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EXISTENCE OF A SOLUTION TO A CLASS OF PSEUDOPARABOLIC PROBLEMS

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Abstract. In this paper we are interested, on the one hand, in problems involving a nonlinearity of form $f(\partial_t u)$; on the other hand, we are interested in Barenblatt's type equations [5] too.

By the way of an implicit time-discretization, we would prove the existence of a solution to the following problem: $f(\partial u_t) - \Delta\phi(u) - \epsilon\Delta\partial u_t = g$ with a Lipschitz-continuous function ϕ.

Keywords: Pseudoparabolic problems, existence results, time-discretization.

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§1. Introduction

In this paper, we are interested in the mathematical analysis of the pseudoparabolic Cauchy problem:

$$f(\partial_t u) - \Delta\phi(u) - \epsilon\Delta\partial_t u = g, \quad u(0,.) = u_0, \tag{1}$$

where f and ϕ are Lipschitz-continuous functions with f non-decreasing.

This study has its roots in the analysis of problems with a nonlinearity of form $f(\partial_t u)$. Such a term has been previously introduced by G. I. Barenblatt in [5] for elasto-plastic porous media. It has been revisited by S. N. Antontsev *et al.* [1, 2, 3, 4] or G. Vallet [8] concerning a constrained stratigraphic models in geology.

An implicit time-discretization scheme is used to prove the existence of a solution in a suitable functional space. As an application, by passing to the limits with respect to ϵ, one proves the existence of a solution to the Barenblatt's equation.

Let us consider in the sequel a bounded domain $\Omega \subset \mathbb{R}^d$ with a Lipschitz-boundary Γ. For any $T > 0$, let us denote Q a cylinder defined by $Q :=]0,T[\times\Omega$.
Moreover, one assumes that:

$$f \text{ is a non-decreasing Lipschitz-continuous function,} \tag{H_1}$$

$$\phi \text{ is a } C^1(\mathbb{R})\text{-Lipschitz-continuous function such that } \phi(0) = 0, \tag{H_2}$$

$$\epsilon > 0 \text{ and } u_0 \in H^1_0(\Omega), \tag{H_3}$$

$$g \in L^2(Q). \tag{H_4}$$

We shall write $M = \|\phi'\|_\infty$.

Let us define now what is a solution to our pseudoparabolic problem.

Definition 1. A solution to (1) is any $u \in H^1(0, T, H_0^1(\Omega))$ such that $u(0, \cdot) = u_0$ and, for all v in $H_0^1(\Omega)$,

$$\int_\Omega \left\{ f(\partial_t u)\, v + \phi'(u)\, \nabla u \nabla v + \epsilon \nabla \partial_t u \nabla v \right\} dx = \int_\Omega g v\, dx. \tag{2}$$

The main result of this paper is that

Theorem 1. *There exists a solution to Problem* (1).

§2. Existence of a solution

2.1. Semi-discretized processes

Consider a positive integer N and denote by $h = T/N$. In this section, we are interested in proving the existence of the sequence of approximation by the way of an implicit semi-discretization scheme.

Each step of the scheme consist in solving a nonlinear elliptic problem. In a first par, the case of a bounded f would be consider. Then, thanks to some truncation arguments, the general case would be obtained.

Proposition 2. *Under the hypothesis* (H_1) *to* (H_3) *and by assuming moreover that* f *is a bounded function, if* h *is small enough* ($h < \epsilon/(M+1)$)*, for any* $g \in L^2(\Omega)$*, there exists an element* u *in* $H_0^1(\Omega)$ *such that, for all* v *in* $H_0^1(\Omega)$*,*

$$\int_\Omega f\left(\frac{u - u_0}{h}\right) v\, dx + \int_\Omega \phi'(u)\, \nabla u \nabla v, dx + \epsilon \int_\Omega \nabla \frac{u - u_0}{h} \nabla v\, dx = \int_\Omega g v\, dx. \tag{3}$$

This element is unique as soon as ϕ' *is a Lipschitz-continuous function.*

Proof. The existence of a solution of (2) is classically obtained by using the Schauder-Tikhonov fixed point theorem in the framework of separable reflexive B-spaces. In order to do it, let us denoted Ψ the mapping defined by $\Psi : H_0^1(\Omega) \to H_0^1(\Omega)$, $S \mapsto u_S$, where u_S is the unique solution of the following linear problem: find $u_S \in H_0^1(\Omega)$ such that, for all $v \in H_0^1(\Omega)$,

$$\int_\Omega \left(\phi'(S) + \frac{\epsilon}{h}\right) \nabla u_S \nabla v\, dx = \int_\Omega g v\, dx - \int_\Omega f\left(\frac{S - u_0}{h}\right) v\, dx + \frac{\epsilon}{h} \int_\Omega \nabla u_0 \nabla v\, dx. \tag{4}$$

As soon as $h < \epsilon/(M+1)$, this linear problem is coercive in $H_0^1(\Omega)$. It is well-posed and Ψ exists. Choosing $v = u_S$ a test function, one gets that

$$\left\| u_{S_n} \right\|_{H_0^1(\Omega)} \leq C_1 = C\left(\Omega, \|f\|_\infty, g, \epsilon, u_0, h\right), \tag{5}$$

and Ψ conserve the closed ball $\bar{B}_{H_0^1(\Omega)}(0, C_1)$.

Let (S_n) be a sequence that converges weakly in $H_0^1(\Omega)$ towards S. Up to a subsequence still denoted in the same way, it can be assumed that S_n converges strongly in $L^2(\Omega)$ and *a.e.*

in Ω. Furthermore, the functions ϕ' and f are continuous and bounded, then owing to the theorem of Lebesgue, we can prove that, for all v in $H_0^1(\Omega)$,

$$\int_\Omega f\left(\frac{S_n - u_0}{h}\right) v\,dx \to \int_\Omega f\left(\frac{S - u_0}{h}\right) v\,dx \quad \text{and} \quad \phi'(S_n)\nabla v \to \phi'(S)\nabla v \quad \left(L^2(\Omega)\right)^d, \quad (6)$$

Moreover, according to (5), the sequence (u_{S_n}) is bounded in $H_0^1(\Omega)$. Thus, χ in $H_0^1(\Omega)$ exists, as well as a subsequence, still indexed by n, extracted from (u_{S_n}), such that, u_{S_n} converges weakly in $H_0^1(\Omega)$ toward χ. Then, we have that

$$\nabla u_{S_n} \rightharpoonup \nabla\chi \quad \text{in } \left(L^2(\Omega)\right)^d \quad \text{and} \quad \nabla\frac{u_{S_n} - u_0}{h} \rightharpoonup \nabla\frac{\chi - u_0}{h} \quad \text{in } \left(L^2(\Omega)\right)^d. \quad (7)$$

Passing to the limits in (4) with S_n by using (6) and (7), we obtain that χ is a solution to problem (4) with S. By uniqueness of such a solution, one gets that $\chi = u_S$.

Thus by a compactness argument, all the sequences converge weakly in $H_0^1(\Omega)$ toward u_S, *i.e.* $u_{S_n} \rightharpoonup u_S$ weakly in $H_0^1(\Omega)$. Then the mapping Ψ is sequentially weakly weakly continuous in $H_0^1(\Omega)$. Thus the fixed point theorem of Schauder-Tikhonov proves that Ψ has at most a fixed point; *i.e.* there exists S in $H_0^1(\Omega)$ such that $u_S = S$ and a solution to (3) exists.

Let us prove now that this solution is unique. Let us consider $\widehat{u}$ another solution of (3). Thus we obtain by subtraction, for all v in $H_0^1(\Omega)$,

$$\begin{aligned} 0 = &\int_\Omega \left[f\left(\frac{u-u_0}{h}\right) - f\left(\frac{\widehat{u}-u_0}{h}\right)\right] v\,dx + \int_\Omega \left(\phi'(u) + \frac{\epsilon}{h}\right)\nabla(u-\widehat{u})\nabla v\,dx \\ &+ \int_\Omega (\phi'(u) - \phi'(\widehat{u}))\nabla\widehat{u}\nabla v\,dx. \end{aligned} \quad (8)$$

For a giving $\mu > 0$, let us denote by $p_\mu(r) = (r-\mu)^+/r$; p_μ is non-decreasing Lipschitz function with $p'_\mu(r) = \frac{\mu}{r^2}\mathbf{1}_{\{r>\mu\}}$.

Therefore, as $v = p_\mu(u - \widehat{u})$ is a suitable test function, its comes that

$$\begin{aligned} 0 = &\int_\Omega \left[f\left(\frac{u-u_0}{h}\right) - f\left(\frac{\widehat{u}-u_0}{h}\right)\right] p_\mu(u-\widehat{u})\,dx + \mu\int_{\{u-\widehat{u}>\mu\}} \left(\phi'(u) + \frac{\epsilon}{h}\right)\frac{\left|\nabla(u-\widehat{u})\right|^2}{|u-\widehat{u}|^2}dx \\ &+ \mu\int_{\{u-\widehat{u}>\mu\}} \frac{\phi'(u) - \phi'(\widehat{u})}{|u-\widehat{u}|^2}\nabla\widehat{u}.\nabla(u-\widehat{u})\,dx. \end{aligned}$$

Since f is a non-decreasing function and as $h \leq \epsilon/(M+1)$, it comes that

$$\begin{aligned} \int_{\{u-\widehat{u}>\mu\}} \frac{\left|\nabla(u-\widehat{u})\right|^2}{|u-\widehat{u}|^2}dx &\leq \int_{\{u-\widehat{u}>\mu\}} \frac{\left|\phi'(u) - \phi'(\widehat{u})\right|^2}{2|u-\widehat{u}|^2}|\nabla\widehat{u}|^2 dx + \int_{\{u-\widehat{u}>\mu\}} \frac{|\nabla(u-\widehat{u}|^2)}{2|u-\widehat{u}|^2}dx \\ &\leq \int_{\{u-\widehat{u}>\mu\}} \frac{\left|\phi'(u) - \phi'(\widehat{u})\right|^2}{|u-\widehat{u}|^2}|\nabla\widehat{u}|^2 dx \leq \|\phi''\|_\infty \int_\Omega |\nabla\widehat{u}|^2 dx. \end{aligned}$$

Let us denote by $F_\mu(r) = \ln(1 + (r-\mu)^+/\mu)$. F_μ is a Lipchitz-continuous function, $F_\mu(u-\widehat{u}) \in H_0^1(\Omega)$ and one gets that

$$\int_\Omega \left|\nabla F_\mu(u-\widehat{u})\right|^2 dx \leq \left\|\phi''\right\|_\infty \int_\Omega |\nabla\widehat{u}|^2\,dx.$$

Thanks to Poincaré inequality, the sequence $\left(F_\mu(u-\widehat{u})\right)_\mu$ is bounded in $L^2(\Omega)$ independently of μ. Note that the sequence $(F_{1/n}(u-\widehat{u}))_n$ is non-decreasing, and converges almost everywhere in $\mathbb{R} \cup \{+\infty\}$ to $+\infty\, \mathbf{1}_{\{u-\widehat{u}>0\}}$. Hence, the theorem of Beppo Levi leads to $\text{meas}(\{u > \widehat{u}\}) = 0$. Then $(u-\widehat{u})^+ = 0$, *i.e* $u \le \widehat{u}$.

Permutating u and $\widehat{u}$ thereinbefore gives $\widehat{u} \le u$ as well and the solution is unique. □

Proposition 3. *Under the hypothesis* (H_1) *to* (H_3)*, if* h *is small enough* ($h < \epsilon/(M+1)$)*, for any* $g \in L^2(\Omega)$*, there exists an element* u *in* $H_0^1(\Omega)$ *such that, for all* v *in* $H_0^1(\Omega)$*,*

$$\int_\Omega f\left(\frac{u-u_0}{h}\right) v\, dx + \int_\Omega \nabla\phi(u)\, \nabla v\, dx + \epsilon \int_\Omega \nabla \frac{u-u_0}{h} \nabla v\, dx = \int_\Omega gv\, dx. \tag{9}$$

This element is unique as soon as ϕ' *is a Lipschitz-continuous function.*

Proof. The proof of the uniqueness result of the solution is identical to the one proposed previously.

Concerning the result of existence, consider for any positive n, $f_n = \max(-n, \min(n, f))$. The corresponding solutions, given by the above proposition, are denoted by u_n. Applying the test function $v = (u_n - u_0)/h$ to (3), one gets that

$$\begin{aligned}
&\int_\Omega \left[f_n\left(\frac{u_n-u_0}{h}\right) - f_n(0)\right] \frac{u_n-u_0}{h}\, dx + \int_\Omega [h\phi'(u_n) + \epsilon] \left|\nabla \frac{u_n-u_0}{h}\right|^2 dx \\
&\quad \le \int_\Omega [g - f_n(0)] \frac{u_n-u_0}{h}\, dx - \int_\Omega \phi'(u_n)\, \nabla u_0 \nabla \frac{u_n-u_0}{h}\, dx \\
&\quad \le \left[\|g - f_n(0)\|_{L^2(\Omega)} + M\, \|u_0\|_{H_0^1(\Omega)}\right] \cdot \left\|\frac{u_n-u_0}{h}\right\|_{H_0^1(\Omega)}.
\end{aligned}$$

Since f is non-decreasing, f_n too, $h < \epsilon/(M+1)$ and thanks to Poincaré's inequality, one gets that

$$\left\|\frac{u_n-u_0}{h}\right\|_{H_0^1(\Omega)} \le \|g\|_{L^2(\Omega)} + |f(0)| \sqrt{\text{meas}(\Omega)} + M\, \|u_0\|_{H_0^1(\Omega)}. \tag{10}$$

Therefore, a sub-sequence still indexed by n can be extracted, such that u_n converges in $H_0^1(\Omega)$ weakly to u, strongly in $L^2(\Omega)$ and *a.e.* in Ω. Moreover, one has that

$$\left\|f_n(\frac{u_n-u_0}{h})\right\|_{H_0^1(\Omega)} \le \left\|f'\right\|_\infty \left[\|g\|_{L^2(\Omega)} + |f(0)| \sqrt{\text{meas}(\Omega)} + M\, \|u_0\|_{H_0^1(\Omega)}\right]. \tag{11}$$

Since $f_n(\frac{u_n-u_0}{h})$ converges a.e. to $f(\frac{u-u_0}{h})$, it ensures that $f(u_n)$ converges in $L^2(\Omega)$ toward $f(u)$ (and weakly in $H^1(\Omega)$). Furthermore, since ϕ is a Lipschitz-continous function, $\phi(u_n)$ converges weakly to $\phi(u)$ in $L^2(\Omega)$, and, passing to the limits in the variational formulation stating u_n, one gets (9). □

Inductively, the following result can be proved:

Theorem 4. *Let us consider $N \in \mathbb{N}^*$ with $N > T(M+1)/\epsilon$, $h = T/N$ and $(g^k) \subset L^2(\Omega)$. Then, under the hypothesis (H_1)–(H_3), there exists a sequence $(u^k)_k$ in $H_0^1(\Omega)$ with $u^0 = u_0$ and such that, for all $v \in H_0^1(\Omega)$,*

$$\int_\Omega f\left(\frac{u^{k+1}-u^k}{h}\right) v\,dx + \int_\Omega \nabla\phi\left(u^{k+1}\right)\nabla v\,dx + \epsilon\int_\Omega \nabla\frac{u^{k+1}-u^k}{h}\nabla v\,dx = \int_\Omega g^{k+1} v\,dx. \quad (12)$$

This sequence is unique as soon as ϕ' is a Lipschitz-continuous function.

2.2. Existence of a solution

In order to prove the existence of a solution, let us introduce some notations. For any sequence v^k, let us denote in the sequel

$$v^h = \sum_{k=0}^{N-1} v^{k+1}\mathbf{1}_{[t_k,t_{k+1}[} \quad \text{and} \quad \widetilde{v}^h = \sum_{k=0}^{N-1}\left[\frac{v^{k+1}-v^k}{h}(t-t_k)+v^k\right]\mathbf{1}_{[t_k,t_{k+1}[},$$

where $t_k = kh$ and

$$g^h = \sum_{k=0}^{N-1}\frac{1}{h}\int_{kh}^{(k+1)h} g(t,\cdot)dt\,\mathbf{1}_{[t_k,t_{k+1}[}.$$

Lemma 5. *Assume that $h < \epsilon/(M+1)$. Then,*

(i) The sequence (u^h) is bounded in $L^\infty(0,T;H_0^1(\Omega))$ and $(\widetilde{u}^h)$ is bounded in $H^1(0,T;H_0^1(\Omega))\cap L^\infty(0,T;H_0^1(\Omega))$.

(ii) There exists $C > 0$ such that for all t in $[0,T[$, $\left\|\widetilde{u}^h(t)-u^h(t)\right\|_{H_0^1(\Omega)} \le C\sqrt{h}$.

(iii) There exists a set Z of full measure in $]0,T[$ such that, for any t in Z, $\partial_t\widetilde{u}^h(t)$ is bounded in $H_0^1(\Omega)$.

Proof. Thanks to (10), one has that

$$\left\|\frac{u^{k+1}-u^k}{h}\right\|_{H_0^1(\Omega)} \le \left\|g^{k+1}\right\|_{L^2(\Omega)} + |f(0)|\sqrt{\text{meas}(\Omega)} + M\left\|u^k\right\|_{H_0^1(\Omega)}, \quad (13)$$

and, if $k > 0$,

$$\left\|\frac{u^{k+1}-u^k}{h}\right\|_{H_0^1(\Omega)} \le \left\|g^{k+1}\right\|_{L^2(\Omega)} + C + M\|u_0\|_{H_0^1(\Omega)} + Mh\sum_{i=0}^{k-1}\left\|\frac{u^{i+1}-u^i}{h}\right\|_{H_0^1(\Omega)}. \quad (14)$$

Then, one gets that

$$\begin{aligned}\sum_{k=0}^{n} h\left\|\frac{u^{k+1}-u^k}{h}\right\|_{H_0^1(\Omega)}^2 &\le 4\sum_{k=0}^{n} h\left\|g^{k+1}\right\|_{L^2(\Omega)}^2 + C(u_0)T + 4M^2h^2\sum_{k=1}^{n} h\left[\sum_{i=0}^{k-1}\left\|\frac{u^{i+1}-u^i}{h}\right\|_{H_0^1(\Omega)}\right]^2 \\ &\le C(g,u_0) + 4M^2Th\sum_{k=1}^{n}\sum_{i=0}^{k-1} h\left\|\frac{u^{i+1}-u^i}{h}\right\|_{H_0^1(\Omega)}^2 \le C(g,u_0)e^{4M^2T},\end{aligned}$$

thanks to the discrete Gronwall lemma. This yields

$$\sum_{k=0}^{N-1} \left\| u^{k+1} - u^k \right\|^2_{H^1_0(\Omega)} \leq hC(g, u_0) e^{4M^2 T}, \tag{15}$$

and (i)–(ii) hold.

Moreover, (14) yields, for any $t \in]t_k, t_{k+1}[$, to

$$\left\| \partial_t \widetilde{u}^h(t) \right\|^2_{H^1_0(\Omega)} \leq 4 \left\| g^h(t) \right\|^2_{L^2(\Omega)} + C(u_0) + 4M^2 C(g, u_0) e^{4M^2 T}. \tag{16}$$

If moreover t belongs to the set of Lebesgue of g in $L^2(0, T; L^2(\Omega))$, $\partial_t \widetilde{u}^h(t)$ is bounded in $H^1_0(\Omega)$ and (iii) holds. □

Theorem 6. *Under the hypotheses* (H_1)–(H_4)*, there exists* u *in* $H^1\left(0, T; H^1_0(\Omega)\right)$ *such that, for all* v *in* $H^1_0(\Omega)$*,*

$$\int_\Omega f(\partial_t u)\, v\, dx + \int_\Omega \nabla\phi(u)\, \nabla v\, dx \epsilon + \int_\Omega \nabla \partial_t u \nabla v dx = \int_\Omega g v\, dx, \tag{17}$$

with $u(0, \cdot) = u_0$*.*

Proof. Leading from Lemma 5-(i), there exists u in $H^1(0, T; H^1_0(\Omega))$, such that, up to a sub-sequences still denoted in the same way, one may assume that $\widetilde{u}^h$ converges to u weakly in $H^1(0, T; H^1_0(\Omega))$. Then, for any t in $[0, T]$, $\widetilde{u}^h(t)$ converges weakly in $H^1_0(\Omega)$ toward $u(t)$. Then, Lemma 5-(ii) ensures that $u^h(t)$ converges weakly to $u(t)$ in $H^1_0(\Omega)$. Moreover, since ϕ is a Lipschitz-countinuous function, $\phi(u^h(t))$ converges weakly to $\phi(u(t))$ in $H^1_0(\Omega)$ too.

Thanks to Lemma 5-(iii), for any t in Z, up to a sub-sequence indexed by h_t, $\partial_t \widetilde{u}^{h_t}(t)$ converges weakly in $H^1_0(\Omega)$ towards a given $\xi(t)$ and strongly in $L^2(\Omega)$.

Then, there exists k such that (12) leads, for any $v \in H^1_0(\Omega)$, to

$$\int_\Omega f\left(\partial_t \widetilde{u}^{h_t}(t)\right) v\, dx + \int_\Omega \nabla\phi\left(u^{h_t}(t)\right) \nabla v\, dx + \epsilon \int_\Omega \nabla \partial_t \widetilde{u}^{h_t}(t) \nabla v\, dx = \int_\Omega g^{h_t}(t) v\, dx. \tag{18}$$

By passing to the limits in the above equation, on gets that $\xi(t)$ is a solution in in $H^1_0(\Omega)$ to the variational problem:

$$\forall v \in H^1_0(\Omega),\ \int_\Omega f(\xi(t))\, v\, dx + \epsilon \int_\Omega \nabla\xi(t) \nabla v\, dx = \int_\Omega g v dx - \int_\Omega \phi'(u(t))\, \nabla u(t) \nabla v\, dx. \tag{19}$$

Then, since f is non-decreasing, this implies that such a solution is unique. As $\partial_t \widetilde{u}^h(t)$ is a bounded sequence in $H^1_0(\Omega)$, one concludes that $\partial_t \widetilde{u}^h(t)$ converges toward $\xi(t)$ weakly in $H^1_0(\Omega)$.

Therefore, $\xi :]0, T[\to H^1_0(\Omega)$ is a weakly measurable function. Then, thanks to the theorem of Pettis ([9, p. 131]), it is a measurable function.

For any v in $H^1_0(\Omega)$, $\int_\Omega \nabla\partial_t u^h(t)\nabla v\,dx$ converges *a.e.* in $]0,T[$ toward $\int_\Omega \nabla\xi(t)\nabla v\,dx$. Since $\left|\int_\Omega \nabla\partial_t\widetilde{u}^h(t)\nabla v\,dx\right| \leq \left\|\partial_t\widetilde{u}^h(t)\right\|_{H^1_0(\Omega)} \|v\|_{H^1_0(\Omega)}$, it is bounded in $L^2(0,T)$ and [7, Lemma 1.3, p.12] ensures that

$$\forall\alpha \in L^2(0,T),\ \int_0^T\int_\Omega \alpha(t)\nabla\partial_t\widetilde{u}^h(t).\nabla v\,dx\,dt \to \int_0^T\int_\Omega \alpha(t)\nabla\xi(t).\nabla v\,dx\,dt.$$

Since $(\partial_t\widetilde{u}^h)$ is bounded in $L^2(0,T;H^1_0(\Omega))$, an argument of density leads to the weak convergence in $L^2(0,T;H^1_0(\Omega))$ of $\partial_t\widetilde{u}^h$ toward ξ. Thus by uniqueness of the weak limit, one obtains that $\partial_t u = \xi$ and that there exists a solution. □

§3. Application to Barenblatt's equation

As an application, let us return to the existence of a solution to Barenblatt's equation:

$$f(\partial_t u) - \Delta u = g,$$

where $f(r) = r$ if $r > 0$ and $f(r) = \alpha r$ $(\alpha > 0)$ if $r \leq 0$, with $\alpha \neq 1$ *a priori*.

Our method consists in passing to the limits in the pseudoparabolic problem (2) with respect to ϵ toward 0, when $\phi = Id$, g in $L^2(Q)$ and u_0 in $H^1_0(\Omega)$.

By using the test function $v = \partial_t u_\epsilon$ in (2), we obtain, for any t, the following estimate:

$$\int_{\Omega\times]0,t[} f(\partial_t u_\epsilon)\partial_t u_\epsilon + \epsilon|\nabla\partial_t u_\epsilon|^2\,dx + \frac{1}{2}\int_\Omega |\nabla u_\epsilon(t)|^2\,dx = \int_{\Omega\times]0,t[} g\partial_t u_\epsilon\,dx + \frac{1}{2}\int_\Omega |\nabla u_0|^2\,dx. \quad (20)$$

Thus, the sequence (u_ϵ) is bounded in $H^1(Q)\cap L^\infty(0,T;H^1_0(\Omega))$ as well as $(f(\partial_t u_\epsilon))$ in $L^2(Q)$. Indeed, for all t,

$$\min(1,\alpha)\int_{]0,t[\times\Omega} |\partial_t u_\epsilon|^2\,dx\,dt + \frac{1}{2}\int_\Omega |\nabla u_\epsilon(t)|^2\,dx \leq \frac{1}{2}\int_\Omega |\nabla u_0|^2\,dx + \int_{]0,t[\times\Omega} g\partial_t u_\epsilon\,dx\,dt.$$

Up to a sub-sequence still indexed by ϵ, one assumes that there exists u in $H^1(Q) \cap L^\infty(0,T;H^1_0(\Omega))$, weak limit in $H^1(Q)$ and weak-* limit in $L^\infty(0,T;H^1_0(\Omega))$ of (u_ϵ); as well as χ, weak limit in $L^2(Q)$ of $f(\partial_t u_\epsilon)$.

On the one hand, one has $\chi - \Delta u = g$, *i.e.* $\partial_t u - \Delta u = g + \partial_t u - \chi := h$. Since $h \in L^2(Q)$ with the initial condition in $H^1_0(\Omega)$, one gets

$$\int_Q |\partial_t u|^2\,dx\,dt + \frac{1}{2}\int_\Omega |\nabla u(T)|^2\,dx = \frac{1}{2}\int_\Omega |\nabla u_0|^2\,dx + \int_Q [g + \partial_t u - \chi]\partial_t u\,dx\,dt. \quad (21)$$

On the other hand, since $(u_\epsilon(T))$ bounded in $H^1_0(\Omega)$ and as $u_\epsilon(T)$ converges toward $u(T)$ in $L^2(\Omega)$, it converges weakly in $H^1_0(\Omega)$ and passing to the limits in (20) yields

$$\limsup_{\epsilon\to 0}\int_Q f(\partial_t u_\epsilon)\partial_t u_\epsilon\,dx\,dt + \frac{1}{2}\int_\Omega |\nabla u(T)|^2\,dx \leq \frac{1}{2}\int_\Omega |\nabla u_0|^2\,dx + \int_Q g\partial_t u\,dx\,dt.$$

Thus, $\limsup \epsilon \to 0 \int_Q f(\partial_t u_\epsilon)\partial_t u_\epsilon\,dx\,dt \leq \int_Q \chi\partial_t u\,dx\,dt$. Then, according to H. Brézis [6, Prop. 2.5, p. 27], $\chi = f(\partial_t u)$ and u is a solution to the problem.

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