline{\Omega_{R,\epsilon}})$. Assume that for all $(s,u) \in \overline{\Omega_{R,\epsilon}}$, $(\partial_s z_0 \partial_s \eta_0)^2 \geq cst > 0$. Then there exists a positive constant $C$ such that for $S$ sufficiently large*

$$S^2\int_{\Omega_{R,\epsilon}} e^{-2S\eta_0}|g|^2 \leq C\int_{\Omega_{R,\epsilon}} |P_0 g|^2 e^{-2S\eta_0} + CS\int_{\Gamma_\epsilon} e^{-2S\eta_0}|g|^2|\partial_s\eta_0\nu_s|$$

*for any $g \in H^1(\Omega_{R,\epsilon})$.*

*Proof.* Let $g \in H^1(\Omega_{R,\epsilon})$. Define $w = e^{-S\eta_0}g$ and $Q_0 w := e^{-S\eta_0}P_0(e^{S\eta_0}w)$. If we set $q_0 = \partial_s z_0 \partial_s \eta_0$, then we get $Q_0 w = Sq_0 w + P_0 w$. Therefore we have:

$$\int_{\Omega_{R,\epsilon}} |Q_0 w|^2 = \int_{\Omega_{R,\epsilon}} |P_0 g|^2 e^{-2S\eta_0} = S^2\int_{\Omega_{R,\epsilon}} q_0^2|w|^2 + \int_{\Omega_{R,\epsilon}} |P_0 w|^2 + 2S\,\mathrm{Re}\int_{\Omega_{R,\epsilon}} q_0 w\overline{P_0 w} \geq S^2\int_{\Omega_{R,\epsilon}} q_0^2|w|^2 + S\int_{\Omega_{R,\epsilon}} q_0\partial_s z_0\partial_s(|w|^2)$$

and so, integrating by parts, since $\nu_s = 0$ on $\Sigma^+_{R,\epsilon}\cup\Sigma^-_{R,\epsilon}$, we get

$$\int_{\Omega_{R,\epsilon}} |P_0 g|^2 e^{-2S\eta_0} \geq S^2\int_{\Omega_{R,\epsilon}} q_0^2 e^{-2S\eta_0}|g|^2 + S(-\int_{\Omega_{R,\epsilon}} \partial_s(q_0\partial_s z_0)e^{-2S\eta_0}|g|^2 + \int_{\Gamma_\epsilon} e^{-2S\eta_0}|g|^2 q_0\partial_s z_0\nu_s).$$

Since $\partial_s(q_0\partial_s z_0)$ is a bounded function in $\overline{\Omega_{R,\epsilon}}$ and $q_0\partial_s z_0\nu_s = (\partial_s z_0)^2\partial_s\eta_0\nu_s$, we can conclude. □

Then, we consider $\gamma$ and $\widetilde{\gamma}$ two functions satisfying Assumption 1.

Let $z$ be a solution of

$$\begin{cases} -i\partial_t z(s,u,t) - \partial_s(c_\gamma(s,u)\partial_s z(s,u,t)) - \partial_u^2 z(s,u,t) + V_\gamma(s,u)z(s,u,t) = 0, \\ (s,u,t) \in \Omega_{R,\epsilon} \times (0,T), \\ z(s,u,t) = l(s,u,t),\ (s,u) \in \partial\Omega_{R,\epsilon},\ t \in (0,T), \\ z(s,u,0) = z_0(s,u),\ (s,u) \in \Omega_{R,\epsilon}, \end{cases} \tag{23}$$

and let $\widetilde{z}$ be a solution of

$$\begin{cases} -i\partial_t \widetilde{z}(s,u,t) - \partial_x(c_{\widetilde{\gamma}}(s,u)\partial_s \widetilde{z}(s,u,t)) - \partial_u^2 \widetilde{z}(s,u,t) + V_{\widetilde{\gamma}}(s,u)\widetilde{z}(s,u,t) = 0, \\ (s,u,t) \in \Omega_{R,\epsilon} \times (0,T), \\ \widetilde{z}(s,u,t) = l(s,u,t),\ (s,u) \in \partial\Omega_{R,\epsilon},\ t \in (0,T), \\ \widetilde{z}(s,u,0) = z_0(s,u),\ (s,u) \in \Omega_{R,\epsilon}. \end{cases} \tag{24}$$

Let $\Lambda_N := \{f \in C^1([R_1,R_2]),\ |f'(s)| \leq N|f(s)| \text{ and } |f(s| \leq N \text{ for all } s \in [R_1,R_2]\}$ with $N$ a positive real given. We obtain the following theorem:

**Theorem 5.** *Let $\gamma$ and $\widetilde{\gamma}$ be functions both satisfying Assumption 1 and such that $(\gamma-\widetilde{\gamma})(s) \neq 0$ and $(\gamma' - \widetilde{\gamma}')(s) \neq 0$ for all $s \in [R_1,R_2]$. Assume that $\beta$ is a function which satisfies Assumption 2 w.r.t. $c_\gamma$ with $c_\gamma$ defined by* (2). *Assume also that*

*(i) $z_0$ is a real function such that $z_0 \in C^2(\overline{\Omega_{R,\epsilon}})$.*

*(ii) For all $(s,u) \in \overline{\Omega_{R,\epsilon}}$, $(\partial_s z_0(s,u)\partial_s\eta(s,u,0))^2 \geq cst > 0$ (where $\eta$ is defined by* (5)*).*

*(iii) $\partial_t\widetilde{z} \in L^\infty(\Omega_{R,\epsilon}\times(0,T))$, $\partial_s(\partial_t\widetilde{z}) \in L^\infty(\Omega_{R,\epsilon}\times(0,T))$, $\partial_s^2(\partial_t\widetilde{z}) \in L^\infty(\Omega_{R,\epsilon}\times(0,T))$, $\partial_\nu(\partial_t(z-\widetilde{z})) \in L^\infty(\Gamma_\epsilon\times(0,T))$ and the $L^\infty$-norm of each of these functions is less than $N$.*

*(iv) $\gamma - \widetilde{\gamma} \in \Lambda_N$ and $\gamma' - \widetilde{\gamma}' \in \Lambda_N$.*

*Then there exists a positive constant $C$, depending upon $N$, $T$, $\|\beta\|_{L^\infty}$, $\|\partial_s\beta\|_{L^\infty}$ such that, for $S$ and $\lambda$ sufficiently large, we have:*

$$\int_{L^2(\Omega_{R,\epsilon})} e^{-2S\eta_0}|\gamma(s)-\widetilde{\gamma}(s)|^2\, dsdu \leq C\int_{\Sigma^+_{R,\epsilon}\times(-T,T)} \phi e^{-2S\eta}|\partial_\nu(\partial_t(z-\widetilde{z}))|^2 + C\epsilon. \tag{25}$$

Note that $\partial_s z_0\partial_s\eta := -\lambda\partial_s z_0(e^{\lambda\beta}/T^2)\,\partial_s\beta$ satisfies the above hypothesis (ii) for any function $z_0$ such that $\partial_s z_0$ is a continuous and non null function in $\overline{\Omega_{R,\epsilon}}$ (by assuming also that $\partial_s\beta$ is a non null function in $\overline{\Omega_{R,\epsilon}}$, which is true for $\beta(s,u) = e^s + e^u$ for example). Note that since $\gamma - \widetilde{\gamma}$ is assumed satisfying $(\gamma-\widetilde{\gamma})(s) \neq 0$ and $(\gamma' - \widetilde{\gamma}')(s) \neq 0$ for all $s \in [R_1,R_2]$, then $\frac{\gamma'-\widetilde{\gamma}'}{\gamma-\widetilde{\gamma}}$ and $\frac{\gamma''-\widetilde{\gamma}''}{\gamma'-\widetilde{\gamma}'}$ are bounded functions in $[R_1,R_2]$ and therefore the previous hypothesis iv) is verified for some $N$. Note also that the above hypothesis (iii) is satisfied for any function $\widetilde{z} \in C^3(\Omega_{R,\epsilon}\times(0,T))$.

*Proof.* Now, recall that $z$ (resp. $\widetilde{z}$) is a solution of (23) (resp. (24)). If we set $w = z - \widetilde{z}$, $v = \partial_t w$, $g = c_\gamma - c_{\widetilde{\gamma}}$ and $h = V_\gamma - V_{\widetilde{\gamma}}$, we get

$$\begin{cases} -i\partial_t w - \partial_s(c_\gamma\partial_s w) - \partial_u^2 w + V_\gamma w = \partial_s(g\partial_s\widetilde{z}) - h\widetilde{z} \text{ in } \Omega_{R,\epsilon}\times(0,T), \\ w = 0 \text{ on } \partial\Omega_{R,\epsilon}\times(0,T), \\ w(s,u,0) = 0,\ (s,u) \in \Omega_{R,\epsilon}, \end{cases}$$

$$\begin{cases} -i\partial_t v - \partial_s(c_\gamma \partial_s v) - \partial_u^2 v + V_\gamma v = \partial_s(g\partial_s(\partial_t \widetilde{z})) - h\partial_t \widetilde{z} \text{ in } \Omega_{R,\epsilon} \times (0,T), \\ v = 0 \ \text{ on } \partial\Omega_{R,\epsilon} \times (0,T), \\ v(s,u,0) = i(\partial_s(g(s,u)\partial_s z_0(s,u)) - hz_0(s,u)), \ (s,u) \in \Omega_{R,\epsilon}. \end{cases}$$

As in [1] or [5], we extend the function $v$ on $\Omega_{R,\epsilon}\times(-T,T)$ by the formula $v(s,u,t) = \overline{v}(s,u,-t)$ for every $(s,u,t) \in \Omega_{R,\epsilon} \times (-T,0)$. Note that this extension is available if the initial data is a real valued function. Note also that this extension satisfies the previous Carleman estimate. We set $\psi = e^{-S\eta} v$ with $\eta$ defined by (5). We recall that $M_1\psi = -i\partial_t\psi - \Delta_c\psi - S^2\lambda^2\phi^2\psi|\nabla_c\beta|^2$ with $c = c_\gamma$.

In a first step, we define $I := \text{Im} \int_{\Omega_{R,\epsilon}\times(-T,0)} M_1\psi\overline{\psi}$. Then by integrations by parts, we obtain: $I = (-1/2)\int_{\Omega_{R,\epsilon}} |\psi(s,u,0)|^2 ds\ du$. If we denote by $\eta_0(s,u) := \eta(s,u,0)$ and by $\phi_0(s,u) := \phi(s,u,0)$, recalling that $\psi = e^{-S\eta}v = e^{-S\eta}\partial_t w$, we get:

$$I = -\frac{1}{2}\int_{\Omega_{R,\epsilon}} e^{-2S\eta_0(s,u)}|\partial_t w(s,u,0)|^2\, ds\, du. \tag{26}$$

Moreover, we have:

$$|I| \le S^{-3/4}\lambda^{-1}\left(\int_Q |M_1\psi|^2\right)^{1/2} S^{3/4}\lambda\left(\int_Q |\psi|^2\right)^{1/2}$$
$$\le \frac{S^{-3/2}\lambda^{-2}}{2}\left(\int_Q |M_1(e^{-S\eta}v)|^2 + S^3\lambda^4\int_Q e^{-2S\eta}|v|^2\right).$$

Since $H_\gamma v = \partial_s(g\partial_s(\partial_t\widetilde{z})) - h\partial_t\widetilde{z}$, applying the Carleman inequality, we get:

$$|I| \le CS^{-3/2}\lambda^{-2}\int_Q e^{-2S\eta}|\partial_s(g\partial_s(\partial_t\widetilde{z})) - h\partial_t\widetilde{z}|^2 + CS^{-1/2}\lambda^{-1}\int_{\Gamma_{R,\epsilon}\times(-T,T)} \phi e^{-2S\eta}|\partial_\nu v|^2,$$

with $C$ a positive constant. Since $\partial_t\widetilde{z} \in L^\infty(\Omega_{R,\epsilon}\times(0,T))$, $\partial_s(\partial_t\widetilde{z}) \in L^\infty(\Omega_{R,\epsilon}\times(0,T))$, $\partial_s^2(\partial_t\widetilde{z}) \in L^\infty(\Omega_{R,\epsilon}\times(0,T))$ and $e^{-2S\eta(s,u,t)} \le e^{-2S\eta(s,u,0)}$ we have:

$$|I| \le CS^{-3/2}\lambda^{-2}\int_Q e^{-2S\eta_0}[|\partial_s g|^2 + |g|^2 + |h|^2] + CS^{-1/2}\lambda^{-1}\int_{\Gamma_{R,\epsilon}\times(-T,T)} \phi e^{-2S\eta}|\partial_\nu v|^2, \tag{27}$$

with $C$ a positive constant depending on $T$. Moreover, from $-i\partial_t w(s,u,0) = \partial_s(g\partial_s z_0) - hz_0 = \partial_s g\partial_s z_0 + g\partial_s^2 z_0 - hz_0$, applying the Lemma 2 for the function $g = c_\gamma - c_{\widetilde{\gamma}}$ and $P_0 g = \partial_s z_0\partial_s g = -i\partial_t w(s,u,0) - g\partial_s^2 z_0 + hz_0$, we obtain:

$$S^2\int_{\Omega_{R,\epsilon}} e^{-2S\eta_0}|g|^2 \le C\int_{\Omega_{R,\epsilon}} |-i\partial_t w(s,u,0) - g\partial_s^2 z_0 + hz_0|^2 e^{-2S\eta_0} + CS\int_{\Gamma_\epsilon} e^{-2S\eta_0}|g|^2|\partial_s\eta_0\nu_s|.$$

And so

$$S^2\int_{\Omega_{R,\epsilon}} e^{-2S\eta_0}|g|^2 \le C\int_{\Omega_{R,\epsilon}} [|\partial_t w(s,u,0)|^2 + |g|^2 + |h|^2]e^{-2S\eta_0} + CS\lambda\int_{\Gamma_\epsilon} e^{-2S\eta_0}|g|^2|\partial_s\beta\phi_0\nu_s|. \tag{28}$$

From (26)–(28) we get:

$$S^2 \int_{\Omega_{R,\epsilon}} e^{-2S\eta_0}|g|^2 \leq CS^{-3/2}\lambda^{-2} \int_{\Omega_{R,\epsilon}} [|\partial_s g|^2 + |g|^2 + |h|^2]e^{-2S\eta_0} + CS\lambda \int_{\Gamma_\epsilon} e^{-2S\eta_0}|g|^2|\partial_s\beta\phi_0\nu_s| + CS^{-1/2}\lambda^{-1} \int_{\Gamma_{R,\epsilon}\times(-T,T)} \phi e^{-2S\eta}|\partial_\nu v|^2 + C\int_{\Omega_{R,\epsilon}} [|g|^2 + |h|^2]e^{-2S\eta_0}. \quad (29)$$

Finally note that

$$0 < cst|\gamma - \widetilde{\gamma}| \leq |g| \leq cst|\gamma - \widetilde{\gamma}|,\ |\partial_s g| \leq cst[|\gamma - \widetilde{\gamma}| + |\gamma' - \widetilde{\gamma}'|], \quad |h| \leq cst[|\gamma - \widetilde{\gamma}| + |\gamma' - \widetilde{\gamma}'| + |\gamma'' - \widetilde{\gamma}''|]. \quad (30)$$

Combining (29) and (30) we can conclude for $S$ sufficiently large. □

*Remark* 1. Such result (25) can be generalized on the whole space $\Omega_1$ ($\int_{\Omega_1} e^{-2S\eta_0}|\gamma - \widetilde{\gamma}|^2 \leq C\int_{\Sigma_R^+\times(-T,T)} \phi e^{-2S\eta}|\partial_\nu(\partial_t(z - \widetilde{z}))|^2$) under the condition that there exists a function $\widetilde{\beta}$ which satisfies Assumption 2 on the whole $\Omega_1$.

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