Monografías Matemáticas *García de Galdeano* **37**, 13–23 (2012)

$\mathbf{Q}$-RESOLUTIONS AND INTERSECTION NUMBERS

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Abstract. In this paper we introduce the notion of embedded $\mathbf{Q}$-resolution, which is a special class of toric resolutions, and explain briefly how to compute it for plane curve singularities and obtain invariants from them. The main difference with standard resolutions is that we allow both the ambient space and the hypersurface to contain quotient singularities in some mild conditions. We develop an intersection theory on V-manifolds that allows us to calculate the intersection numbers of the exceptional divisors of the weighted blow-ups. An illustrative example is given at the end showing that the intersection matrix has the expected properties.

Keywords: Quotient singularity, intersection number, embedded $\mathbf{Q}$-resolution.

AMS classification: 32S25, 32S45.

Introduction

In Singularity Theory, resolution is the most important tool. In the embedded case, the starting point is a singular hypersurface; after a sequence of suitable blow-ups this hypersurface is replaced by a long list of smooth hypersurfaces (the strict transform and the exceptional divisors) which intersect in the simplest way (at any point we see coordinate hyperplanes for suitable local coordinates). This process can be very expensive from the computational point of view and, moreover, only a few amount of the obtained data is used for the understanding of the singularity. The experimental work shows that most of these data can be recovered if we allow some mild singularities to survive in the process (the quotient singularities). These *partial* resolutions, denoted as $\mathbf{Q}$-resolutions, can be obtained as a sequence of weighted blow-ups and their computational complexity is extremely lower compared with standard resolutions. Moreover, the process is optimal in the sense that we do not obtain useless data. To do this, we develop an intersection theory on varieties with quotient singularities and study weighted blow-ups at points. By using these tools we will be able to get a big amount of information about the singularity.

The paper is organized as follows. In §1 we give a general presentation of varieties with quotient singularities and list their basic properties; we introduce the main example, the weighted projective spaces. In §2 we describe the rational Weil and Cartier divisors on V-varieties and §3 introduces their intersection numbers. We discuss briefly in §4 the concepts of weighted blow-ups and embedded $\mathbf{Q}$-resolutions and their relationship with intersection theory. Finally an example on how to use $\mathbf{Q}$-resolutions to compute intersection numbers is given. Detailed proofs and further application will appear in a forthcoming work, see [1, 2, 6].

§1. V-manifolds and quotient singularities

Definition 1. A V-manifold of dimension n is a complex analytic space which admits an open covering $\{U_i\}$ such that U_i is analytically isomorphic to B_i/G_i where $B_i \subset \mathbb{C}^n$ is an open ball and G_i is a finite subgroup of $GL(n,\mathbb{C})$.

V-manifolds were introduced in [9] and have the same homological properties over $\mathbb{Q}$ as manifolds. For instance, they admit a Poincaré duality if they are compact and carry a pure Hodge structure if they are compact and Kähler, see [3]. They have been classified locally by Prill [8]. In this paper special attention is paid to V-manifolds where all groups G_i are abelian. In particular, the following notation is used.

Let $G := \mu_{d_1} \times \cdots \times \mu_{d_r}$ be an arbitrary finite abelian group written as a product of finite cyclic groups, that is, μ_{d_i} is the cyclic group of d_i-th roots of unity. Consider a matrix of weight vectors $A := (a_{ij})_{i,j} = [a_1 \,|\, \cdots \,|\, a_n] \in Mat(r \times n, \mathbb{Z})$ and the action

$$\begin{aligned} (\mu_{d_1} \times \cdots \times \mu_{d_r}) \times \mathbb{C}^n &\longrightarrow \mathbb{C}^n, \\ ((\xi_{d_1}, \ldots, \xi_{d_r}), (x_1, \ldots, x_n)) &\longmapsto (\xi_{d_1}^{a_{11}} \cdots \xi_{d_r}^{a_{r1}}\, x_1, \ldots, \xi_{d_1}^{a_{1n}} \cdots \xi_{d_r}^{a_{rn}}\, x_n). \end{aligned}$$

Note that the i-th row of the matrix A can be considered modulo d_i. The set of all orbits $\mathbb{C}^n/G$ is called *(cyclic) quotient space of type* $(d; A)$ and is denoted by

$$X(d;A) := X\left(\begin{array}{c|ccc} d_1 & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ d_r & a_{r1} & \cdots & a_{rn} \end{array}\right).$$

The following result shows that the family of varieties which can locally be written like $X(d,A)$ is exactly the same as the family of V-manifolds with abelian quotient singularities.

Lemma 1. *Let G be a finite abelian subgroup of $GL(n,\mathbb{C})$. Then $\mathbb{C}^n/G$ is isomorphic to some quotient space $X(d;A)$.*

We finish this section with one of the classical examples of V-manifold, cf. [4], the *weighted projective spaces*.

Let $\omega = (q_0, \ldots, q_n)$ be a weight vector, that is, a finite set of positive integers. There is a natural action of the multiplicative group $\mathbb{C}^*$ on $\mathbb{C}^{n+1} \setminus \{0\}$ given by

$$(x_0, \ldots, x_n) \longmapsto (t^{q_0} x_0, \ldots, t^{q_n} x_n).$$

The set of orbits $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$ under this action is denoted by $\mathbb{P}^n_\omega$ and is called the *weighted projective space* of type ω. The class of a nonzero element $(x_0, \ldots, x_n) \in \mathbb{C}^{n+1}$ is denoted by $[x_0 : \ldots : x_n]_\omega$ and the weight vector is omitted if no ambiguity seems likely to arise. When $(q_0, \ldots, q_n) = (1, \ldots, 1)$ one obtains the usual projective space and the weight vector is always omitted.

As in the classical case, the weighted projective spaces can be endowed with an analytic structure. However, in general they contain cyclic quotient singularities. Consider the decomposition $\mathbb{P}^n_\omega = U_0 \cup \cdots \cup U_n$, where U_i is the open set consisting of all elements $[x_0 : \ldots : x_n]_\omega$ with $x_i \neq 0$. The map

$$\widetilde{\psi}_0 : \mathbb{C}^n \longrightarrow U_0, \quad \widetilde{\psi}_0(x_1, \ldots, x_n) := [1 : x_1 : \ldots : x_n]_\omega$$

is clearly a surjective analytic map but it is not a chart since injectivity fails. In fact, $[1 : x_1 : \ldots : x_n]_\omega = [1 : x'_1 : \ldots, x'_n]_\omega$ if and only if there exists $\xi \in \mu_{q_0}$ such that $x'_i = \xi^{q_i} x_i$ for all $i = 1, \ldots, n$. Hence the map above induces the isomorphism

$$\begin{aligned} \psi_0 : X(q_0; q_1, \ldots, q_n) &\longrightarrow U_0, \\ [(x_1, \ldots, x_n)] &\longmapsto [1 : x_1 : \ldots : x_n]_\omega. \end{aligned}$$

Analogously, $X(q_i; q_0, \ldots, \widehat{q_i}, \ldots, q_n) \cong U_i$ under the obvious analytic map. Therefore $\mathbb{P}^n_\omega$ is an analytic space with cyclic quotient singularities as claimed.

§2. Cartier and Weil divisors on V-manifolds: $\mathbb{Q}$-divisors

Given X a complex analytic surface, the intersection product $D \cdot E$ is well understood whenever D is a compact Weil divisor on X and E is a Cartier divisor on X. Over varieties with quotient singularities the notion of Cartier and Weil divisor coincide after tensoring with $\mathbb{Q}$, see Theorem 2 below. A rational intersection theory can be defined on this kind of varieties.

Let us start with X an irreducible complex analytic variety. As usual, consider $\mathcal{O}_X$ the structure sheaf of X and $\mathcal{K}_X$ the sheaf of total quotient rings of $\mathcal{O}_X$. Denote by $\mathcal{K}^*_X$ the (multiplicative) sheaf of invertible elements in $\mathcal{K}_X$. Similarly $\mathcal{O}^*_X$ is the sheaf of invertible elements in $\mathcal{O}_X$.

Definition 2. A *Cartier divisor* on X is a global section of the sheaf $\mathcal{K}^*_X/\mathcal{O}^*_X$, that is, an element in $\Gamma(X, \mathcal{K}^*_X/\mathcal{O}^*_X) = H^0(X, \mathcal{K}^*_X/\mathcal{O}^*_X)$. Any Cartier divisor can be represented by giving an open covering $\{U_i\}_{i\in I}$ of X and, for all $i \in I$, an element $f_i \in \Gamma(U_i, \mathcal{K}^*_X)$ such that

$$\frac{f_i}{f_j} \in \Gamma(U_i \cap U_j, \mathcal{O}^*_X), \quad \forall i, j \in I.$$

Two systems $\{(U_i, f_i)\}_{i\in I}$ and $\{(V_j, g_j)\}_{j\in J}$ represent the same Cartier divisor if and only if on $U_i \cap V_j$, f_i and g_j differ by a multiplicative factor in $\mathcal{O}_X(U_i \cap V_j)^*$. The abelian group of Cartier divisors on X is denoted by $\mathrm{CaDiv}(X)$. If $D := \{(U_i, f_i)\}_{i\in I}$ and $E := \{(V_j, g_j)\}_{j\in J}$ then $D + E = \{(U_i \cap V_j, f_i g_j)\}_{i\in I, j\in J}$.

Definition 3. A *Weil divisor* on X is a locally finite linear combination with integral coefficients of irreducible subvarieties of codimension one. The abelian group of Weil divisors on X is denoted by $\mathrm{WeDiv}(X)$.

Let $V \subset X$ be an irreducible subvariety of codimension one. It corresponds to a prime ideal in the ring of sections of any local complex model space meeting V. The *local ring of X along V*, denoted by $\mathcal{O}_{X,V}$, is the localization of such ring of sections at the corresponding prime ideal; it is a one-dimensional local domain. For a given $f \in \mathcal{O}_{X,V}$ define $\mathrm{ord}_V(f)$ to be $\mathrm{ord}_V(f) := \mathrm{length}_{\mathcal{O}_{X,V}}(\mathcal{O}_{X,V}/\langle f\rangle)$, where $\mathrm{length}_{\mathcal{O}_{X,V}}$ denotes the length as an $\mathcal{O}_{X,V}$-module.

Now if D is a Cartier divisor on X, one writes $\mathrm{ord}_V(D) = \mathrm{ord}_V(f_i)$ where f_i is a local equation of D on any open set U_i with $U_i \cap V \neq \emptyset$. This is well defined since f_i is uniquely determined up to multiplication by units and the order function is a homomorphism. Define the *associated Weil divisor* of a Cartier divisor D by setting

$$T_X : \mathrm{CaDiv}(X) \longrightarrow \mathrm{WeDiv}(X) : \quad D \longmapsto \sum_{V\subset X} \mathrm{ord}_V(D) \cdot [V],$$

where the sum is taken over all codimension one irreducible subvarieties V of X. By the additivity of the order function, the mapping T_X is a homomorphism of abelian groups.

Example 1. Let X be the surface in $\mathbb{C}^3$ defined by the equation $z^2 = xy$. The line $V = \{x = z = 0\}$ defines a Weil divisor which is not a Cartier divisor. The associated Weil divisor of $\{(X, x)\}$ is $T_X(\{(X, x)\}) = \sum_{Z \subset X} \mathrm{ord}_Z(x) \cdot [Z] = 2[V]$. Thus $[V]$ is principal as an element in $\mathrm{WeDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and corresponds to the $\mathbb{Q}$-Cartier divisor $\frac{1}{2}\{(X, x)\}$.

This fact can be interpreted as follows. First note that identifying our surface X with $X(2; 1, 1)$ under $[(x, y)] \mapsto (x^2, y^2, xy)$, the previous Weil divisor corresponds to $D = \{x = 0\}$. Although $f = x$ defines a zero set on $X(2; 1, 1)$, it does not induce a function on the quotient space. However, $x^2 : X(2; 1, 1) \to \mathbb{C}$ is a well-defined function and gives the same zero set as f. Hence as $\mathbb{Q}$-Cartier divisors one writes $D = \frac{1}{2}\{(X(2; 1, 1), x^2)\}$.

The preceding example illustrates the general behaviour of Cartier and Weil divisors on V-manifolds as the following result shows.

Theorem 2. *Let X be a V-manifold. Then the notion of Cartier and Weil divisor coincide over $\mathbb{Q}$. More precisely, the linear map*

$$T_X \otimes 1 : \mathrm{CaDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathrm{WeDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism of $\mathbb{Q}$-vector spaces. In particular, for a given Weil divisor D on X there exists $k \in \mathbb{Z}$ such that $kD \in$ CaDiv(X).

Definition 4. Let X be a V-manifold. The vector space of $\mathbb{Q}$-Cartier divisors is identified under T_X with the vector space of $\mathbb{Q}$-Weil divisors. A *$\mathbb{Q}$-divisor* on X is an element in $\mathrm{CaDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{WeDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. The set of all $\mathbb{Q}$-divisors on X is denoted by $\mathbb{Q}$-Div(X).

The proof of the previous result is constructive. Let us summarize here how to write a Weil divisor as an element in $\mathrm{CaDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ where X is an algebraic V-manifold.

1. Write $D = \sum_{i \in I} a_i [V_i] \in \mathrm{WeDiv}(X)$, where $a_i \in \mathbb{Z}$ and $V_i \subset X$ irreducible. Also choose $\{U_j\}_{j \in J}$ an open covering of X such that $U_j = B_j / G_j$ where $B_j \subset \mathbb{C}^n$ is an open ball and G_j is a **small**[1] finite subgroup of $GL(n, \mathbb{C})$.
2. For each $(i, j) \in I \times J$ choose a polynomial function $f_{i,j} : U_j \to \mathbb{C}$ satisfying the condition $[(f_{i,j})_x \in \mathcal{O}_{B_j, x}$ **reduced** $\forall x \in B_j]$ and such that $V_i \cap U_j = \{f_{i,j} = 0\}$. Then,
$$[V_i|_{U_j}] = \frac{1}{|G_j|} \left\{ \left(U_j, f_{i,j}^{|G_j|} \right) \right\}.$$
3. Identifying $\{(U_j, f_{i,j}^{|G_j|})\}$ with its image under $\mathrm{CaDiv}(U_j) \hookrightarrow \mathrm{CaDiv}(X)$, one finally writes D as a sum of locally principal Cartier divisors over $\mathbb{Q}$,
$$D = \sum_{(i,j) \in I \times J} \frac{a_i}{|G_j|} \left\{ \left(U_j, f_{i,j}^{|G_j|} \right) \right\}.$$

[1]A finite subgroup G of $GL(n, \mathbb{C})$ is called *small* if no element of G has 1 as an eigenvalue of multiplicity precisely $n - 1$, that is, G does not contain rotations around hyperplanes other than the identity.

§3. Rational intersection number on V-surfaces

Now we are going to develop an intersection theory on varieties with quotient singularities, without getting into technical details.

Definition 5. Let C be an irreducible analytic curve. Given a Weil divisor on C with finite support, $D = \sum_{i=1}^{r} n_i \cdot [P_i]$, its *degree* is defined to be $\deg(D) = \sum_{i=1}^{r} n_i \in \mathbb{Z}$. The *degree of a Cartier divisor* is the degree of its associated Weil divisor.

Definition 6. Let X be an analytic surface and consider $D_1 \in \text{WeDiv}(X)$ and $D_2 \in \text{CaDiv}(X)$. If D_1 is irreducible then the *intersection number* is defined as $D_1 \cdot D_2 := \deg\left(j_{D_1}^{*} D_2\right) \in \mathbb{Z}$, where $j_{D_1} : D_1 \hookrightarrow X$ denotes the inclusion and $j_{D_1}^{*}$ its pull-back functor. The expression above extends linearly if D_1 is a finite sum of irreducible divisors. This number is only well defined if either $D_1 \not\subseteq D_2$ and $D_1 \cap D_2$ is finite, or the divisor D_1 is compact, cf. [5, Ch. 2].

In the case $D_1 \not\subseteq D_2$, the number $(D_1 \cdot D_2)_P := \text{ord}_P(j_{D_1}^{*} D_2)$ with $P \in D_1 \cap D_2$ is well defined too and it is called *local intersection number* at P.

Definition 7. Let X be a V-manifold of dimension 2 and consider $D_1, D_2 \in \mathbb{Q}\text{-Div}(X)$. The *intersection number* is defined as $D_1 \cdot D_2 := (k_1 k_2)^{-1} (k_1 D_1 \cdot k_2 D_2) \in \mathbb{Q}$, where $k_1, k_2 \in \mathbb{Z}$ are chosen so that $k_1 D_1 \in \text{WeDiv}(X)$ and $k_2 D_2 \in \text{CaDiv}(X)$. Analogously, it is defined the *local intersection number* at $P \in D_1 \cap D_2$, if the condition $D_1 \not\subseteq D_2$ is satisfied.

In the following result the main usual properties of the intersection product are collected. Their proofs are straightforward since they are well known for the classical case (i.e. without tensoring with $\mathbb{Q}$), cf. [5], and our generalization is based on extending the classical definition to rational coefficients.

Proposition 3. *Let X be a V-manifold of dimension* 2 *and* $D_1, D_2, D_3 \in \mathbb{Q} - \text{Div}(X)$. *Then the local and the global intersection numbers, provided the indicated operations make sense according to Definition 7, satisfy the following properties: ($\alpha \in \mathbb{Q}$, $P \in X$)*

1. *The intersection product is* ***bilinear*** *over* $\mathbb{Q}$.
2. ***Commutative:*** *If $D_1 \cdot D_2$ and $D_2 \cdot D_1$ are both defined, then $D_1 \cdot D_2 = D_2 \cdot D_1$. Analogously $(D_1 \cdot D_2)_P = (D_2 \cdot D_1)_P$ if both local numbers are defined.*
3. ***Non-negative:*** *Assume D_1 and D_2 are effective, irreducible and distinct. Then $D_1 \cdot D_2$ and $(D_1 \cdot D_2)_P$ are greater than or equal to zero. Moreover, $(D_1 \cdot D_2)_P = 0$ if and only if $P \notin |D_1| \cap |D_2|$, and hence $D_1 \cdot D_2 = 0$ if and only if $|D_1| \cap |D_2| = \emptyset$.*
4. ***Non-rational:*** *If $D_2 \in \text{CaDiv}(X)$, $D_1 \in \text{WeDiv}(X)$ then $D_1 \cdot D_2$ and $(D_1 \cdot D_2)_P$ are integral numbers. By the commutative property, the same holds if D_1 is a Cartier divisor and D_2 is a Weil divisor.*
5. ***$\mathbb{Q}$-Linear equivalence:*** *Assume D_1 has compact support. If D_2 and D_3 are $\mathbb{Q}$-linearly equivalent, i.e. $[D_2] = [D_3] \in \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, then $D_1 \cdot D_2 = D_1 \cdot D_3$. Due to the commutativity, the roles of D_1 and D_2 can be exchanged.*
6. ***Normalization:*** *Let $\nu : \widetilde{|D_1|} \to |D_1|$ be the normalization of the support of D_1 and $j_{D_1} : |D_1| \hookrightarrow X$ the inclusion. Then $D_1 \cdot D_2 = \deg\left(j_{D_1} \circ \nu\right)^{*} D_2$. Observe that in this situation the normalization is a smooth complex analytic curve.*

7. ***Pull-back:*** *Let Y be another irreducible V-surface and let $F : Y \to X$ be a proper morphism. Given $D_1, D_2 \in \mathbb{Q} - \mathrm{Div}(X)$, if the intersection product $D_1 \cdot D_2$ is defined, then so is $F^*(D_1) \cdot F^*(D_2)$ and one has $F^*(D_1) \cdot F^*(D_2) = \deg(F)(D \cdot E)$.*

Remark 1. This rational intersection product was first introduced by Mumford for normal surfaces, see [7, pag. 17]. Our Definition 7 coincides with Mumford's because it has good behavior with respect to the pull-back. The main advantage is that ours does not involve a resolution of the ambient space and, for instance, this allowed us to easily find formulas for the self-intersection numbers of the exceptional divisors of weighted blow-ups, without computing any resolution, see Proposition 4 below.

The rest of this section is devoted to reviewing some classical results concerning the intersection multiplicity.

Classical blow-up at a smooth point. Let X be a smooth analytic surface. Let $\pi : \widehat{X} \to X$ be the classical blow-up at a (smooth) point P. Consider C and D two Cartier or Weil divisors on X with multiplicities m_C and m_D at P. Denote by E the exceptional divisor of π, and by $\widehat{C}$ (resp. $\widehat{D}$) the strict transform of C (resp. D). Then there are following equalities:

1. $E \cdot \pi^*(C) = 0, \quad \pi^*(C) = \widehat{C} + m_C E, \quad E \cdot \widehat{C} = m_C.$
2. $E^2 = -1, \quad \widehat{C} \cdot \widehat{D} = C \cdot D - m_C m_D, \quad \widehat{D}^2 = D^2 - m_D^2$ (when D is compact).

Note that the exceptional divisor has multiplicity 1 at every point. This is why one only has to subtract 1 for the self-intersection number of the exceptional divisors every time we blow up a point on them, when computing an embedded resolution on a plane curve.

Bézout's Theorem on $\mathbb{P}^2$. Every analytic Cartier or Weil divisor on $\mathbb{P}^2$ is algebraic and thus can be written as a difference of two effective divisors. On the other hand, every effective divisor is defined by a homogeneous polynomial. The *degree of an effective divisor on* $\mathbb{P}^2$ is the degree, $\deg(F)$, of the corresponding homogeneous polynomial. This degree map is extended linearly yielding a group homomorphism $\deg : \mathrm{Div}(\mathbb{P}^2) \to \mathbb{Z}$.

Let D_1, D_2 be two divisors on $\mathbb{P}^2$, then $D_1 \cdot D_2 = \deg(D_1)\deg(D_2)$. In particular the self-intersection number of a divisor D on $\mathbb{P}^2$ is $D^2 = \deg(D)^2$.

The rest of this paper is devoted to generalizing the classical results above to V-manifolds of dimension 2, weighted blow-ups, and quotient weighted projective planes, respectively.

§4. Weighted blow-ups and embedded Q-resolutions

Classically an embedded resolution of $\{f = 0\} \subset \mathbb{C}^n$ is a proper map $\pi : X \to (\mathbb{C}^n, 0)$ from a smooth variety X satisfying, among other conditions, that $\pi^{-1}(\{f = 0\})$ is a normal crossing divisor. To weaken the condition on the preimage of the singularity we allow the new ambient space X to contain abelian quotient singularities and the divisor $\pi^{-1}(\{f = 0\})$ to have "normal crossings" over this kind of varieties. This notion of normal crossing divisor on V-manifolds was first introduced by Steenbrink in [10].

Definition 8. A hypersurface D on a V-manifold X with abelian quotient singularities is said to be with $\mathbb{Q}$-*normal crossings* if it is locally isomorphic to the quotient of a normal crossing divisor under a group action of type $(d;A)$. That is, given $x \in X$, there is an isomorphism of germs $(X,x) \simeq (X(d;A),[\mathbf{0}])$ such that $(D,x) \subset (X,x)$ is identified under this morphism with a germ of the form $\left(\{[\mathbf{x}] \in X(d;A) \mid x_1^{m_1} \cdots x_k^{m_k} = 0\},\ [(0,\ldots,0)]\right)$.

Definition 9. Let $M = \mathbb{C}^{n+1}/G$ be an abelian quotient space. Consider $H \subset M$ an analytic subvariety of codimension one. An embedded **Q**-resolution of $(H,0) \subset (M,0)$ is a proper analytic map $\pi : X \to (M,0)$ such that:

1. X is a V-manifold with abelian quotient singularities.
2. π is an isomorphism over $X \setminus \pi^{-1}(\mathrm{Sing}(H))$.
3. $\pi^{-1}(H)$ is a hypersurface with $\mathbb{Q}$-normal crossings on X.

Usually one uses weighted or toric blow-ups with smooth center as a tool for finding embedded **Q**-resolutions. Here we only discuss briefly the surface case. Let X be an analytic surface with abelian quotient singularities. Let us define the weighted blow-up $\pi : \widehat{X} \to X$ at a point $P \in X$ with respect to $\omega = (p,q)$. We distinguish two different situations.

(i) The point P is smooth. Assume $X = \mathbb{C}^2$ and $\pi = \pi_\omega : \widehat{\mathbb{C}}^2_\omega \to \mathbb{C}^2$ the weighted blow-up at the origin with respect to $\omega = (p,q)$,

$$\widehat{\mathbb{C}}^2_\omega := \left\{((x,y),[u:v]_\omega) \in \mathbb{C}^2 \times \mathbb{P}^1_\omega \mid (x,y) \in \overline{[u:v]_\omega}\right\}.$$

Here the condition about the closure means that $\exists t \in \mathbb{C}$, $x = t^p u$, $y = t^q v$. The new ambient space is covered as $\widehat{\mathbb{C}}^2_\omega = U_1 \cup U_2 = X(p;-1,q) \cup X(q;\ p,-1)$ and the charts are given by

$$\begin{array}{rl|rl} X(p;-1,q) \longrightarrow U_1, & & X(q;\ p,-1) \longrightarrow U_2, & \\ [(x,y)] \longmapsto ((x^p,x^q y),[1:y]_\omega); & & [(x,y)] \longmapsto ((xy^p,y^q),[x:1]_\omega). & \end{array}$$

The exceptional divisor $E = \pi_\omega^{-1}(0)$ is isomorphic to $\mathbb{P}^1_\omega$ which is in turn isomorphic to $\mathbb{P}^1$ under the map $[x:y]_\omega \mapsto [x^q : y^p]$. The singular points of $\widehat{\mathbb{C}}^2_\omega$ are cyclic quotient singularities located at the exceptional divisor. They actually coincide with the origins of the two charts.

(ii) The point P is of type $(d;a,b)$. Assume that $X = X(d;a,b)$. The group μ_d acts also on $\widehat{\mathbb{C}}^2_\omega$ and passes to the quotient yielding a map $\pi = \pi_{(d;a,b),\omega} : \widehat{X(d;a,b)}_\omega \to X(d;a,b)$, where by definition $\widehat{X(d;a,b)}_\omega := \widehat{\mathbb{C}}^2_\omega/\mu_d$. The new space is covered as

$$\widehat{X(d;a,b)}_\omega = \widehat{U}_1 \cup \widehat{U}_2 = X\left(\begin{matrix} p \\ pd \end{matrix}\,\middle|\,\begin{matrix} -1 & q \\ a & pb-qa \end{matrix}\right) \cup X\left(\begin{matrix} q \\ qd \end{matrix}\,\middle|\,\begin{matrix} p & -1 \\ qa-pb & b \end{matrix}\right)$$

and the charts are given by

$$\begin{array}{c|c} X\left(\begin{matrix} p \\ pd \end{matrix}\,\middle|\,\begin{matrix} -1 & q \\ a & pb-qa \end{matrix}\right) \longrightarrow \widehat{U}_1, & X\left(\begin{matrix} q \\ qd \end{matrix}\,\middle|\,\begin{matrix} p & -1 \\ qa-pb & b \end{matrix}\right) \longrightarrow \widehat{U}_2, \\ [(x,y)] \longmapsto [((x^p,x^q y),[1:y]_\omega)]_{(d;a,b)}; & [(x,y)] \longmapsto [((xy^p,y^q),[x:1]_\omega)]_{(d;a,b)}. \end{array}$$

The exceptional divisor $E = \pi^{-1}_{(d;a,b),\omega}(0)$ is identified with $\mathbb{P}^1_\omega(d;a,b) := \mathbb{P}^1_\omega/\mu_d$. Again the singular points are cyclic and correspond to the origins of the two charts.

Proposition 4. *Let X be a surface with abelian quotient singularities. Let $\pi : \widehat{X} \to X$ be the weighted blow-up at a point of type $(d; a, b)$ with respect to $\omega = (p, q)$. Assume $(d, a) = (d, b) = (p, q) = 1$ and write $e = \gcd(d, pb - qa)$.*

Consider two $\mathbb{Q}$-divisors C and D on X and, as usual, denote by E the exceptional divisor of π, and by $\widehat{C}$ (resp. $\widehat{D}$) the strict transform of C (resp. D). Let ν and μ the (p, q)-multiplicities of C and D at P, i.e. x (resp. y) has (p, q)-multiplicity p (resp. q). Then there are the following equalities:

1. $E \cdot \pi^*(C) = 0, \quad \pi^*(C) = \widehat{C} + \frac{\nu}{e} E, \quad E \cdot \widehat{C} = \frac{e\nu}{pqd}.$
2. $E^2 = -\frac{e^2}{pqd}, \quad \widehat{C} \cdot \widehat{D} = C \cdot D - \frac{\nu\mu}{pqd}, \quad \widehat{D}^2 = D^2 - \frac{\mu^2}{pqd}$ *(when D is compact).*

§5. Bézout's Theorem for Quotient Weighted Projective Planes

For a given weight vector $\omega = (p, q, r) \in \mathbb{N}^3$ and an action on $\mathbb{C}^3$ of type $(d; a, b, c)$, consider the quotient weighted projective plane $\mathbb{P}^2_\omega(d; a, b, c) := \mathbb{P}^2_\omega/\mu_d$ and the corresponding morphism $\tau_{(d;a,b,c),\omega} : \mathbb{P}^2 \to \mathbb{P}^2_\omega(d; a, b, c)$ defined by $\tau_{(d;a,b,c),\omega}([x : y : z]) = [x^p : y^q : z^r]_\omega$.

The space $\mathbb{P}^2_\omega(d; a, b, c)$ is a V-manifold with abelian quotient singularities; its charts are obtained as in Section 1. The *degree of a $\mathbb{Q}$-divisor on $\mathbb{P}^2_\omega(d; a, b, c)$* is the degree of its pull-back under the map $\tau_{(d;a,b,c),\omega}$, that is, by definition,

$$D \in \mathbb{Q}\text{-Div}\Big(\mathbb{P}^2_\omega(d; a, b, c)\Big), \quad \deg_\omega(D) := \deg\Big(\tau^*_{(d;a,b,c),\omega}(D)\Big).$$

Thus if $D = \{F = 0\}$ is a $\mathbb{Q}$-divisor on $\mathbb{P}^2_\omega(d; a, b, c)$ given by a ω-homogeneous polynomial that indeed defines a zero set on the quotient projective space, then $\deg_\omega(D)$ is the classical degree, denoted by $\deg_\omega(F)$, of a quasi-homogeneous polynomial.

Proposition 5. *Let us denote by m_1, m_2, m_3 the determinants of the three minors of order* 2 *of the matrix $\left(\begin{smallmatrix} p & q & r \\ a & b & c \end{smallmatrix}\right)$. Assume $\gcd(p, q, r) = 1$ and write $e = \gcd(d, m_1, m_2, m_3)$. Then the intersection number of two $\mathbb{Q}$-divisors, D_1 and D_2, on $\mathbb{P}^2_\omega(d; a, b, c)$ is*

$$D_1 \cdot D_2 = \frac{e}{dpqr} \deg_\omega(D_1) \deg_\omega(D_2) \in \mathbb{Q}.$$

Corollary 6. *Let X, Y, Z be the Weil divisors on $\mathbb{P}^2_\omega(d; a, b, c)$ given by $\{x = 0\}$, $\{y = 0\}$ and $\{z = 0\}$, respectively. Using the notation above one has:*

$$X^2 = \frac{ep}{dqr}, \quad Y^2 = \frac{eq}{dpr}, \quad Z = \frac{er}{dpq}, \quad X \cdot Y = \frac{e}{dr}, \quad X \cdot Z = \frac{e}{dq}, \quad Y \cdot Z = \frac{e}{dp}.$$

Remark 2. If $d = 1$, then e equals one too and the formulas become a bit simpler.

§6. Example of an Embedded Q-Resolution

Let us consider the following divisors on $\mathbb{C}^2$: $C_1 = \{((x^3 - y^2)^2 - x^4 y^3) = 0\}$, $C_2 = \{x^3 - y^2 = 0\}$, $C_3 = \{x^3 + y^2 = 0\}$, $C_4 = \{x = 0\}$ and $C_5 = \{y = 0\}$. We shall see that the local intersection

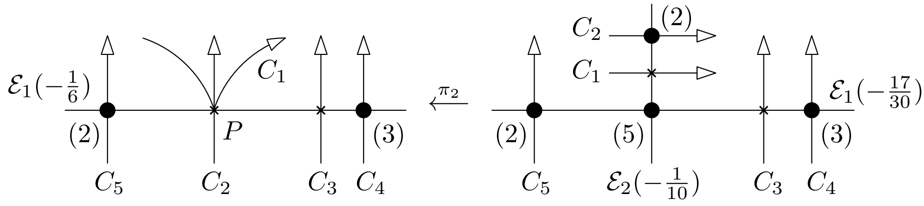

Figure 1: Embedded **Q**-resolution of $C = \bigcup_{i=1}^{5} C_i \subset \mathbb{C}^2$.

numbers $(C_i \cdot C_j)_0$, $i, j \in \{1, \ldots, 5\}$, $i \neq j$, are encoded in the intersection matrix associated with any embedded **Q**-resolution of $C = \bigcup_{i=1}^{5} C_i$.

Let $\pi_1 : \mathbb{C}^2_{(2,3)} \to \mathbb{C}^2$ be the $(2,3)$-weighted blow-up at the origin. The new space has two cyclic quotient singular points of type $(2; 1, 1)$ and $(3; 1, 1)$ located at the exceptional divisor $\mathcal{E}_1$. The local equation of the total transform in the first chart is given by the function

$$x^{29}\, ((1 - y^2)^2 - x^5 y^3)\, (1 - y^2)\, (1 + y^2)\, y : X(2; 1, 1) \longrightarrow \mathbb{C},$$

where $x = 0$ is the equation of the exceptional divisor and the other factors correspond in the same order to the strict transform of C_1, C_2, C_3, C_5 (denoted again by the same symbol). To study the strict transform of C_4 one needs the second chart, the details are left to the reader.

Hence $\mathcal{E}_1$ has multiplicity 29 and self-intersection number $-1/6$; it intersects transversally C_3, C_4 and C_5 at three different points, while it intersects C_1 and C_2 at the same smooth point P, different from the other three. The local equation of the divisor $\mathcal{E}_1 \cup C_2 \cup C_1$ at this point P is $x^{29}\, y\, (x^5 - y^2) = 0$, see Figure 1 below.

Let π_2 be the $(2,5)$-weighted blow-up at the point P above. The new ambient space has two singular points of type $(2; 1, 1)$ and $(5; 1, 2)$. The local equations of the total transform of $\mathcal{E}_1 \cup C_2 \cup C_1$ are given by the following two functions.

1st chart	2nd chart
$\underbrace{x^{73}}_{\mathcal{E}_2} \cdot \underbrace{y}_{C_2} \cdot \underbrace{(1 - y^2)}_{C_1} : X(2; 1, 1) \longrightarrow \mathbb{C}$	$\underbrace{x^{29}}_{\mathcal{E}_1} \cdot \underbrace{y^{73}}_{\mathcal{E}_2} \cdot \underbrace{(x^5 - 1)}_{C_1} : X(2; 1, 1) \longrightarrow \mathbb{C}$

Thus the new exceptional divisor $\mathcal{E}_2$ has multiplicity 73 and intersects transversally the strict transform of C_1, C_2 and $\mathcal{E}_1$. Hence the composition $\pi_2 \circ \pi_1$ is an embedded **Q**-resolution of $C = \bigcup_{i=1}^{5} C_i \subset \mathbb{C}^2$. As for the self-intersection numbers, $\mathcal{E}_2^2 = -1/10$ and $\mathcal{E}_1^2 = -1/6 - 2^2/(1 \cdot 2 \cdot 5) = -17/30$. The following figure illustrates the whole process. The intersection matrix associated with the embedded **Q**-resolution obtained is $A = \begin{pmatrix} -17/30 & 1/5 \\ 1/5 & -1/10 \end{pmatrix}$ and $B = -A^{-1} = \begin{pmatrix} 6 & 12 \\ 12 & 34 \end{pmatrix}$.

Now one observes the intersection number is encoded in B as follows. For $i = 1, \ldots, 5$, set $k_i \in \{1, \ldots, 5\}$ such that $\emptyset \neq C_i \cap \mathcal{E}_{k_i} =: \{P_i\}$. Denote by $O(C_i)$ the order of the cyclic group acting on P_i. Then,

$$\left(C_i \cdot C_j\right)_0 = \frac{b_{k_i,k_j}}{O(C_i)\, O(C_j)}.$$

Looking at the figure one sees that $(k_1, \ldots, k_5) = (2, 2, 1, 1, 1)$ and $(O(C_1), \ldots, O(C_5)) = (1, 2, 1, 3, 2)$. Hence, for instance,

$$(C_1 \cdot C_2)_0 = \frac{b_{k_1,k_2}}{O(C_1)\, O(C_2)} = \frac{b_{22}}{1 \cdot 2} = \frac{34}{2} = 17,$$

which is indeed the intersection multiplicity at the origin of C_1 and C_2. Analogously for the other indices.

Remark 3. Consider the group action of type $(5; 2, 3)$ on $\mathbb{C}^2$. The previous plane curve C is invariant under this action and then it makes sense to compute an embedded **Q**-resolution of $\overline{C} := C/\mu_5 \subset X(5; 2, 3)$. Similar calculations as in the previous example, lead to a figure as the one obtained above with the following relevant differences:

- $\mathcal{E}_1 \cap \mathcal{E}_2$ is a smooth point.
- $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) has self-intersection number $-17/6$ (resp. $-1/2$).
- The intersection matrix is $A' = \begin{pmatrix} -17/6 & 1 \\ 1 & -1/2 \end{pmatrix}$ and $B' = -(A')^{-1} = \begin{pmatrix} 6/5 & 12/5 \\ 12/5 & 34/5 \end{pmatrix}$.

Hence, for instance, $(\overline{C}_1 \cdot \overline{C}_2)_0 = b'_{22}/(1 \cdot 2) = (34/5)/2 = 17/5$, which is exactly the intersection number of the two curves, since that local number can also be computed as $(\overline{C}_1 \cdot \overline{C}_2)_0 = 5^{-1}(C_1 \cdot C_2)_0$.

Conclusion. The combinatorial and computational complexity of embedded **Q**-resolutions is much simpler than the one of the classical embedded resolutions, but they keep as much information as needed for the comprehension of the topology of the singularity. This will become clear in the second author's Ph.D. thesis. We will prove in a forthcoming paper another advantages of these embedded **Q**-resolutions, e.g. in the computation of abstract resolutions of surfaces via Jung method, see [1, 2, 6].

Acknowledgements

We wish to express our gratitude to J. I. Cogolludo for his constant support and motivation in our work over the years. All authors are partially supported by the Spanish projects MTM2010-2010-21740-C02-02 and "E15 Grupo Consolidado GeometrÌa" from the Government of Aragón. Also, the second author is partially supported by FQM-333.

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