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SOLVING ONE-DIMENSIONAL LINEAR BOUNDARY VALUE PROBLEMS BY MULTI-POINT TAYLOR POLYNOMIALS. APPLICATIONS TO SPECIAL FUNCTIONS

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Abstract. We consider second order linear differential equations of the form $\varphi(x)y'' + f(x)y' + g(x)y = h(x)$ in a real finite interval I with mixed Dirichlet and Neumann boundary data and a representation of its solution $y(x)$ by a multi-point Taylor expansion. The number and location of the base points of that expansion are conveniently chosen to guarantee that the expansion is uniformly convergent $\forall x \in I$. We propose several algorithms to approximate the multi-point Taylor polynomials of the solution based on the power series method for initial value problems. We show that multi-point Taylor polynomials are adequate to approximate the solution when the singularities of the coefficient functions of the differential equation are close to the interval I. We apply this technique to the approximation of several special functions.

Keywords: Second order linear differential equations, boundary value problem, multi-point Taylor expansions, special functions.

AMS classification: 34A25, 34B05, 41A58.

§1. Introduction

Let us consider boundary value problems of the form

$$\begin{cases} \varphi(x)y'' + f(x)y' + g(x)y = h(x) & \text{in } (-1,1), \\ MY = N, \end{cases} \tag{1}$$

where

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{33} \end{pmatrix}, \quad N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}, \quad Y^T = \left(y(-1), y'(-1), y(1), y'(1)\right),$$

M_{ij} and N_i are real numbers and rank$(M) = 2$. We assume that (1) has a unique solution.

Different methods for approximating the solution of this kind of problems have been developed in the literature. Among these methods, the Taylor polynomial method is one of the most used tools. In the last few years, several authors have revisited this method and proposed new algorithms ([1, 6]). In the case in which it is possible to find a disk of convergence where the coefficient functions φ, f, g and h are analytic, the interval $[-1, 1]$ is contained inside that disk and $\varphi(x)$ does not vanish in that disk, the basic idea of the method proposed by Sezer and Kesan ([1, 6]) is the following. We consider the finite part of the Taylor expansion of the solution y at $x = c$:

$$y(x) \simeq y_n(x) := \sum_{k=0}^{n} a_k(x-c)^k$$

and equate to zero the Taylor coefficients at $x = c$ of $R(x) := \varphi(x)y_n''(x) + f(x)y_n'(x) + g(x)y_n(x) - h(x)$ up to the order $n-2$. Thus, we obtain a system of $n-1$ linear equations for the $n+1$ unknowns $a_0, a_1, a_2, \ldots, a_n$. The system is complemented with the two linear equations $MY = N$. We obtain then a linear system of $n+1$ equations and $n+1$ unknowns $a_0, a_1, a_2, \ldots, a_n$, whose solution gives an approximation to the Taylor polynomial $y_n(x)$, and then an approximation of the solution $y(x)$ of (1) ([1, 6]).

When $[-1, 1]$ is not included in the disk $D_r(c)$, we can take several points c_k (typically along the interval $[-1, 1]$) in such a way that $[-1, 1] \subset \cup_k D_{r_k}(c_k)$. Then, we use a Taylor expansion of the solution at every such point $x = c_k$ and match these expansions at intersecting disks $D_{r_k}(c_k)$ [5, Sec. 7]. In this way, we obtain an approximation of the solution of (1) in the form of a piecewise polynomial in several subintervals of $[-1, 1]$. Although this method gives an analytic approximation to the solution, this approximation is not uniform in the whole interval $[-1, 1]$ because it has a different polynomial representation over different subintervals $[-1, 1] \cap D_{r_k}(c_k)$. Besides this, the coefficients of the Taylor polynomial in every subinterval are determined by the coefficients of the Taylor polynomial in the adjacent subintervals, and this matching of expansions translates into numerical errors.

Thus, our purpose in this work is to show that multi-point Taylor polynomials [3, 4] combined with the method proposed in [1, 6] are adequate to approximate the solution of these equations in the case in which it is not possible to find a disk of convergence containing the interval of integration. Besides this, we show that this approximation provides a convergent expansion of the solution uniformly valid in the whole interval.

The paper is organized as follows. Section 2 presents a new method by considering two-point Taylor expansions instead of the classical Taylor expansion. We illustrate the technique with different examples. As a straightforward generalization of the two-point Taylor approximation, Section 3 includes an approximation by an $n-$point Taylor expansion.

§2. A Taylor expansion of the solution at the two extreme points

Let us consider a two-point Taylor expansion of the solution of (1) at the base points $x = \pm 1$ ([3]):

$$y(x) = \sum_{k=0}^{\infty} [a_k + b_k x](x^2 - 1)^k, \tag{2}$$

where the (unique) two-point Taylor coefficients a_k and b_k are related to the derivatives of y at $x = \pm 1$ ([3]).

We denote the Cassini oval in the complex plane with foci at $x = \pm 1$ and Cassini radius r by $O_r = \left\{ z \in \mathbb{C} \mid |z^2 - 1| = r \right\}$ and the Cassini disk by $\mathcal{D}_r = \left\{ z \in \mathbb{C} \mid |z^2 - 1| < r \right\}$. When $r > 1$, O_r is a single oval, when $r = 1$ it is a lemniscate, and when $r < 1$ it consists of two

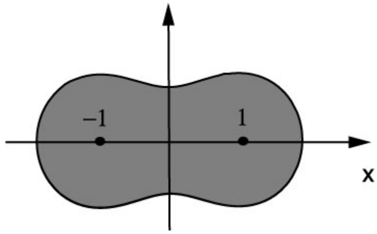

Figure 1: Graph of the Cassini disk when $r > 1$.

small ovals around the points ± 1. When we assume $r > 1$, the interval $[-1, 1]$ is lying inside $\mathcal{D}_r$ (see Figure 1).

Suppose that the functions φ, f, g and h are analytic in the Cassini disk $\mathcal{D}_r$, $r > 1$, and $\varphi \neq 0$ in $\mathcal{D}_r$. We propose the following algorithm to approximate the unique solution y of (1).

Algorithm 1. The method of Frobenius assures that the unique solution y of (1) is analytic in the Cassini disk $\mathcal{D}_r$. Then, it is shown in [3] and [4] that y admits a two-point Taylor expansion of the form (2). From (2) we have

$$y'(x) = \sum_{k=0}^{\infty}\Big\{\big[(2k+1)b_k + 2(k+1)b_{k+1}\big] + 2(k+1)a_{k+1}x\Big\}(x^2-1)^k,$$

$$\begin{aligned} y'(x) &= \sum_{k=0}^{\infty}\Big\{\big[(2k+1)b_k + 2(k+1)b_{k+1}\big] + 2(k+1)a_{k+1}x\Big\}(x^2-1)^k, \\ y''(x) &= \sum_{k=0}^{\infty} 2(k+1)\Big\{\big[(2k+1)a_{k+1} + 2(k+2)a_{k+2}\big] + \big[(2k+3)b_{k+1} + 2(k+2)b_{k+2}\big]x\Big\} \\ &\quad \times (x^2-1)^k. \end{aligned} \tag{3}$$

Using the above two-point Taylor expansions of y, y' and y'', we equate to zero the two-point Taylor coefficients of $R(x) := \varphi(x)y'' + f(x)y' + g(x)y - h(x)$ at $x = \pm 1$. We obtain in this way a_k and b_k, $k = 2, 3, 4, \ldots$, from a system of two recursions of the form:

$$\begin{cases} a_k = \displaystyle\sum_{j=0}^{k-1}\big[\alpha_{k,j}a_j + \beta_{k,j}b_j\big] + \gamma_k, \\ b_k = \displaystyle\sum_{j=0}^{k-1}\big[\alpha'_{k,j}a_j + \beta'_{k,j}b_j\big] + \gamma'_k, \end{cases} \qquad k = 2, 3, 4, \ldots, \tag{4}$$

where the coefficients $\alpha_{k,j}$, $\beta_{k,j}$, γ_k, $\alpha'_{k,j}$, $\beta'_{k,j}$, γ'_k depend on the two-point Taylor coefficients of φ, f, g and h at $x = \pm 1$. The computation of the coefficients a_k, b_k, $k = 2, 3, 4, \ldots$, requires the initial seed a_0, b_0, a_1 and b_1. From these recurrence relations we obtain the two-point Taylor coefficients a_k and b_k, $k = 2, 3, 4, \ldots$, of y at $x = \pm 1$ as an affine combination of the four first coefficients a_0, b_0, a_1 and b_1. We have

$$\begin{cases} a_k = A_k a_0 + B_k b_0 + C_k a_1 + D_k b_1 + E_k, \\ b_k = F_k a_0 + G_k b_0 + H_k a_1 + I_k b_1 + J_k, \end{cases} \qquad k = 2, 3, 4, \ldots, \tag{5}$$

where the coefficients A_k, $B_k, \ldots, J_k$ are functions of $\alpha_{k,j}$, $\beta_{k,j}$, γ_k, $\alpha'_{k,j}$, $\beta'_{k,j}$, γ'_k. The parameters a_0, b_0, a_1 and b_1 are linked by the equations $MY = N$, with

$$Y^T = (a_0 - b_0, b_0 + 2b_1 - 2a_1, a_0 + b_0, b_0 + 2b_1 + 2a_1). \tag{6}$$

This means that only two of these four parameters are free; suppose, for example, that a_1 and b_1 are free (if we choose another pair of parameters as free parameters we can proceed in a similar manner). Then, every two-point Taylor coefficient a_k and b_k, $k = 2, 3, 4, \ldots$, is an affine combination of only a_1 and b_1.

Every pair (a_1, b_1) gives rise to a different function y given by (2)-(5). Formally, all of these functions y are solutions of (1). But this problem has a unique solution, and then it must happen that the series (2) is convergent only for one pair (a_1, b_1), the one that gives rise to the unique solution of (1). The series (2) must be divergent for any other pair (a_1, b_1).

The correct values (a_1, b_1) may be then obtained by imposing the convergence of (2). In practice, we obtain an approximation $(\tilde{a}_1, \tilde{b}_1)$ of (a_1, b_1) by solving the two linear equations $a_{n+1} = b_{n+1} = 0$ (a_{n+1} and b_{n+1} are affine combinations of a_1 and b_1). Doing this we are imposing implicitly that (2) is convergent when we approximate this infinite series by

$$y_n(x) := \sum_{k=0}^{n} [a_k + b_k x](x^2 - 1)^k. \tag{7}$$

Once we have obtained the approximation $(\tilde{a}_1, \tilde{b}_1)$, we obtain from $MY = N$ an approximation $(\tilde{a}_0, \tilde{b}_0)$ of (a_0, b_0) and then, from (5), we obtain the approximations $\tilde{a}_k$ and $\tilde{b}_k$, $k = 2, 3, 4, \ldots$ of a_k and b_k as affine combinations of $\tilde{a}_1$ and $\tilde{b}_1$ and hence, the approximate two-point Taylor polynomial

$$\tilde{y}_n(x) := \sum_{k=0}^{n} [\tilde{a}_k + \tilde{b}_k x](x^2 - 1)^k. \tag{8}$$

Algorithm 1 can be reformulated in a more appropriate computational form. For further information, we refer to [2].

Example 1. Consider the boundary value problem

$$\begin{cases} \left(x^2 + \dfrac{1}{4}\right) y''(x) + i\left[c - (a+b+1)\left(\dfrac{1}{2} + ix\right)\right] y'(x) + ab\, y(x) = 0, \quad x \in (-1, 1), \\ y(-1) = {}_2F_1(a, b; c; 1/2 - i), \quad y(1) = {}_2F_1(a, b; c; 1/2 + i). \end{cases}$$

We have $M_{11} = M_{23} = 1$ and the remaining $M_{ij} = 0$; $N_1 = {}_2F_1(a, b; c; 1/2 - i)$, $N_2 = {}_2F_1(a, b; c; 1/2 + i)$, $\varphi(x) = x^2 + 1/4$, $f(x) = i(c - (a+b+1)(1/2 + ix))$, $g(x) = ab$ and $h(x) = 0$. The unique solution of this problem is the hypergeometric function: $y(x) = {}_2F_1(a, b; c; 1/2 + ix)$.

The coefficient functions are entire functions, but the function $\varphi(x) = (x^2 + \frac{1}{4})$ vanishes at $x = \pm 1/2i$. Thus, this function is nonvanishing in the Cassini disk $\mathcal{D}_r$ with foci at $x = \pm 1$ for any $1 < r < \sqrt{5}/2$.

We have

$$\begin{cases} y(-1) = a_0 - b_0 = {}_2F_1(a, b; c; 1/2 - i), \\ y(1) = a_0 + b_0 = {}_2F_1(a, b; c; 1/2 + i), \end{cases}$$

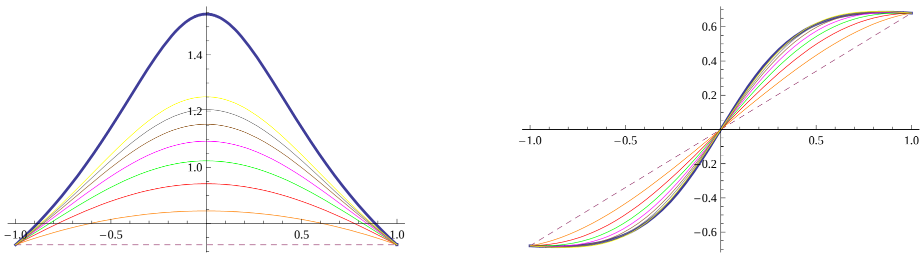

Figure 2: Graph of the real part and the imaginary part of the exact solution ${}_2F_1(a, b; c; 1/2 + ix)$ (blue) and the approximations $\widetilde{y}_n$, $n = 0, 1, \ldots, 7$ for $a = 1$, $b = 2$ and $c = 3$.

thus,

$$\begin{cases} a_0 = \dfrac{{}_2F_1(a, b; c; 1/2 + i) + {}_2F_1(a, b; c; 1/2 - i)}{2}, \\ b_0 = \dfrac{{}_2F_1(a, b; c; 1/2 + i) - {}_2F_1(a, b; c; 1/2 - i)}{2}. \end{cases}$$

The two-point Taylor expansions of the coefficient functions are finite in this example:

$$\varphi(x) = \left[\frac{5}{4} + 0 \cdot x\right] + [1 + 0 \cdot x]\,(x^2 - 1),$$

$$f(x) = \left[i\left(c - \frac{a + b + 1}{2}\right) + (a + b + 1)x\right], \qquad g(x) = [ab + 0 \cdot x]\,,$$

and then, the recursions are, for $k = 0, 1, 2, \ldots$,

$$\begin{aligned} &5(k+1)(k+2)a_{k+2} + \frac{1}{2}(k+1)(4a + 4b + 9 + 18k)a_{k+1} - i(k+1)(a + b + 1 - 2c)b_{k+1} \\ &\quad + (a + 2k)(b + 2k)a_k - \frac{i}{2}(2k+1)(a + b + 1 - 2c)b_k = 0, \\ &5(k+1)(k+2)b_{k+2} + \frac{1}{2}(k+1)(4a + 4b + 19 + 18k)b_{k+1} - i(k+1)(a + b + 1 - 2c)a_{k+1} \\ &\quad + (1 + a + 2k)(1 + b + 2k)b_k = 0, \end{aligned}$$

with a_0 and b_0 given above and a_1 and b_1 free.

For several values of $n \in \mathbb{N}$, we solve the equations $a_{n+1} = b_{n+1} = 0$ for a_1 and b_1 and obtain the approximate values $\widetilde{a}_1$ and $\widetilde{b}_1$. From the above recursions and using the exact values of a_0 and b_0 and the approximate $\widetilde{a}_1$ and $\widetilde{b}_1$ we obtain the approximate Taylor polynomial. Figure 2 shows the approximation $\widetilde{y}_n(x)$ of $y(x)$ for some values of n and a, b and c.

Example 2. As an example of an oscillatory function we consider the boundary value problem

$$\begin{cases} [a + b + (b - a)x]^2\, y'' + (b - a)\,[a + b + (b - a)x]\, y' \\ \qquad + (b - a)^2\left[\left(\dfrac{a + b + (b - a)x}{2}\right)^2 - \alpha^2\right] y = 0, \quad x \in (-1, 1), \\ y(-1) = J_\alpha(a), \qquad y(1) = J_\alpha(b), \end{cases} \tag{9}$$

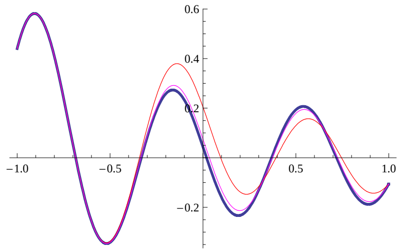

Figure 3: Plot of the exact solution $y(x) = J_\alpha(\frac{a+b+(b-a)x}{2})$ (thick blue) of (9) and the approximations $\tilde{y}_{10}(x)$ (red) and $\tilde{y}_{11}(x)$ (magenta) with $a = 1$, $b = 19$ and $\alpha = 1$.

with $0 < a < b$. We have $M_{11} = M_{23} = 1$ and the remaining $M_{ij} = 0$; $N_1 = J_\alpha(a)$, $N_2 = J_\alpha(b)$, $\varphi(x) = (a+b+(b-a)x)^2$, $f(x) = (b-a)[a+b+(b-a)x]$, $g(x) = (b-a)^2[(\frac{a+b+(b-a)x}{2})^2 - \alpha^2]$ and $h(x) = 0$. We consider the base points ± 1. The unique solution of this problem is the Bessel function: $y(x) = J_\alpha(\frac{a+b+(b-a)x}{2})$.

For several $n \in \mathbb{N}$, we seek for an approximation $\tilde{y}_n(x)$ of the two-point Taylor polynomial $y_n(x)$ of $y(x)$ using Algorithm 1. Figure 3 illustrates the approximation $y(x) \simeq \tilde{y}_n(x)$ for some values of n, a, b and α.

Other examples in which two-point Taylor polynomials may be applied can be found in [2].

§3. A Taylor expansion of the solution at n points

When the Cassini disk $\mathcal{D}_r$ of analyticity of the coefficient functions of (1) with foci $x = \pm 1$ does not contain the interval $[-1, 1]$, we may consider an $n-$point Taylor expansion with $n > 2$ (see [4]). When those base points are conveniently chosen, we facilitate the inclusion of the interval $[-1, 1]$ in the generalized Cassini disk of convergence of the $n-$point Taylor expansion.

In general, if the coefficient functions have more singular points P_1, P_2, $P_3, \ldots,$ close to the interval $[-1, 1]$, then we should consider a multi-point Taylor expansion with more base points such that the region of convergence avoids those singular points and contains the interval $[-1, 1]$. When we take more base points for the multi-point Taylor expansion, we squeeze the convergence region of the expansion avoiding the singular points P_k and including the interval $[-1, 1]$ in this region [4] (see Figure 4). The generalization of Algorithm 1 from two-point Taylor expansions to the $n-$point Taylor expansion case is straightforward. For further information, we refer to [2].

We illustrate the idea for $n = 3$ with the following example.

Example 3. Consider the boundary value problem

$$\begin{cases} \left[1 - (x-ia)^2\right] y'' - 2(x-ia)y' + \left[\nu(\nu+1) - \dfrac{\mu^2}{1-(x-ia)^2}\right] y(x) = 0, \qquad x \in (-1, 1), \\ y(-1) = P_\nu^\mu(-1-ia), \qquad y(1) = P_\nu^\mu(1-ia), \end{cases} \tag{10}$$

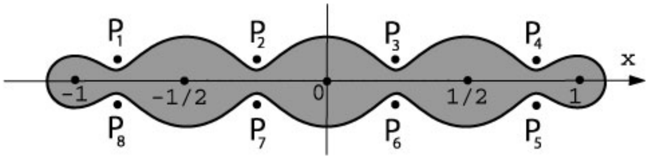

Figure 4: Typical portrait of the convergence region of a five-point Taylor expansion at the five base points $x = \pm 1$, $x = \pm 1/2$ and $x = 0$.

with $0 < a$ and $P_\nu^\mu(z)$ the Legendre function of the second kind with $\nu, \mu \in \mathbb{C}$. The coefficient functions are entire functions, but the function $\varphi(x) = [1 - (x - ia)^2]$ vanishes at $x = \pm 1 - ia$. If $a < (\sqrt{5} - 2)^{1/2}$, we cannot find a Cassini oval with foci at $x = \pm 1$ that contains the interval $[-1, 1]$ and that does not contain the points $x = \pm 1 - ia$. Hence, we cannot apply the method of Section 2. We consider then a three-point Taylor approximation for y with base points $x = \pm 1$ and $x = 0$ (see [4]) in the form

$$y(x) = \sum_{k=0}^{\infty} [a_k + b_k x + c_k x^2] x^k (x^2 - 1)^k.$$

This expansion is convergent in the region (see [4]) $\mathcal{E}_r = \left\{ z \in \mathbb{C} \,|\, |z(z^2 - 1)| < r \right\}$ with $r \le a\sqrt{a^4 + 5a^2 + 4}$, that does not contain the points $x = \pm 1 - ia$. Moreover, this region contains the interval $[-1, 1]$ when $r > 2/(3\sqrt{3})$ (see Figure 5).

Figure 6 shows the approximation for some values of ν, μ and a.

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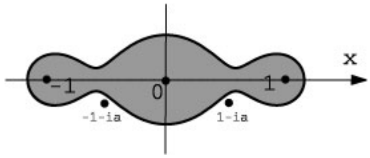

Figure 5: The region $\mathcal{E}_r$ of convergence of Example 3 contains the real interval $[-1, 1]$ and it does not contain the zeros $\pm 1 - ia$ of the function φ if $2/(3\sqrt{3}) < r \le a\sqrt{a^4 + 5a^2 + 4}$.

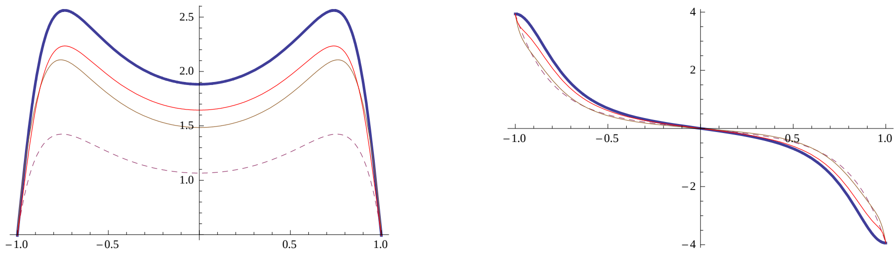

Figure 6: Plot of the exact solution $y(x) = P_\nu^\mu(x - ia)$ (thick blue) of (10) and the approximations $\tilde{y}_3(x)$ (dashed), $\tilde{y}_{12}(x)$ (brown) and $\tilde{y}_{20}(x)$ (red) for $a = 1/4$, $\nu = 1$ and $\mu = 2$.

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