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SYMMETRIC AND ROW SCALES PARTIAL PIVOTING STRATEGIES

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Abstract. Row and symmetric scaled partial pivoting strategies present nice stability properties for some classes of matrices. In this paper both kinds of strategies are compared. Following [17], the average normalized growth factor for random matrices associated to Gauss elimination with scaled partial pivoting strategies for several norms is approximated by power functions. For nonsingular M-matrices, an economic implementation of the symmetric scaled partial pivoting for the 1-norm is presented.

Keywords: Gauss elimination, scaled pivoting, growth factor, conditioning, M-matrices.

AMS classification: 65F05, 65F35, 65G50.

§1. Introduction

Several pivoting strategies for Gauss elimination, such as partial and complete pivoting, have been deeply studied. Their growth factor has been analyzed from several points of view. The nice behaviour of a pivoting strategy introduced recently, and called rook pivoting, has been analyzed in several papers (see, for instance, [4] and [13]–[15]). This paper considers scaled partial pivoting strategies, which present very nice properties when dealing with some important classes of matrices, as we shall recall and show in this paper. These pivoting strategies have been frequently used in the literature and even in basic books such as [3]. In [14], it is established that row scaled partial pivoting is generally successful when the larger elements of the coefficient matrix of a linear system $Ax = b$ are uniformly distributed across its rows and columns. One of the attractive features of scaled partial pivoting (SPP) is that the accuracy of the computed solution of a linear system by SPP is essentially independent of row scaling of the coefficient matrix. Thus, if the matrix is ill-conditioned due to bad row scaling, then a highly accurate solution can usually be obtained with SPP. A nice explanation of the advantages of SPP comes from the underlying hyperplane geometry of Gauss elimination (see [7] and [14]), as recalled in Section 2.

There are two types of SPP strategies: row SPP and symmetric SPP strategies (see definitions in Section 2). In this paper we compare the properties of these two types of strategies. Besides, there is scarce literature about their stability properties when applied to general or random matrices. This is another topic considered in this paper. Rice (see [16, p. 44]) and Poole and Neal [13] noted that if the elements of the coefficient matrix of a linear system are of uniform size, the computations are more robust. In [17] and [5], the average normalized growth factor for random matrices has been analyzed for several pivoting strategies different from SPP. We analyze the average normalized growth factor for SPP strategies.

In general, a drawback of SPP is its high computational cost. It requires $O(n^3)$ elementary operations in addition to the computational cost of the Gauss elimination of an $n \times n$ matrix.

However, its implementation for special classes of matrices can have lower computational cost.

Let us now mention two classes of matrices playing an important role in many applications where SPP strategies present very nice properties. In the first case (with sign regular matrices), the implementation of the pivoting strategy was performed in [10] and the good properties correspond to row SPP. The second case (with M-matrices) is a novelty of this paper and now the good properties correspond to symmetric SPP. The two classes of matrices are:

- Nonsingular sign regular matrices. An $n \times n$ matrix A is sign regular if, for each k with $1 \leq k \leq n$, all minors of order k have the same sign. Due to their variation diminishing properties, these matrices present important applications in many fields, such as Approximation Theory, Statistics or Computer Aided Geometric Design (see references in [10]). In [10] it was proved that row SPP for any strictly monotone vector norm can be implemented increasing the computational cost of Gauss elimination with $O(n)$ elementary operations, a cost considerably lower than that of partial pivoting. In addition the growth factor is optimal (see Corollary 2.4).
- Nonsingular M-matrices. A nonsingular matrix A is an M-matrix if it has positive diagonal entries, nonpositive off-diagonal entries and A^{-1} is nonnegative. Nonsingular M-matrices present many applications to Numerical Analysis, Dynamic Systems, Economics and Linear Programming, among other fields. In Section 3, we show how to implement with low computational cost (increasing the computational cost of Gauss elimination with $O(n^2)$ elementary operations) the symmetric SPP for $\|\cdot\|_1$ in the class of nonsingular M-matrices. We also show that the growth factor is optimal.

We now introduce some basic notations. Given $k, l \in \{1, 2, \dots, n\}$, let α (resp., β) be any increasing sequence of k (resp., l) positive integers less than or equal to n. Let A be a real square matrix of order n. Then we denote by $A[\alpha|\beta]$ the $k \times l$ submatrix of A containing rows numbered by α and columns numbered by β. Besides let $A[\alpha] := A[\alpha|\alpha]$. Gauss elimination transforms a linear system $Ax = b$ into an equivalent upper triangular linear system $Ux = c$. Gauss elimination with a given pivoting strategy, for nonsingular matrices A, consists of a succession of at most $n-1$ major steps resulting in a sequence of matrices as follows:

$$A = A^{(1)} \longrightarrow \tilde{A}^{(1)} \longrightarrow A^{(2)} \longrightarrow \tilde{A}^{(2)} \longrightarrow \cdots \longrightarrow A^{(n)} = \tilde{A}^{(n)} = U,$$

where $A^{(t)} = (a_{ij}^{(t)})_{1 \leq i,j \leq n}$ has zeros below its main diagonal in the first $t-1$ columns. The matrix $\tilde{A}^{(t)} = (\tilde{a}_{ij}^{(t)})_{1 \leq i,j \leq n}$ is obtained from the matrix $A^{(t)}$ by reordering the rows and/or columns $t, t+1, \dots, n$ of $A^{(t)}$ according to the given pivoting strategy and satisfying $\tilde{a}_{tt}^{(t)} \neq 0$. To obtain $A^{(t+1)}$ from $\tilde{A}^{(t)}$ we produce zeros in column t below the *pivot element* $\tilde{a}_{tt}^{(t)}$ by subtracting multiples of row t from the rows beneath it. Rows $1, 2, \dots, t$ are not altered. If P is the permutation matrix associated to the pivoting strategy and $B := PA$, then the Gauss elimination of B can be performed without row exchanges and we say that we have performed a *row pivoting strategy*. Finally, if $B = P^T AP$ we say that we have performed a *symmetric pivoting strategy*. In Section 4.2.9 of [8], symmetric pivoting strategies are applied to symmetric matrices. However, in this paper we can apply them to unsymmetric matrices.

§2. Row SPP strategies versus symmetric SPP strategies

A *row* (resp., *symmetric*) *scaled partial pivoting* strategy for a norm $\|\cdot\|$ consists of an implicit scaling by the norm $\|\cdot\|$ followed by partial (resp., symmetric and partial) pivoting. Let $r_i^{(t)}$ denote the ith row ($t \leq i \leq n$) of the submatrix $A^{(t)}[t, t+1, \ldots, n]$. For each t ($1 \leq t \leq n-1$), these strategies look for the first integer i_t ($t \leq i_t \leq n$) satisfying

$$\frac{|a_{i_t t}^{(t)}|}{\|r_{i_t}^{(t)}\|} = \max_{t \leq i \leq n} \frac{|a_{it}^{(t)}|}{\|r_i^{(t)}\|}$$

(resp.,

$$\frac{|a_{i_t i_t}^{(t)}|}{\|r_{i_t}^{(t)}\|} = \max_{t \leq i \leq n} \frac{|a_{ii}^{(t)}|}{\|r_i^{(t)}\|}).$$

We shall deal with monotone vector norms. As examples of monotone vector norms, we can consider the vector norms $\|\cdot\|_2, \|\cdot\|_1, \|\cdot\|_\infty$. In the particular case of $\|\cdot\|_2$, the associated SPP strategy for Gauss elimination is called Euclidean scaled partial pivoting (ESPP) and has a nice geometric interpretation remarked in [13]. This strategy leads to a triangular system where the hyperplane of $\mathbf{R^n}$ associated to its ith equation ($i = 1, 2, \ldots, n$) is well oriented with respect to the x_i-axis. We mean that, in step i, we select as the ith hyperplane the one which is the most orthogonal to the x_i-axis (observe that the strategy is based on direction cosines).

Let us compare row SPP and symmetric SPP strategies with respect to theoretical bounds for the growth factor. Given a matrix M, $|M|$ will denote the matrix whose entries are given by the absolute values of the entries of M. The growth factor is an indicator of the stability of Gauss elimination. Given an $n \times n$ nonsingular matrix A, let us consider the growth factor given by

$$\rho_n(A) := \frac{\||L|\,|U|\|_\infty}{\|A\|_\infty}, \tag{1}$$

where LU is the triangular factorization of the matrix $B = PAQ$ and P, Q are the permutation matrices associated to the pivoting strategy. Amodio and Mazzia (see [2] p. 398) introduced the number

$$\rho_n^N(A) := \frac{\max_t \|A^{(t)}\|_\infty}{\|A\|_\infty} \tag{2}$$

and have shown its nice behavior for the error analysis of Gauss elimination.

In Corollary 4.2 of [12] it was found an upper bound for the growth factor associated to row SPP strategy for $\|\cdot\|_1$: it satisfies $\rho_n^N(A) \leq 2^{n-1}$, analogously to the theoretical bound satisfied by partial pivoting (see [2]). The following example shows that, in contrast, the growth factor of symmetric SPP strategies can be arbitrarily large even for 2×2 matrices.

Example 1. Let us consider $\varepsilon > 0$, the matrix A and the upper triangular matrix U obtaining after applying Gauss elimination with any symmetric SPP strategy (which does not produce row and column exchanges):

$$A = \begin{pmatrix} \varepsilon & 1 \\ 1 & \varepsilon \end{pmatrix}, \quad U = \begin{pmatrix} \varepsilon & 1 \\ 0 & \varepsilon - 1/\varepsilon \end{pmatrix}.$$

Observe that $\rho_2^N(A) = \frac{(1/\varepsilon)-\varepsilon}{1+\varepsilon} = \frac{1-\varepsilon}{\varepsilon}$ is arbitrarily large.

Let us mention that we shall see at the end of this section that the growth factor of symmetric SPP strategies for random matrices is not as catastrophic as in the previous example.

In Theorem 2.2 of [9] it was proved that, given a nonsingular matrix A, if there exists a permutation matrix P such that the LU-factorization of the matrix $B = PA$ satisfies $|LU| = |L|\,|U|$, then P is associated with the row scaled partial pivoting for any strictly monotone vector norm. This can be used to derive nice backward error bounds (see [9]). This happens, for instance, with the class of sign-regular matrices, as shown in [10]. The following result shows that the growth factors defined in (1) and (2) are optimal under the previous hypothesis.

Proposition 1. *Let A be an $n \times n$ nonsingular matrix. If there exists a permutation matrix P such that the LU-factorization of the matrix PA satisfies $|LU| = |L|\,|U|$, then P is associated to the row SPP for a strictly monotone vector norm $\|\cdot\|$ and this strategy satisfies*

$$\rho_n(A) = 1, \qquad \rho_n^N(A) = 1.$$

Proof. The first part of the proposition is consequence of Theorem 2.2 of [9]. The result $\rho_n(A) = 1$ follows from the hypothesis $|LU| = |L|\,|U|$.

Since P is the permutation matrix associated to the row SPP strategy, the Gauss elimination of $B := PA$ can be performed without row exchanges and so, if $B = LU$ with L a lower triangular matrix with unit diagonal and U a nonsingular upper triangular matrix, then

$$B^{(t)}[t, \dots, n] = L[t, \dots, n]U[t, \dots, n]$$

and

$$B^{(t)}[1, \dots, t-1|1, \dots, n] = U[1, \dots, t-1|1, \dots, n].$$

From the previous formulas, we can conclude that $\|A^{(t)}\|_\infty = \|B^{(t)}\|_\infty \le \||L|\,|U|\|_\infty = \|A\|_\infty$ for all $t = 1, \dots, n-1$. Thus, $\rho_n^N(A) = 1$. □

An analogous result to Proposition 1 does not hold for symmetric SPP strategies.

Example 2. The following nonsingular matrix A has associated a permutation matrix P such that the LU-factorization of the matrix PAP^T satisfies $|PAP^T| = |L|\,|U|$:

$$A = \begin{pmatrix} 1 & 1 & 4 \\ 1/2 & 2 & 3 \\ 4 & 3 & 20 \end{pmatrix}, \ P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ L = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & 13/3 & 1 \end{pmatrix}, \ U = \begin{pmatrix} 2 & 1/2 & 3 \\ 0 & 3/4 & 5/2 \\ 0 & 0 & 14/3 \end{pmatrix}.$$

However, P is not associated to the symmetric SPP strategy for $\|\cdot\|_1$. This strategy is associated to the permutation matrix Q and it can also be checked that $|QAQ^T| \neq |\tilde{L}||\tilde{U}|$, where $\tilde{L}\tilde{U}$ is the LU-factorization of the matrix QAQ^T:

$$Q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ \tilde{L} = \begin{pmatrix} 1 & 0 & 0 \\ 3/20 & 1 & 0 \\ 1/5 & 8/31 & 1 \end{pmatrix}, \ \tilde{U} = \begin{pmatrix} 20 & 3 & 4 \\ 0 & 31/20 & -1/10 \\ 0 & 0 & 7/31 \end{pmatrix}.$$

As an application of Proposition 1, let us see that the pivoting strategy introduced in [10] for nonsingular sign regular matrices and called first-last pivoting presents optimal growth factors (1) and (2). Let us also recall that this pivoting strategy increases the computational cost of Gauss elimination in only $O(n)$ elementary operations.

Corollary 2. *Let A be an $n \times n$ nonsingular sign regular matrix. The growth factors (1) and (2) of the first-last pivoting satisfy*

$$\rho_n(A) = 1, \qquad \rho_n^N(A) = 1. \tag{3}$$

Proof. By Corollary 3.5 of [10], the permutation matrix P associated with the first-last pivoting strategy satisfies that PA admits an LU-decomposition $PA = LU$ with $|PA| = |L|\,|U|$. By Corollary 3.6 of [9], this matrix P coincides with the permutation matrix associated with any scaled partial pivoting strategy for a strictly monotone vector norm. Then, by Proposition 2.2, the growth factors (1) and (2) of the first-last pivoting satisfy (3). □

Another good property for backward stability is diagonal dominance. In fact, nice stability properties satisfied when the resultant matrix U is diagonally dominant by rows are described in [11]. In Section 3, we shall see that this happens when we apply symmetric SPP to a nonsingular M-matrix.

As commented in the introduction, SPP strategies present very good stability properties for some special classes of matrices capable of good properties in this sense. Here we analyze the behavior for random matrices. The behavior is worse than with partial pivoting but better than with other strategies considered in [17].

Stability of Gauss elimination with partial pivoting *on average* was analyzed through numerical experiments in [17]. In [5], the stability on average was studied for some pivoting strategies intermediate between partial pivoting and rook pivoting (see [15], [4]). Here we consider the stability on average of row SPP strategies. Following [17], we have considered matrices whose elements are independent samples of the standard normal distribution of mean 0 and variance 1. In the numerical experiments these $n \times n$ matrices are selected at random, with the sample size N diminishing with n to keep the computing time within reasonable bounds. A typical set of dimensions and sample sizes are listed below, although for some of our experiments the samples were larger:

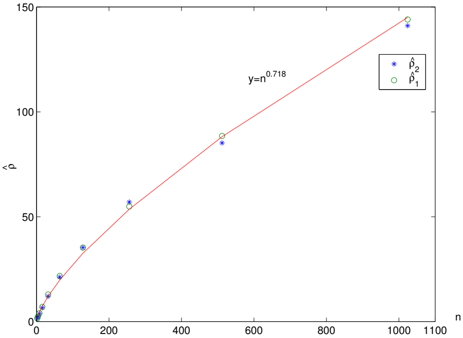
dimension n	2	4	8	16	32	64	128	256	512	1024
sample size N	4096	2048	1024	512	256	128	64	32	20	10

We also modify the classical definition of growth factor due to Wilkinson dividing by the standard deviation σ_A of the initial element distribution:

$$\hat{\rho} := \frac{\max_{i,j,k} |a_{ij}^{(k)}|}{\sigma_A},$$

which will be called the *average normalized growth factor*.

In [17] it was shown that the average normalized growth factor of the partial pivoting and complete pivoting for random $n \times n$ matrices was very close to $n^{2/3}$ and $n^{1/2}$, respectively. Now, let us show in Figure 1 and Table 1 the average normalized growth factor of some row scaled partial pivoting strategies: $\hat{\rho}_2$ (corresponding to row SPP for $\|\cdot\|_2$), $\hat{\rho}_1$ (corresponding to row SPP for $\|\cdot\|_1$) and $\hat{\rho}_\infty$ (corresponding to row SPP for $\|\cdot\|_\infty$). The calculations were performed with MATLAB.



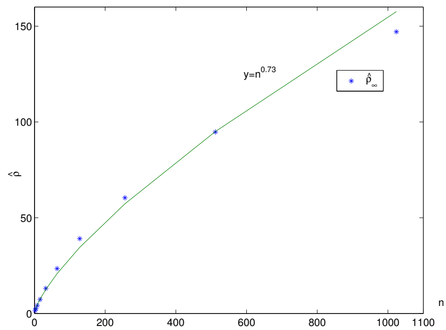


Figure 1: Approximations for $\hat{\rho}_2$ and $\hat{\rho}_1$ (left) and for $\hat{\rho}_\infty$ (right)

n	$\hat{\rho}_2$	$\hat{\rho}_1$	$n^{0.718}$
2	1.5695	1.5763	1.6449
4	2.4725	2.4966	2.7057
8	3.8001	4.0399	4.4506
16	6.6124	7.0371	7.3208
32	12.0579	12.9469	12.0420
64	21.1924	21.7841	19.8078
128	35.3150	35.2997	32.5819
256	57.0067	54.9150	53.5940
512	85.2017	88.5132	88.1568
1024	141.0891	144.0571	145.0091

n	$\hat{\rho}_\infty$	$n^{0.73}$
2	1.5534	1.6586
4	2.5188	2.7511
8	4.1648	4.5631
16	7.3439	7.5685
32	13.1250	12.5533
64	23.4446	20.8215
128	39.0697	34.5353
256	60.4346	57.2816
512	94.7647	95.0095
1024	147.0777	157.5865

Table 1: Approximations for $\hat{\rho}_2$ and $\hat{\rho}_1$ (left) and for $\hat{\rho}_\infty$ (right)

n	$\tilde{\rho}_1$	$n^{0.73}$	$\tilde{\rho}_2$	$n^{0.728}$	$\tilde{\rho}_\infty$	$n^{0.734}$
2	2.9984	1.6586	2.9867	1.6563	2.9397	1.6632
4	5.0854	2.7511	5.0893	2.7435	4.8774	2.7664
8	8.0621	4.5631	7.6822	4.5441	8.0890	4.6012
16	12.4662	7.5685	12.5019	7.5266	12.4775	7.6529
32	19.9420	12.5533	19.2773	12.4666	19.6611	12.7286
64	29.4259	20.8215	29.7842	20.6490	28.8675	21.1707
128	47.7391	34.5353	45.2706	34.2018	48.2033	35.2121
256	75.1349	57.2816	73.1623	56.6498	72.1990	58.5663
512	107.4418	95.0095	104.2726	93.8315	101.2777	97.4101
1024	139.9168	157.5865	139.6121	155.4169	149.4781	162.0168

Table 2: Approximation for $\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_\infty$

In Figure 1, we observe a very slightly better behavior in the cases of norms $\|\cdot\|_1$ and $\|\cdot\|_2$ than with $\|\cdot\|_\infty$, and slightly worse bounds than with partial pivoting. The average normalized growth factor can be approximated by $n^{0.718}$ for the two first norms and by $n^{0.73}$ for $\|\cdot\|_\infty$.

Now, let us calculate the average normalized growth factor of some symmetric scaled partial pivoting strategies: $\tilde{\rho}_1$ (corresponding to symmetric SPP for $\|\cdot\|_1$), $\tilde{\rho}_2$ (corresponding to symmetric SPP for $\|\cdot\|_2$) and $\tilde{\rho}_\infty$ (corresponding to symmetric SPP for $\|\cdot\|_\infty$). The results are given in Table 2. We note that, in the numerical experiments with symmetric scaled partial pivoting, we have refused the test matrices that have some submatrix $A^{(k)}[k,\dots,n]$ of their elimination process with null diagonal.

In Tables 1 and 2, we also observe that the behavior of the average normalized growth factor for SPP strategies is nice as we previously expected because of the introduction comments for matrices with uniform elements (see Rice (see [16] p. 44) and Poole and Neal [13]). However, if we analyze the obtained approximations of the average normalized growth factor, then we can say that row SPP strategies work better than symmetric SPP strategies for random matrices.

§3. An economic implementation of symmetric SPP for nonsingular M-matrices

In general, a disadvantage of SPP versus PP is the computational cost because SPP pivoting strategies require $O(n^3)$ (instead of $O(n^2)$) elementary operations in addition to the cost of Gauss elimination. However, for special classes of matrices SPP strategies can require lower computational cost. This already happened with the class of sign regular matrices, for which an implementation of row SPP for $\|\cdot\|_1$ with less computational cost than PP was presented in [10]. This section is devoted to the important class of M-matrices.

A nonsingular $n\times n$ matrix A is an *M-matrix* if it has positive diagonal entries, nonpositive off-diagonal entries and A^{-1} is nonnegative. M-matrices have very important applications, for instance, in iterative methods in numerical analysis, in the analysis of dynamical systems, in economics and in mathematical programming. Let us see that we can implement for a nonsingular $n \times n$ M-matrix the symmetric SPP for $\|\cdot\|_1$ increasing the computational cost of Gauss elimination in only $O(n^2)$ elementary operations, and that the corresponding growth factor (2) is optimal. In Proposition 4.7 of [11], a similar computational cost was obtained but with a pivoting strategy which was not a SPP strategy.

Theorem 3. *Let A be a nonsingular $n\times n$ M-matrix and let us consider the linear system $Ax = b$. Then the symmetric scaled partial pivoting for the norm $\|\cdot\|_1$ leads to an upper triangular matrix diagonally dominant by rows and can be implemented with a computational cost which adds $\frac{3}{2}(n^2-n)$ sums, $\frac{1}{2}(n^2-n)$ multiplications, $\frac{1}{2}(n^2-n)$ divisions and $\frac{1}{2}(n^2+n)$ comparisons to the computational cost of Gauss elimination without row or column exchanges. Moreover, the growth factor (2) of this strategy satisfies that*

$$\rho_n^N(A) = 1. \tag{4}$$

Proof. By Proposition 4.3 (i) of [11] and Proposition 4.5 of [11], symmetric SPP for $\|\cdot\|_1$ applied to a nonsingular M-matrix leads to an upper triangular matrix U diagonally dominant by rows. Then, by Proposition 3.1 of [11], $\|A^{(t)}\|_\infty \le \|A\|_\infty$ for all $t \in \{1,\ldots,n\}$. So, (4) holds.

For each t ($1 \le t \le n-1$), the symmetric scaled partial pivoting for $\|\cdot\|_1$ chooses as pivot of the tth step the first integer i_t ($t \le i_t \le n$) such that

$$|a_{i_t i_t}^{(t)}|/(\textstyle\sum_{j\ge t} |a_{i_t j}^{(t)}|) = \max_{t\le i\le n}(|a_{ii}^{(t)}|/(\textstyle\sum_{j\ge t}|a_{it}^{(t)}|)).$$

Let us recall that if we perform a row permutation and the same column permutation in a nonsingular M-matrix we again obtain a nonsingular M-matrix and that the Schur complements of nonsingular M-matrices are also M-matrices (cf. [6]). So, when applying a symmetric pivoting strategy to a nonsingular M-matrix A the resulting matrices $A^{(t)}[t,\ldots,n]$ are also nonsingular M-matrices and have positive diagonal entries. Clearly, i_t is also the first integer between t and n such that

$$\frac{\sum_{j\ge t}|a_{i_t j}^{(t)}|}{a_{i_t i_t}^{(t)}} = \min_{t\le i\le n}\frac{\sum_{j\ge t}|a_{ij}^{(t)}}{a_{ii}^{(t)}} = \min_{t\le i\le n}\left(1+\frac{\sum_{j\ge t}^{j\ne i}|a_{ij}^{(t)}|}{a_{ii}^{(t)}}\right),$$

which in turn coincides with the first integer between t and n such that

$$1-\frac{\sum_{j\ge t}^{j\ne i_t}|a_{ij}^{(t)}|}{a_{i_t i_t}^{(t)}} = \max_{t\le i\le n}\left(1-\frac{\sum_{j\ge t}^{j\ne i}|a_{ij}^{(t)}|}{a_{ii}^{(t)}}\right) = \max_{t\le i\le n}\frac{a_{ii}^{(t)}-\sum_{j\ge t}^{j\ne i}|a_{ij}^{(t)}|}{a_{ii}^{(t)}}. \tag{5}$$

Taking into account that each matrix $A^{(t)}[t,\ldots,n]$ is a nonsingular M-matrix, we know that it has positive diagonal elements and nonpositive off-diagonal entries and so we conclude from (5) that i_t is also the first integer between t and n such that

$$\frac{\sum_{j\ge t}a_{i_t j}^{(t)}}{a_{i_t i_t}^{(t)}} = \max_{t\le i\le n}\frac{\sum_{j\ge t}a_{ij}^{(t)}}{a_{ii}^{(t)}}. \tag{6}$$

It remains to see that we can calculate the indices i_t ($1 \leq t \leq n-1$) with the number of additional elementary operations mentioned above. Let $e := (1, \ldots, 1)^T$ and $z := Ae$. As usual, we also denote $A^{(1)} := A$, $z^{(1)} := z$. By (6) for $t = 1$, the first index i_1 such that

$$\frac{z_{i_1}}{a_{i_1 i_1}} = \max_{t \leq i \leq n} \frac{z_i}{a_{ii}}$$

determines the pivot row i_1 and the permutation matrix P_1 such that $\tilde{A}^{(1)} = P_1^T A P_1$. The solution of the augmented matrix $(\tilde{A}^{(1)}; P_1^T b, P_1^T z)$ is $(P_1^T x, e)$. If we perform one step of Gauss elimination we arrive at the augmented matrix $(A^{(2)}; b^{(2)}, z^{(2)})$ and we have that $A^{(2)} e = z^{(2)}$. Then, by (6) for $t = 2$, the first index $i_2 \in \{2, \ldots, n\}$ such that

$$\frac{z_{i_2}^{(2)}}{a_{i_2 i_2}^{(2)}} = \max_{2 \leq i \leq n} \frac{z_i^{(2)}}{a_{ii}^{(2)}}$$

determines the pivot row i_2. Iterating this procedure, we conclude that the computational cost of the pivoting strategy corresponds to the extra calculations for obtaining the right side z (given by the row sums of A), for transforming it by Gauss elimination into

$$z^{(2)}[2, \ldots, n], \ldots, z^{(n-1)}[n-1, n],$$

for calculating the quotients $z_i^{(k)}/a_{ii}^{(k)}$ ($k = 1, \ldots, n-1$) and for choosing the largest component

$$\frac{z_{i_k}^{(k)}}{a_{i_k i_k}^{(k)}} = \max_{k \leq i \leq n} \frac{z_i^{(k)}}{a_{ii}^{(k)}}$$

in each step k. □

Let us observe that, in many applications (as shown in [1]), the row sums (that is, the vector z of the proof of the previous theorem) are natural parameters. In this case, we even can reduce the computational cost of the pivoting strategy in $n^2 - n$ sums.

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