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On a stochastic nonlinear conservation law

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Abstract. In this paper, we are interested in the stochastic viscous Buckley-Leverett equation with a Hölder continuous nonlinear function.

Keywords: Viscous Buckley-Leverett, stochastic perturbation, well-posedness.

AMS classification: 35K60, 35K65, 60H15.

§1. Introduction

In our presentation "On stochastic nonlinear conservation laws", at the Ninth International Conference Zaragoza-Pau on Applied Mathematics and Statistics, we have presented results of existence and uniqueness for the solution to parabolic and hyperbolic problems. These results were extracted from the publications G. Vallet [10] and G. Vallet and P. Wittbold [11]. In this paper, we would like to revisit the example of the formal stochastic viscous Buckley-Leverett equation

$$du - \epsilon\Delta u\, dt - \operatorname{div}(f(u)\vec{B})\, dt = h dw \quad \text{ in } D \times]0, T[\times \Omega,$$

where f is assumed to be a Hölder continuous function.

In the sequel, one assumes that D is a bounded Lipschitz domain of $\mathbb{R}^d$, that T is a positive number and one denotes by $Q =]0, T[\times D$. Then, homogeneous Dirichlet would be considered.

Thereafter, $W = \{w_t, \mathcal{F}_t\, ;\, 0 \leq t \leq T\}$ denotes a standard adapted one-dimensional continuous Brownian motion, defined on some probability space $(\Omega, \mathcal{F}, P)$, with the property that $w_0 = 0$. This assumption on W is made for convenience. Our aim is to adapt known methods for nonlinear PDE to noise perturbed ones.

Usually, the Buckley-Leverett equation is a transport equation used to model two-phase flow in porous media ($\epsilon = 0$). It can be obtained as the limit, when ϵ goes to 0, of the above viscous equation. Such a result can be found in G. Vallet and P. Wittbold [11] for a regular function f, but one needs the notion of entropy solution. Note that the model corresponds to a generalization to $d > 1$ of the Burger's equation too: *i.e.* $d = 1$ and $f(x) = x^2$.

The Burger's equation has been intensively studied in the literature with many extensions. Usually, the stochastic convolution is used. Let us mention, without exhaustiveness, G. Da Prato *et al.* [2, 3], W. Grecksch *et al.* [4] or I. Gyongy *et al.* [5] and M. Röckner *et al.* in [9] for a generalization of the classical.

Usually, Lipschitz or local-Lipschitz conditions are assumed on the function f. In this application we consider that f is a $1/2$-Hölder-continuous function (with $f(0) = 0$ since $\operatorname{div} \vec{B} = 0$). The method consists in using a Lipschitz-approximation of f and passing to the limits with respect to this approximation.

§2. Assumptions, definition of a solution and the main result

Let us assume in the sequel that
- $\vec{B} \in (L^\infty(D))^d$ with div $\vec{B} = 0$ a.e. in D,
- $f : \mathbb{R} \to \mathbb{R}$ is a 1/2 Hölder-continuous function with $f(0) = 0$,
- $h \in L^2(Q)$ and $u_0 \in L^2(D)$.

Denote by $V = H_0^1(D)$, endowed with $\|u\|_V = \left(\int_D |\nabla u|^2\, dx\right)^{1/2}$ the norm of Poincaré (*cf.* R. Adams [1, Th. 6.28, p.159]), by C_p the Poincaré's constant, *i.e.*, for all $v \in V$, $\|v\|_{L^2(D)} \leq C_p\|v\|_V$.

Our aim is then to give a result of existence and uniqueness of the variational solution to the above-mentioned problem. Let us fix in what sense such a solution is understood.

Definition 1. Any function u of $L^2(\Omega \times]0, T[\,; V)$ such that $\frac{\partial}{\partial t}\left[u - \int_0^t h(s, .)\, dw(s)\right]$, taken in the sense of the vectorial V'-valued distributions, belongs to $L^2(\Omega \times]0, T[\,; V')$ is a solution to our stochastic problem if u is $L^2(D)$-valued progressively measurable and if for t a.e. in $]0, T[$ and any test-function v of V, the variational formulation holds

$$0 = \left\langle \frac{\partial}{\partial t}\left[u - \int_0^t h(s, .,)\, dw(s)\right], v\right\rangle_{V',V} + \int_D \{\epsilon \nabla u.\nabla v + f(u)\vec{B}.\nabla v\}\, dx,$$

with the initial condition $u(0, .) = u_0$.

The results we want to prove is:

Theorem 1. *A unique solution in the sense of the above definition exists to the above stochastic Buckley-Leverett equation.*

§3. Proof of the result

For any positive M, consider $f_M = (f * \rho_M) \circ T_M$ where ρ_M denotes the usual mollifier sequence of support $1/M$ and $T_M(x) = \max[\min(x, M), -M]$. Then, f_M is a bounded, Lipschitz-continuous function and classical results yield the existence and uniqueness of the solution, denote by u_M, to the problem:

$$du_M - \epsilon\Delta u_M\, dt - \mathrm{div}(f_M(u_M)\vec{B})dt = hdw \quad \text{in } D \times]0, T[\times \Omega$$

for the same initial condition and the regularity required in the previous definition.

Such a result would be admitted; refer *e.g.* to G. Da Prato *et al.* [3], W. Grecksch *et al.* [4] or G. Vallet [10].

Thanks to the stochastic-energy equality, one has that a positive constant C exists such that, for any t,

$$E\int_D u_M^2(t)\, dx + 2E\int_0^t\int_D |\nabla u_M|^2\, dx\, ds + 2E\int_0^t\int_D f_M(u_M)\vec{B}.\nabla u_M\, dx\, ds = \int_0^t\int_D h^2\, dx\, ds.$$

Thus, one deduces that

$$E\int_D u_M^2(t)\, dx + 2E\int_Q |\nabla u_M|^2\, dx\, ds \leq C(h). \tag{1}$$

Moreover, for any v in $V \setminus \{0\}$,

$$\frac{\left|\left\langle \frac{\partial}{\partial t}\left[u_M - \int_0^t h\, dw(s)\right], v\right\rangle_{V',V}\right|}{\|v\|_V} \le \|\nabla u_M\|_{L^2(D)} + \|\vec{B}\|_\infty c_P \|f_M(u_M)\|_{L^2(D)}.$$

Since

$$\begin{aligned}|f_M(u_M)|^2 &= \left|\int_{\mathbb{R}} f(T_M(u_M) - y)\rho_M(y)\, dy\right|^2 \le \int_{\mathbb{R}} |f(T_M(u_M) - y)|^2 \rho_M(y)\, dy \\ &\le c(f)\int_{\mathbb{R}} |T_M(u_M) - y|\rho_M(y)\, dy \le c(f)(|T_M(u_M)| + 1) \le c_1 u_M^2 + c_2,\end{aligned}$$

one deduces that

$$\frac{\left|\left\langle \frac{\partial}{\partial t}\left[u_M - \int_0^t h\, dw(s)\right], v\right\rangle_{V',V}\right|^2}{\|v\|_V^2} \le 2\|\nabla u_M\|^2_{L^2(D)} + 2\|\vec{B}\|^2_\infty c_P^2 \left[c_1 \|u_M\|^2_{L^2(D)} + c_2 \,\mathrm{meas}(D)\right]$$

and that

$$E\int_0^T \left\|\frac{\partial}{\partial t}\left[u_M - \int_0^t h\, dw(s)\right]\right\|^2_{V'} dt \le C(h). \tag{2}$$

Thus, one is able to assert the

Lemma 2. *Uniformly with respect to M and for any $t \in [0, T]$, the sequences $u_M(t)$, u_M and $\frac{\partial}{\partial t}\left[u_M - \int_0^t h\, dw(s)\right]$ are bounded respectively in $L^2(\Omega \times D)$, $L^2(\Omega \times]0, T[\,, V)$ and $L^2(\Omega \times]0, T[\,, V')$.*

Following J. U. Kim [6], denote, for any t, by

$$\Theta(u_M, t) = \sup_{s\in[0,t]} \|u_M(s)\|^2_{L^2(D)} + \|u_M\|^2_{L^2(0,t;V)} + \left\|\frac{\partial}{\partial t}\left[u_M - \int_0^{\cdot} h\, dw(s)\right]\right\|^2_{L^2(0,t,V')},$$
$$\widetilde{\Omega}(t) = \bigcup_{L\ge 2}\bigcup_{M\ge 1}\bigcup_{k\ge m}\{\Theta(u_M, t) \le L\} \qquad \text{and} \qquad \widetilde{\Omega} = \widetilde{\Omega}(T).$$

Thanks to the above lemma, one deduces that $P(\widetilde{\Omega}) = 1$. Then, for P-a.s. ω, a positive constant $L(\omega)$ and a sub-sequence denoted by u_{M_ω} exist such that $\{\Theta(u_{M_\omega}, T) \le L(\omega)\}$. Therefore, there exist $u = u(\omega)$ in $L^2(0, T; V)$ with moreover $\frac{\partial}{\partial t}[u - \int_0^t h\, dw(s)]$ in $L^2(0, T, V')$ and a sub-sequence denoted by (u_k) such that u_k converges weakly to u in $L^2(0, T; V)$ and that $\frac{\partial}{\partial t}[u_k - \int_0^t h\, dw(s)]$ converges weakly to $\frac{\partial}{\partial t}[u - \int_0^t h\, dw(s)]$ in $L^2(0, T, V')$.

Moreover, thanks to Corollary 4, (u_k) converges in $L^2(0, T; L^2(D))$ and a.e. in Q since sub-sequences are considered, and in $C([0, T]; H^{-1}(D))$. In particular, $u_0 = u_k(0)$ converges to $u(0)$ in V'.

Since $f_k^2(u_k) \le c_1 u_k^2 + c_2$, it can be assumed, up to a sub-sequence denoted in the same way, that $f_k(u_k)$ converges weakly to some f_u in $L^2(Q)$. Note that, by construction, $f_k(u_k)$ converges a.e. in Q to $f(u)$. Then, it converges weakly to $f(u)$ in $L^2(Q)$ (Cf. J.-L. Lions [7, lemma 1.3, p.12]).

It follows that, for any v in V and t a.e. in $]0,T[$,

$$\left\langle \frac{\partial}{\partial t}[u - \int_0^t h\,dw], v\right\rangle_{V',V} + \int_D \nabla u.\nabla v + f(u)\vec{B}.\nabla v\,dx = 0.$$

If one denotes by $\hat{u}$ an other solution, for any v in V, one gets that

$$\left\langle \frac{\partial}{\partial t}[u - \hat{u}], v\right\rangle_{V',V} + \int_D \nabla[u-\hat{u}].\nabla v + [f(u) - f(\hat{u})]\vec{B}.\nabla v\,dx = 0.$$

For a given $\mu > 0$, set $v = p_\mu[u - \hat{u}]$ where $p_\mu(x) = 0$ if $x < \mu/e$, 1 if $x > \mu$ and $\ln(ex/\mu)$ else. Note that p_μ is a Lipschitz-continuous function and denote by $P_\mu = \int_0^x p_\mu(s)\,ds$. Then,

$$0 = \frac{d}{dt}\int_D P_\mu\,[u-\hat{u}]\,dx + \int_D p'_\mu\,[u-\hat{u}]\,|\nabla[u-\hat{u}]|^2 + [f(u) - f(\hat{u})]p'_\mu[u-\hat{u}]\vec{B}.\nabla[u-\hat{u}]\,dx.$$

And by construction,

$$\frac{d}{dt}\int_D P_\mu\,[u-\hat{u}]\,dx + \frac{1}{2}\int_D p'_\mu[u-\hat{u}]|\nabla[u-\hat{u}]|^2 \leq C\int_{\{\mu/e<u-\hat{u}<\mu\}} |u-\hat{u}|p'_\mu\,[u-\hat{u}]\,dx.$$

Thus,

$$\int_D P_\mu[u-\hat{u}]\,dx \leq C\,\mathrm{meas}\,(\{\mu/e < u-\hat{u} < \mu\}) + \int_D P_\mu[0]\,dx.$$

Passing to the limits leads to $u \leq \hat{u}$.

Since one is able to prove in the same way that $u \geq \hat{u}$, the solution to the above problem is unique and all the sequence (u_{M_ω}) converges.

Now, one needs to prove that u, generated by sub-sequences depending on ω, is adapted to the filtration and belongs to the stated spaces. In order to prove this, we propose to follow J. U. Kim's [6] arguments. Consider a closed ball B in $H^{-1}(D)$ and, for any positive integer n, $B_n = \bigcup_{v\in B}\bar{B}_{H^{-1}(D)}(v, 1/n)$. For a fixed t^*, note that

$$\widetilde{\Omega}\cap\{u(t^*)\in B\} = \widetilde{\Omega}\cap\Big[\bigcup_{L>0}\bigcap_{n>0}\bigcap_{k>0}\bigcup_{M\geq k}\{u_M(t^*)\in B_n\}\cap\{\Theta(u_M,t^*)\leq L\}\Big]. \tag{3}$$

Indeed, for any $\omega \in \widetilde{\Omega}\cap\{u(t^*)\in B\}$, (u_{M_ω}) satisfies $\Theta(u_{M_\omega}, t^*) \leq \Theta(u_{M_\omega}, T) \leq L(\omega)$. Moreover, since u_{M_ω} converges in $C([0,T], H^{-1}(D))$, ω belongs to the right hand side set.

Conversely, if ω belongs to the right hand side set, there exists $\bar{L}(\omega) > 0$ such that for any positive integer n, one is able to construct a sub-sequence $u_{\bar{M}_{\omega,n}}$ with $u_{\bar{M}_{\omega,n}}(t^*) \in B_n$ and $\Theta(u_{\bar{M}_{\omega,n}}, t^*) \leq \bar{L}(\omega)$.

Since, what has been done with u_{M_ω} in $]0,T[$ can be done again with $u_{\bar{M}_{\omega,n}}$ in $]0,t^*[$, the uniqueness result proved above yields the convergence of $u_{\bar{M}_{\omega,n}}$ to u. Therefore, $u(t^*) \in B_n$ for any n, and the result holds.

Thanks to the regularity of u_M, the left hand side of (3) is $\mathcal{F}_{t^*}$-measurable and $\{u(t^*)\in B\}$ is in $\mathcal{F}_{t^*}$. More generally, for any t in $[0,T]$, $\{u(t^*)\in F\}\in\mathcal{F}_t$ for any Borel subset F of $H^{-1}(D)$. Since u belongs to $C([0,T], H^{-1}(D))$, $\{(t,\omega),\ 0\leq t\leq t^*,\ u(t,\omega)\in F\}\in([0,T])\times\mathcal{F}_{t^*}$ for each $F\in\mathcal{B}(H^{-1}(D))$ and any $t\in\,]0,T]$.

Since u belongs to $C_s([0,T], L^2(D))$, $u(t) \in L^2(D)$ for any t and thanks to lemmata 5 and 7 in the annexes, the same result of measurability holds for any $F \in \mathcal{B}(L^2(D))$; and u is progressively measurable as a $L^2(D)$ valued process.

Note that a similar argument could be used in $L^2(D)$ with the weak topology since u belongs to $C_s([0,T], L^2(D))$ with values in a fixed bounded subset of $L^2(D)$ and thanks to lemma 6 in the annexes.

Then, thanks to (1), (2) and the lemma of Fatou, on gets that

$$E\int_D u^2(t)\,dx + 2E\int_Q |\nabla u|^2\,dxds + E\int_0^T \left\| \frac{\partial}{\partial t}\left[u - \int_0^t h\,dw(s)\right]\right\|_{V'}^2 dt \le C(h),$$

and a solution exists in the sense of the definition 1.

For the uniqueness of the solution, one has just to use the same method than the one given above, based on the approximation of the sgn^+ function by p_μ.

§4. Annexes

In this section we propose some classical tools used in this paper.

First, let us remind the theorem on Aubin-Simon:

Theorem 3 ([7, Th. 5.1, Th. 12.1 and (12.10)])**.** *Let us consider* $1 < p \le +\infty$*,* $1 \le q \le +\infty$*,* B_0*,* B_1 *and* B_2 *three B-spaces such that the embedding of* B_0 *in* B_1 *is compact and the embedding of* B_1 *in* B_2 *is continuous. If* (u_n) *is a bounded sequence in* $L^q(0,T;B_0)$ *such that* (du_n/dt) *(the derivation is taken in the sense of vectorial distributions) is a bounded sequence in* $L^p(0,T;B_2)$*, then there exists a subsequence that converges in* $L^q(0,T;B_1)$ *and in* $C([0,T];B_2)$*.*

The following corollary is the main tool of compactness used in the paper:

Corollary 4. *Let* (u_n) *be a bounded sequence in* $L^2(0,T;H^1_0(D)) \cap L^\infty(0;T;L^2(D))$ *and* $H \in C([0,T];L^2(D))$*. If* $(d(u_n - H)/dt)$ *(the derivation is taken in the sense of vectorial distributions) is a bounded sequence in* $L^2(0,T;H^{-1}(D))$ *then there exists a subsequence* (u_{n_k}) *that converges in* $L^2(0,T;L^2(D))$ *and in* $C([0,T];H^{-1}(D))$*.*

Moreover, the limit is $C_s([0,T];L^2(D))$[1]*.*

Proof. Since the embedding of $L^2(D)$ in $H^{-1}(D)$ is compact and since $(u_n - H)$ is bounded in $L^\infty(0;T;L^2(D))$, thanks to Aubin-Simon's theorem, there exists a subsequence $(u_{n_k} - H)$ that converges in $C([0,T];H^{-1}(D))$. In particular, (u_{n_k}) converges in $C([0,T];H^{-1}(D))$ too.

Thanks to the lemma of Lions ([7, Lemma 5.1]), for any positive ϵ, there exists a positive d_ϵ such that, for any n, p,

$$\|u_{n+p} - u_n\|_{L^2(0,T;L^2(D))} \le \epsilon\|u_{n+p} - u_n\|_{L^2(0,T;H^1_0(D))} + d_\epsilon\|u_{n+p} - u_n\|_{L^2(0,T;H^{-1}(D))}.$$

Thus, since $\|u_n\|_{L^2(0,T;H^1_0(D))}$ is bounded, one gets that for any positive ϵ, there exists a positive d_ϵ such that, for any n, p,

$$\|u_{n+p} - u_n\|_{L^2(0,T;L^2(D))} \le \frac{\epsilon}{2} + d_\epsilon\|u_{n+p} - u_n\|_{L^2(0,T;H^{-1}(D))}.$$

[1] $u \in C_s([0,T];X)$ if, for any $x^* \in X'$, $t \mapsto \langle x^*, u(t)\rangle$ is continuous.

Since (u_{n_k}) is a Cauchy sequence in $L^2(0,T;H^{-1}(D))$, a positive integer N exists such that $d_\epsilon\|u_{n_{k+p}} - u_{n_k}\|_{L^2(0,T;H^{-1}(D))} \leq \epsilon/2$ as soon as $n_k \geq N$. Then, (u_{n_k}) is a Cauchy sequence in $L^2(0,T;L^2(D))$ and it converges.

Obviously, the limit belongs to $L^\infty(0;T;L^2(D)) \cap C([0,T];H^{-1}(D))$, thus it belongs to $C_s([0,T];L^2(D))$ thanks to [8, Lemma 8.1, p.297]. □

Let us give now some lemmata concerning the measurability of vector-valued functions.

Lemma 5. *Assume that u is a function with values in $L^2(D)$ and $H^{-1}(D)$-measurable, then it is $L^2(D)$-measurable.*

Proof. Our argument is based on the theorem of Pettis in separable B-spaces [12].

If u is $H^{-1}(D)$-measurable, then it is weakly measurable. Thus, for any v in $H_0^1(D)$, $\langle u,v\rangle_{H^{-1},H_0^1}$ is a scalar measurable function. As u is a function with values in $L^2(D)$, $\langle u,v\rangle_{H^{-1},H_0^1} = \int_D uv\,dx$ and it is a scalar measurable function. Note that for any $v \in L^2(D)$, there exists $(v_n) \subset H_0^1(D)$ that converges toward v in $L^2(D)$. Thus, $\int_D uv_n\,dx$ converges a.e. toward $\int_D uv\,dx$ and it is a scalar measurable function. Therefore, u is weakly $L^2(D)$-measurable, thus $L^2(D)$-measurable. □

I would like to present an generalisation proposed by L. Thibault (personal communication) and based on the two following lemmata:

Lemma 6. *Let Y be a separable B-space. Then, the Borel sigma-algebra $\mathcal{B}(Y)$ when Y is endowed with the strong topology is the same than the Borel sigma-algebra $\mathcal{B}_w(Y)$ when Y is endowed with the weak topology. Moreover, $\mathcal{B}(Y)$ is the sigma-algebra generated by the closed balls of Y.*

Proof. First $\mathcal{B}_w(Y) \subset \mathcal{B}(Y)$ is obvious since the same inclusion holds for the topologies.

On the other hand, any closed ball $\bar{B}(a,r)$ in Y is $\sigma(Y,Y^*)$-closed since it is convex. In particular, $\bar{B}(a,r) \in \mathcal{B}_w(Y)$.

As any open ball is a countable reunion of closed ones, any open ball belongs to $\mathcal{B}_w(Y)$. Now, thanks to the separability of Y, any open subset of Y is a countable reunion of open balls. Then, any open subset of Y is an element of $\mathcal{B}_w(Y)$ and $\mathcal{B}(Y) \subset \mathcal{B}_w(Y)$. Note that this prove that $\mathcal{B}(Y)$ is generated by the closed balls too. □

Lemma 7. *Assume that $X \subset Y$ are separable B-spaces with X reflexive. If the embedding i of X in Y is continuous, then $\mathcal{B}(X) \subset \mathcal{B}(Y)$, where $\mathcal{B}(X)$ (resp. Y) denotes the Borel σ-algebra of X (resp. Y).*

Proof. Consider A a closed ball in X. Since X is assumed to be reflexive, A is $\sigma(X,X^*)$ compact. Moreover, the application i is $\sigma(X,X^*)$-$\sigma(Y,Y^*)$ continuous and then A is a compact set of Y for the topology $\sigma(Y,Y^*)$. Therefore, A is weakly closed in Y, it is closed and it belongs to $B(Y)$. The conclusion comes from the remark that $B(X)$ is generated by the closed balls of X. □

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