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On the hydrostatic Stokes approximation with non homogeneous Dirichlet conditions

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Abstract. We deal with the hydrostatic Stokes approximation with non homogeneous Dirichlet boundary conditions. While investigated the homogeneous case, we build a shifting operator of boundary values related to the divergence operator, and solve the non homogeneous problem in a domain with sidewalls.

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§1. Introduction

Let us consider $\Omega \subset \mathbb{R}^3$ a bounded domain defined by

$$\Omega = \{\, x = (x',\, x_3) \in \mathbb{R}^3 \mid x' \in \omega \ \text{ and } \ -h(x') < x_3 < 0 \,\}, \tag{1}$$

where $\omega \subset \mathbb{R}^2$ is a bounded Lipschitz-continuous domain and h, defined in ω, is a mapping satisfying the following assumption.

Assumption 1. *The mapping h is positive and Lipschitz-continuous on ω. Besides, there is a constant $\alpha > 0$ such that*

$$\inf_{x' \in \omega} h(x') \geqslant \alpha. \tag{2}$$

Therefore, Ω has a Lipschitz-continuous boundary Γ splitted into three parts, each one with a positive measure: the surface Γ_S, the bottom Γ_B , and sidewalls Γ_L, defined by:

$$\Gamma_S = \omega \times \{0\}, \quad \Gamma_B = \{(x', -h(x')) \mid x' \in \omega\},$$
$$\Gamma_L = \{\, x \in \mathbb{R}^3 \mid x' \in \partial\omega \ \text{ and } \ -h(x') < x_3 < 0 \,\}.$$

Finally, we denote by $\boldsymbol{n}$ the unit external vector normal to Γ. Below, the drawing of the domain Ω.

Let $\boldsymbol{f}' = (f_1,\, f_2) : \Omega \to \mathbb{R}^2$, $\Phi : \Omega \to \mathbb{R}$, and $\boldsymbol{g} = (\boldsymbol{g}',\, g_3) : \Gamma \to \mathbb{R}^3$ be given functions, Φ and $\boldsymbol{g}$ satisfying adequate compatibility conditions (see (7)). In this paper, we study the hydrostatic Stokes approximation consisting in seeking $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ and $p : \omega \to \mathbb{R}$

$$(\mathcal{SH}) \begin{cases} -\Delta \boldsymbol{u}' + \nabla' p = \boldsymbol{f}', \quad \partial_3 p = 0, \quad \nabla \cdot \boldsymbol{u} = \Phi \quad \text{in } \Omega, \\ \qquad\qquad\qquad\qquad \boldsymbol{u}' = \boldsymbol{g}', \quad u_3 n_3 = g_3 \quad \text{on } \Gamma. \end{cases}$$

Here $\nabla' = (\partial_{x_1}, \partial_{x_2})$ denotes the gradient operator with respect to the variables x_1 and x_2.

When Φ and g_3 are identically equal to 0, some authors have considered $(\mathcal{SH})$ as a reduced Stokes-type system. Indeed, let us consider the case of homogeneous conditions. The simplifications of $(\mathcal{SH})$ come from the hydrostatic pressure hypothesis:

$$\frac{\partial p}{\partial x_3} = 0 \text{ in } \Omega, \tag{3}$$

ensuring that p_S, the pressure at $x_3 = 0$, is in fact the real unknown. Moreover, by integrating with respect to x_3 the incompressibility equation:

$$\nabla \cdot \boldsymbol{u} = 0 \text{ in } \Omega, \tag{4}$$

and taking into account the boundary conditions over u_3, it appears that the vertical velocity u_3 is given by the horizontal velocity $\boldsymbol{u}'$. In this case, the equations of $(\mathcal{SH})$ can be reduced to the following system:

$$\begin{cases} -\Delta \boldsymbol{u}' + \nabla' p_S = \boldsymbol{f}' & \text{in } \Omega, \\ \nabla' \cdot \displaystyle\int_{-h(x')}^{0} \boldsymbol{u}'(x', x_3)\, dx_3 = 0 & \text{in } \omega, \\ \boldsymbol{u}' = 0 & \text{on } \Gamma. \end{cases} \tag{5}$$

Then, we get back to u_3 and the global pressure p by setting

$$x \in \Omega, \quad u_3(x) = \int_{x_3}^{0} \nabla' \cdot \boldsymbol{u}'(x', \xi)\, d\xi, \quad p(x) = p_S(x'). \tag{6}$$

However, studying (5) yields real difficulties when the mapping h vanishes on $\partial\omega$. Previous works dealing with (5) use assumption (2). Weak solutions to (5) was investigated in [5, 4]. Results of [5, 4] are then reviewed in [3], where the author deals with some models close to (5).

The purpose of the paper is to present a proof of the following thoerem, in a simplified case. The complete proof is given in [1]. Before, we introduce the space

$$\boldsymbol{X} = H^1(\Omega)^2 \times H(\partial_{x_3}, \Omega),$$

and its hilbertian norm $\|\boldsymbol{u}\|_X = \left(\|\boldsymbol{u}'\|^2_{H^1(\Omega)^2} + \|u_3\|^2_{H(\partial_{x_3},\Omega)}\right)^{1/2}$, where $H(\partial_{x_3}, \Omega)$ is defined in Subsection 2.2.

Theorem 2. *Assume assumption* (2). *Let* $\boldsymbol{f}' \in H^{-1}(\Omega)^2$, $\Phi \in L^2(\Omega)$, $\boldsymbol{g}' \in H^{1/2}(\Gamma)^2$ *and* $g_3 \in L^2(\Gamma)$ *such that* $g_3 = 0$ *on* Γ_L, *and satisfying the following compatibility condition:*

$$\int_\Gamma \boldsymbol{g}' \cdot \boldsymbol{n}'\, d\sigma + \int_\Gamma g_3\, d\sigma = \int_\Omega \Phi\, dx. \tag{7}$$

Then, there is a unique pair $(\boldsymbol{u}, p) \in \boldsymbol{X} \times (L^2(\Omega)/\mathbb{R})$ *solution to Problem* $(\mathcal{SH})$ *and satisfying the estimate,*

$$\|\boldsymbol{u}\|_X + \|p\|_{L^2(\Omega)/\mathbb{R}} \leqslant C\left\{\|\boldsymbol{f}'\|_{H^{-1}(\Omega)^2} + \|\Phi\|_{L^2(\Omega)} + \|\boldsymbol{g}'\|_{H^{1/2}(\Gamma)^2} + \|g_3\|_{L^2(\Gamma)}\right\}, \tag{8}$$

where $C > 0$ *is a constant depending only on* Ω.

The outline of the paper is as follows. In Section 2 we set the appropriate functional framework. In particular, we recall the definition and structure of the anisotropic space $H(\partial_{x_3}, \Omega)$, which is the adapted space for u_3. Moreover, we introduce the usual integration operators M and F (see (14) and (15)), useful in our study, to provide an adapted lemma of De Rham (see Lemma 7). Finally, we prove Theorem 2 in Section 3.

§2. Functional framework

We assume the reader to be familiar with the classical notations and properties of Lebesgue and Sobolev spaces on a regular open set.

2.1. Computations of surface integrals

For any function $\mu : \Gamma \to \mathbb{R}$, we define the functions μ_S or $(\mu)_S$ and μ_B or $(\mu)_B$ by setting

$$x' \in \omega, \quad \mu_S(x') = \mu(x', 0), \quad \mu_B(x') = \mu(x', -h(x')).$$

We start with an important tool which enables us to replace any integrals defined on Γ_S and Γ_B by one defined on ω.

Lemma 3. *The mapping $\mu \mapsto (\mu_S, \mu_B)$ is linear and continuous from $L^2(\Gamma)$ into $L^2(\omega)^2$. Moreover, one has by definition of the measure $d\sigma$:*

$$\int_{\Gamma_S} \mu\, d\sigma = \int_\omega \mu_S\, dx' \quad \textit{and} \quad \int_{\Gamma_B} \mu\, d\sigma = \int_\omega \mu_B \sqrt{1 + |\nabla h|^2}\, dx'. \tag{9}$$

Proof. This result follows from straightforward calculating. □

Remark 1. Notice that the integrals in (9) are well defined since ω is bounded. Next, the third component of the normal n_3 satisfies $n_3 = 1$ on Γ_S, $n_3 = 0$ on Γ_L and $(n_3)_B(1 + |\nabla h|^2)^{1/2} = -1$ on ω. Moreover, $(n_i)_B(1 + |\nabla h|^2)^{1/2} = -\partial_{x_i} h$ in ω. Therefore,

$$\forall \mu \in L^2(\Gamma), \quad \int_\Gamma \mu n_3\, d\sigma = \int_\omega \mu_S\, dx' - \int_\omega \mu_B\, dx'. \tag{10}$$

$$\int_{\Gamma_B} \mu n_i\, d\sigma = -\int_\omega \mu \frac{\partial h}{\partial x_i}\, dx'. \tag{11}$$

2.2. The anisotropic space $H(\partial_{x_3}, \Omega)$

Let us recall here some useful results that can be found in [6]. Set

$$H(\partial_{x_3}, \Omega) = \left\{ u \in L^2(\Omega) \;\Big|\; \frac{\partial u}{\partial x_3} \in L^2(\Omega) \right\},$$

which is a Hilbert space endowed with norm $\|u\|_{H(\partial_{x_3}, \Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \left\|\partial_{x_3} u\right\|^2_{L^2(\Omega)} \right)^{1/2}$. For any $u \in H(\partial_{x_3}, \Omega)$, we have $u n_3 \in H^{-1/2}(\Gamma)$. Then, setting

$$H_0(\partial_{x_3}, \Omega) = \left\{ u \in L^2(\Omega) \;\Big|\; \frac{\partial u}{\partial x_3} \in L^2(\Omega) \text{ and } u n_3 = 0 \right\},$$

the following Green's formula holds

$$\forall u \in H(\partial_{x_3}, \Omega),\ \forall v \in H_0(\partial_{x_3}, \Omega), \qquad \int_\Omega u \frac{\partial v}{\partial x_3}\, dx = -\int_\Omega v \frac{\partial u}{\partial x_3}\, dx, \tag{12}$$

as well as the Poincaré's Inequality

$$\forall u \in H_0(\partial_{x_3}, \Omega), \quad \|u\|_{L^2(\Omega)} \leqslant \|h\|_{L^\infty(\omega)} \left\| \frac{\partial u}{\partial x_3} \right\|_{L^2(\Omega)}. \tag{13}$$

2.3. Definition and properties of the operators M and F.

Let u be a function defined in Ω. We consider the following operators

$$x' \in \omega, \quad Mu(x') = \int_{-h(x')}^{0} u(x',\, x_3)\, dx_3, \tag{14}$$

$$x = (x',\, x_3) \in \Omega, \quad Fu(x) = \int_{x_3}^{0} u(x',\, \xi)\, d\xi, \quad Gu(x) = \int_{-h(x')}^{x_3} u(x',\, \xi)\, d\xi. \tag{15}$$

Proposition 4. *The operator M is linear and continuous from $L^2(\Omega)$ into $L^2(\omega)$, and from $H^1(\Omega)$ into $H^1(\omega)$. Then, one has for $i = 1,\, 2$:*

$$\forall u \in H^1(\Omega), \quad \frac{\partial}{\partial x_i}(Mu) = M\Big(\frac{\partial u}{\partial x_i}\Big) + \frac{\partial h}{\partial x_i} u_B \ \text{in } \omega; \tag{16}$$

$$\forall u \in H_0^1(\Omega), \quad \frac{\partial}{\partial x_i}(Mu) = M\Big(\frac{\partial u}{\partial x_i}\Big) \ \text{in } \omega. \tag{17}$$

Moreover, the following relation holds:

$$\forall u \in H_0(\partial_{x_3}, \Omega), \quad M\Big(\frac{\partial u}{\partial x_3}\Big) = 0 \ \text{in } \omega. \tag{18}$$

Proof. Let $u \in L^2(\Omega)$. By applying Fubini's Theorem, we deduce that $Mu \in L^2(\omega)$ and $\|Mu\|_{L^2(\omega)} \leqslant \|h\|_{L^\infty(\omega)} \|u\|_{L^2(\Omega)}$. Therefore, the mapping M is linear and continuous from $L^2(\Omega)$ into $L^2(\omega)$. Next, for u in $H^1(\Omega)$ and $i = 1, 2$, one has for any $\psi \in \mathcal{D}(\omega)$:

$$\int_\omega Mu \frac{\partial \psi}{\partial x_i}\, dx' = \int_\Omega u \frac{\partial \psi}{\partial x_i}\, dx = -\int_\Omega \frac{\partial u}{\partial x_i} \psi\, dx + \int_\Gamma u\psi\, n_i\, d\sigma.$$

Then, (11) gives

$$\int_{\Gamma_B} u\psi\, n_i\, d\sigma = -\int_\omega u_B \psi \frac{\partial h}{\partial x_i}\, dx', \tag{19}$$

since ψ does not depend on x_3 and since $\psi = 0$ on Γ_L. Thus

$$\int_\omega Mu \frac{\partial \psi}{\partial x_i}\, dx' = -\int_\omega \left[M\Big(\frac{\partial u}{\partial x_i}\Big) + u_B \frac{\partial h}{\partial x_i} \right] \psi\, dx'.$$

Thus (16) holds in $\mathcal{D}'(\omega)$. From Proposition 3 and the fact that h is Lipschitz-continuous, (16) holds in $L^2(\omega)$. The same arguments prove that M is a linear mapping from $H^1(\Omega)$ in $H^1(\omega)$. When u belongs to $H_0^1(\Omega)$, the function u_B vanishes on ω. Therefore, we get (17). Finally, (18) follows from a computation using relation (12). □

Proposition 5. *The operator F is linear and continuous from $L^2(\Omega)$ into $L^2(\Omega)$ and G is the adjoint operator to F. Next, the operator F is continuous from $L^2(\Omega)$ into $H(\partial_{x_3}, \Omega)$, and*

$$\forall u \in L^2(\Omega), \quad \frac{\partial}{\partial x_3}(Fu) = -u \text{ in } \Omega. \tag{20}$$

Moreover, the following relation holds:

$$\forall u \in H_0(\partial_{x_3}, \Omega), \quad F\Big(\frac{\partial u}{\partial x_3}\Big) = -u \text{ in } \Omega. \tag{21}$$

Proof. Let $u \in L^2(\Omega)$. Thanks to Fubini's Theorem, we deduce that $Fu \in L^2(\Omega)$ and from Poincaré's Inequality we have $\|Fu\|_{L^2(\Omega)} \leqslant \|h\|_\infty \|u\|_{L^2(\Omega)}$ by . Hence F is linear and continuous from $L^2(\Omega)$ into $L^2(\Omega)$. Again Fubini's Theorem ensures that

$$\forall u,\, v \in L^2(\Omega), \quad \int_\Omega v\, Fu\, dx = \int_\Omega u\, Gv\, dx. \tag{22}$$

Next, (22) gives that for any $\varphi \in \mathcal{D}(\Omega)$,

$$\int_\Omega \frac{\partial \varphi}{\partial x_3}\, Fu\, dx = \int_\Omega u\, G\Big(\frac{\partial \varphi}{\partial x_3}\Big)\, dx = \int_\Omega u\varphi\, dx.$$

Hence (20) holds in $\mathcal{D}'(\Omega)$ and $\partial_{x_3}(Fu) \in L^2(\Omega)$. Moreover, we deduce from above that the operator F is continuous from $L^2(\Omega)$ into $H(\partial_{x_3}, \Omega)$. Finally, we use the same arguments as above and relation (12) to prove (21). □

Remark 2. Let $u \in H^1(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$. Thanks to Proposition 5 and (10), one gets:

$$\begin{aligned}\int_\Omega G\Big(\frac{\partial u}{\partial x_3}\Big)\varphi\, dx &= \int_\Omega u\varphi\, dx + \int_{\Gamma_S \cup \Gamma_B} u n_3\, F\varphi\, d\sigma \\ &= \int_\Omega u\varphi\, dx + \int_\omega u_S\, (F\varphi)_S\, dx' - \int_\omega u_B (F\varphi)_B\, dx'.\end{aligned}$$

By observing that $(F\varphi)_S = 0$ and $(F\varphi)_B = M\varphi$ in ω, one has

$$\int_\Omega G\Big(\frac{\partial u}{\partial x_3}\Big)\varphi\, dx = \int_\Omega u\varphi\, dx - \int_\Omega u_B \varphi\, dx,$$

which provides that,

$$\forall u \in H^1(\Omega), \quad G\Big(\frac{\partial u}{\partial x_3}\Big) = u - \widetilde{u_B} \text{ in } \Omega. \tag{23}$$

We conclude this subsection by giving additional properties on M and F. Precisely, we prove the following relation between the operators M and F.

Proposition 6. *Let $u \in L^2(\Omega)$. Then, the following assertions are equivalent:*

(i) $Mu = 0$ *in* $L^2(\omega)$.

(ii) $(Fu)n_3 = 0$ *in* $H^{-1/2}(\Gamma)$.

Proof. Given $u \in L^2(\Omega)$, Proposition 5 ensure that $(Fu)n_3$ is in $H^{-1/2}(\Gamma)$. Next, (23) gives for any $v \in H^1(\Omega)$:

$$\begin{aligned}\langle (Fu)\, n_3, v\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} = \int_\Omega \frac{\partial v}{\partial x_3}\, Fu\, dx - \int_\Omega uv\, dx &= \int_\Omega u\, G\Big(\frac{\partial v}{\partial x_3}\Big)\, dx - \int_\Omega uv\, dx \\ &= \int_\Omega u\,(v - \widetilde{v_B})\, dx - \int_\Omega uv\, dx.\end{aligned}$$

Therefore, one obtains a relation between F and M:

$$\forall (u, v) \in L^2(\Omega) \times H^1(\Omega), \quad \langle (Fu)\, n_3, v\rangle = -\int_\omega v_B\, Mu\, dx', \tag{24}$$

which proves that (i) implies (ii). Conversely, for any ψ in $\mathcal{D}(\omega)$ and applying (24) with $v = \psi$, we get

$$\int_\omega \psi\, Mu\, dx' = \int_\omega v_B\, Mu\, dx' = -\langle (Fu)\, n_3, v\rangle = 0.$$

Then (ii) implies (i): this completes the proof of Proposition 6. □

2.4. Some properties related to the mean divergence operator

For any vector field $\boldsymbol{v} = (v_1,\, v_2,\, v_3)$, we define

$$\nabla' \cdot M\boldsymbol{u}' = \sum_{i=1,2} \partial_{x_i}(Mu_i),$$

and the corresponding space $\boldsymbol{V}_M = \left\{\boldsymbol{v}' \in H_0^1(\Omega)^2 \mid \nabla' \cdot M\boldsymbol{v}' = 0 \text{ in } \omega\right\}$.

Lemma 7. *If $\boldsymbol{f}' \in H^{-1}(\Omega)^2$ satisfies*

$$\forall \boldsymbol{v}' \in \boldsymbol{V}_M, \quad \langle \boldsymbol{f}',\, \boldsymbol{v}'\rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2} = 0,$$

then, there is $q \in L^2(\omega)/\mathbb{R}$ such that $\nabla'\widetilde{q} = \boldsymbol{f}'$ in Ω. Moreover, there is a constant $C > 0$ depending only on Ω such that

$$\|q\|_{L^2(\omega)/\mathbb{R}} \leqslant C\, \|\nabla\widetilde{q}\|_{H^{-1}(\Omega)}\,. \tag{25}$$

Proof. Let us set $\boldsymbol{f} = (\boldsymbol{f}',\, 0)$. Let $\boldsymbol{v} \in H_0^1(\Omega)^3$ such that $\nabla \cdot \boldsymbol{v} = 0$. Thanks to (17) and (18) one has $\boldsymbol{v}' \in \boldsymbol{V}_M$. Therefore, using results from [2] from pages 22-25, there is a unique function p in $L^2(\Omega)/\mathbb{R}$ such that $\nabla p = \boldsymbol{f}$. Then, since $\partial_{x_3} p = 0$ in Ω, there is $q \in L^2(\omega)/\mathbb{R}$, such that $p = \widetilde{q}$ in Ω. Thus q satisfies $\nabla'\widetilde{q} = \boldsymbol{f}'$ in Ω. □

§3. Resolution of Problem $(\mathcal{SH})$ with homogeneous Dirichlet conditions

Proposition 8. *Let $\boldsymbol{f}' \in L^2(\Omega)^2$ and assume that Φ and $\boldsymbol{g}$ are identically equal to 0. Then, Problem $(\mathcal{SH})$ has a at least solution $(\boldsymbol{u}, p)$ in the space $\boldsymbol{X} \times (L^2(\Omega)/\mathbb{R})$.*

Proof. Let us consider the solution $(\boldsymbol{u}, p)$ related to the data $\boldsymbol{f}' = 0$. We multiply the first equation of $(\mathcal{SH})$ by $\boldsymbol{u}'$. Then, using (12) and since $\nabla \cdot \boldsymbol{u} = 0$ and $\partial_{x_3} p = 0$ in Ω, one has

$$\int_\Omega \nabla \boldsymbol{u}' : \nabla \boldsymbol{u}' \, dx = \int_\Omega p \, \nabla' \cdot \boldsymbol{u}' \, dx = -\int_\Omega p \, \frac{\partial u_3}{\partial x_3} \, dx = \int_\Omega u_3 \, \frac{\partial p}{\partial x_3} \, dx = 0.$$

Therefore $\nabla \boldsymbol{u}' = 0$ in Ω and, since Ω is connected, $\boldsymbol{u}' = 0$ in Ω. As $\nabla \cdot \boldsymbol{u} = 0$ in Ω, we deduce that $\partial_{x_3} u_3 = 0$ in Ω, and from the inequality (13) we get $u_3 = 0$ in Ω. Next, since $\nabla' p = \Delta \boldsymbol{u}' = 0$ in Ω, one obtains that $\nabla p = 0$ in Ω, hence $p = 0$ in Ω. Finally, the solution related to the data $\boldsymbol{f}' = 0$ is $\boldsymbol{u} = 0$ and $p = 0$, which proves that Problem $(\mathcal{SH})$ has at least one solution in $\boldsymbol{X} \times (L^2(\Omega)/\mathbb{R})$. □

Theorem 9. *Let $\boldsymbol{f}'$ in $H^{-1}(\Omega)^2$ and assume that Φ and $\boldsymbol{g}$ are identically equal to 0. Then, Problem $(\mathcal{SH})$ has a unique solution $(\boldsymbol{u}, p)$ in the space $\boldsymbol{X} \times (L^2(\Omega)/\mathbb{R})$. Moreover, there is a constant $C > 0$ such that*

$$\|\boldsymbol{u}'\|_{H^1(\Omega)^2} + \|u_3\|_{H(\partial_{x_3},\Omega)} + \|p\|_{L^2(\Omega)/\mathbb{R}} \leqslant C \|\boldsymbol{f}'\|_{H^{-1}(\Omega)^2}. \tag{26}$$

To prove Theorem 9, we need Lemma 7 and the proposition stated below.

Lemma 10. *Let $\boldsymbol{u} = (\boldsymbol{u}', u_3)$ with $\boldsymbol{u}'$ in $H_0^1(\Omega)^2$ and u_3 in $H(\partial_{x_3}, \Omega)$. Then the following assertions are equivalent*

(i) $\nabla \cdot \boldsymbol{u} = 0$ *in* Ω, $\quad u_3 n_3 = 0$ *in* $H^{-1/2}(\Gamma)$.

(ii) $\nabla' \cdot (M\boldsymbol{u}') = 0$ *in* ω, $\quad u_3 = F(\nabla' \cdot \boldsymbol{u}')$ *in* Ω.

Proof. Assume that (i) holds. Then, (18) and (21) yield

$$M(\nabla' \cdot \boldsymbol{u}') = 0 \quad \text{and} \quad u_3 = F(\nabla' \cdot \boldsymbol{u}').$$

Moreover, thanks to (17) one has $M(\nabla' \cdot \boldsymbol{u}') = \nabla' \cdot M\boldsymbol{u}'$, from which follows (ii). Conversely, one has by (20), $\nabla \cdot \boldsymbol{u} = 0$. Since $M(\nabla' \cdot \boldsymbol{u}') = 0$, Proposition 6 ensures that $n_3 F(\nabla' \cdot \boldsymbol{u}') = 0$ in $H^{-1/2}(\Gamma)$. Hence $u_3 n_3 = 0$ in $H^{-1/2}(\Gamma)$. □

From Lemma 10 and the fact that p does not depend on x_3, solving Problem $(\mathcal{SH})$ reduces to solve the following problem:

$$\text{Find } (\boldsymbol{u}', p_S) \in H_0^1(\Omega)^2 \times (L^2(\omega)/\mathbb{R}) \text{ such that:}$$
$$\begin{cases} -\Delta \boldsymbol{u}' + \nabla' p_S = \boldsymbol{f}' & \text{in } \Omega, \\ \nabla' \cdot M\boldsymbol{u}' = 0 & \text{in } \omega, \\ \boldsymbol{u}' = 0 & \text{on } \Gamma. \end{cases} \tag{27}$$

We get back to p and u_3 thanks to (6). The existence and uniqueness of the solution to (27) is given by the following proposition.

Proposition 11. *Let $\boldsymbol{f}'$ in $H^{-1}(\Omega)^2$. There is a unique solution $(\boldsymbol{u}', p_S)$ in the space $H_0^1(\Omega)^2 \times (L^2(\omega)/\mathbb{R})$ to Problem* (27). *Moreover, there is a constant $C > 0$ such that*

$$\|\boldsymbol{u}'\|_{H^1(\Omega)^2} + \|p_S\|_{L^2(\omega)/\mathbb{R}} \leqslant C\,\|\boldsymbol{f}'\|_{H^{-1}(\Omega)^2}. \tag{28}$$

Proof. Any solution $(\boldsymbol{u}', p_S)$ in the space $H_0^1(\Omega)^2 \times (L^2(\omega)/\mathbb{R})$ satisfies the following variational formulation:

$$\forall \boldsymbol{v}' \in \boldsymbol{V}_M, \int_\Omega \nabla \boldsymbol{u}' : \nabla \boldsymbol{v}'\, dx = \langle \boldsymbol{f}', \boldsymbol{v}' \rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2}\,. \tag{29}$$

Conversely, any solution $\boldsymbol{u}' \in \boldsymbol{V}_M$ to (29) is such that

$$\forall \boldsymbol{v}' \in \boldsymbol{V}_M, \quad \langle -\Delta \boldsymbol{u}' - \boldsymbol{f}', \boldsymbol{v}' \rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2} = 0.$$

Therefore, Lemma 7 provides a unique p_S in $(L^2(\omega)/\mathbb{R})$ such that $(\boldsymbol{u}', p_S)$ is a solution to (27). Then, by Lax-Milgram's lemma, there is a unique $\boldsymbol{u}'$ in $\boldsymbol{V}_M$ satisfying (29) and $\|\nabla \boldsymbol{u}'\|_{L^2(\Omega)} \leqslant C\|\boldsymbol{f}'\|_{H^{-1}(\Omega)^2}$, hence $\|\boldsymbol{u}'\|_{H^1(\Omega)^2} \leqslant C\,\|\boldsymbol{f}'\|_{H^{-1}(\Omega)^2}$ by Poincaré's Inequality, where $C > 0$ denotes is a constant depending only on Ω. To finish, we deduce (28) from (25) since

$$\|p_S\|_{L^2(\omega)/\mathbb{R}} \leqslant C\,\left\|\nabla \widetilde{p_S}\right\|_{L^2(\Omega)} \leqslant C\|\boldsymbol{f}'\|_{H^{-1}(\Omega)^2}. \qquad \square$$

Thanks to Proposition 11 and Lemma 10, $(\mathcal{SH})$ admits a unique solution $(\boldsymbol{u}, p) \in \boldsymbol{X} \times (L^2(\Omega)/\mathbb{R})$. Combining results from Proposition 11 and Proposition 5, we get (8). This complete the proof of Theorem 8.

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