Monografías Matemáticas *García de Galdeano* **35**, 91–98 (2010)

Optimal bases of spaces with trigonometric functions

J. M. Carnicer, E. Mainar and J. M. Peña

Abstract. The normalized B-basis of a space has optimal shape preserving properties. We present a procedure to construct the normalized B-basis. We illustrate this construction in the space $\bar{T}_{1/2}$ generated by 1, t, $\cos t$, $\sin t$, $\cos(t/2)$, $\sin(t/2)$. This space can be used to represent exactly the following remarkable curves: complete cycloidal arcs, cardioids, deltoids and Descartes' trifolium.

Keywords: Normalized B-basis, trigonometric functions, shape preserving representations.

AMS classification: 65D17, 42A10.

§1. Introduction

The Bernstein basis is optimal among all other shape preserving bases of the space of polynomials of degree not greater than n on a given compact interval [1]. Roughly speaking, this means that the curve represented by this basis is closer to its control polygon than with other kinds of representations for polynomial curves. In [2] it was proved that each space of functions admitting shape preserving representations (in the sense of [5]) always has an optimal shape preserving basis called *the normalized B-basis*. For polynomial curves, the shape preserving representations exist on intervals of any length. However, for more general spaces interesting in curve design, it is not clear whether there exist shape preserving representations on a given interval (see [6]). In the last years, a growing interest in the design of curves in spaces mixing algebraic, trigonometric and hyperbolic functions has arisen.

It is desirable to represent motions of objects with its natural velocity, which eliminates the freedom in the parameterization. In particular, in order to obtain uniform circular motions, it is necessary to represent a circle with its arc length parameterization. It is also convenient that all types of curves which have to be used in the design process can be obtained with the same kind of representation. This may imply to use a representation of curves in a computer graphics system working simultaneously with algebraic and transcendent curves.

In [4] we have found all six dimensional spaces invariant under translations and reflections containing the first degree polynomials and the trigonometric functions $\cos t$, $\sin t$ and admitting shape preserving representations on the interval $[0, 2\pi]$ (this implies that a complete circular arc can be represented using a single control polygon). Among these spaces, we find spaces mixing trigonometric functions with two angular frequencies

$$\bar{T}_w := \langle 1, t, \cos t, \sin t, \cos(wt), \sin(wt)\rangle, \quad 0 < w < 1.$$

We have chosen the space $\bar{T}_{1/2}$ to show how to obtain the optimal bases. This space allows us to represent exactly not only circles but also complete cycloidal arcs, cardioids, deltoids and Descartes' trifolium among other remarkable curves.

Section 2 presents the procedure to obtain the normalized B-basis of a space. Section 3 is devoted to the construction of the normalized B-basis of the space $\bar{T}_{1/2}$ and the obtention of control polygons for the representation of remarkable curves.

§2. Construction of the optimal shape preserving basis

Let us recall that an *extended Chebyshev space* of functions F defined on an interval I is a space such that each nonzero function of F has at most $\dim F - 1$ zeros (counting multiplicities) in I. In order to check the existence of normalized B-bases, we shall use Theorem 4.1 of [3] restated below.

Theorem 1. *Let F be an $(n+1)$-dimensional subspace of $C^n[a,b]$ such that $1 \in F$. Then F is an extended Chebyshev space with a normalized B-basis on $[a,b]$ if and only if the space of the derivatives*

$$F' := \{f' \mid f \in F\}$$

is an extended Chebyshev space.

Assuming that we have shown the existence of a normalized B-basis, we may proceed to its construction. First we construct a B-basis following the method suggested in Remark 2.3 and Theorem 2.4 of [3] and then, we shall normalize it following Remark 4.1 of [3]. Let us describe the steps of this construction.

Step 1. We start with a basis such $(u_0, \dots, u_n)$ such that the wronskian matrix at the left end of the interval

$$W(u_0, \dots, u_n)(a) = \begin{pmatrix} u_0(a) & u_1(a) & \cdots & u_n(a) \\ u_0'(a) & u_1'(a) & \cdots & u_n'(a) \\ \vdots & \vdots & \ddots & \vdots \\ u_0^{(n)}(a) & u_1^{(n)}(a) & \cdots & u_n^{(n)}(a) \end{pmatrix}$$

is a lower triangular matrix with nonzero diagonal entries.

Step 2. We compute $W(u_n, \dots, u_0)(b)$, the wronskian matrix of the basis $(u_n, \dots, u_0)$, where the ordering of the functions has been reversed, at the right end of the interval and obtain its LU factorization with L a lower triangular matrix with unit diagonal and U a nonsingular upper triangular matrix.

Step 3. We construct the basis $(b_0, \dots, b_n)$ defined by

$$(b_n, -b_{n-1}, \dots, (-1)^n b_0) := (u_n, u_{n-1}, \dots, u_0) U^{-1}.$$

This basis is a Bernstein-like basis in the sense that $W(b_0, \dots, b_n)(a)$ and $W(b_n, \dots, b_0)(b)$ are lower triangular matrices.

Step 4. In order to normalize the obtained basis, we solve the linear system

$$L(c_n, c_{n-1}, \dots, c_0)^T = (1, 0, \dots, 0)^T,$$

and then the normalized B-basis is

$$(B_0, \dots, B_n) := (c_0 b_0, \dots, c_n b_n).$$

Remark 1. Since the space is invariant under reflections we have for the functions of the normalized B-basis

$$B_i(t) = B_{n-i}(a + b - t), \quad t \in [a, b], \quad i = 0, \dots, n.$$

So we only need to compute half of the basis functions B_i, $0 \leq i \leq n/2$.

§3. Designing with two angular frequencies

This section is devoted to the design of curves in the space

$$\bar{T}_{1/2} = \langle 1, t, \cos t, \sin t, \cos(t/2), \sin(t/2) \rangle,$$

on the interval $t \in [0, 2\pi]$. First we shall find the optimal basis (normalized B-basis) of $\bar{T}_{1/2}$ on $[0, 2\pi]$ and later we shall use it for the design of some remarkable curves.

It is a well-known fact that the space $\langle 1, \cos s, \sin s, \cos(2s), \sin(2s) \rangle$ of trigonometric polynomials of degree 2 is an extended Chebyshev space on $[0, 2\pi)$, that is, any nonzero function of the space has at most $\dim \bar{T}'_{1/2} - 1 = 4$ zeros (counting multiplicities).

By Theorem 1, there exists a normalized B-basis on the space $\bar{T}_{1/2}$ on $[0, 2\pi]$ if and only if the space of the derivatives

$$\bar{T}'_{1/2} = \langle 1, \cos t, \sin t, \cos(t/2), \sin(t/2) \rangle$$

is an extended Chebyshev space. Taking $s = t/2$, the space is transformed into the space of trigonometric polynomials of degree 2 on the interval $[0, \pi]$. As mentioned above, this space is extended Chebyshev on each interval contained in $[0, 2\pi)$.

Once we have shown the existence of a normalized B-basis, we proceed to its construction following the steps described in Section 2.

Step 1. We start with the basis $(u_0, \dots, u_5)$ given by

$$\left(1, t, 1 - \cos t, t - \sin t, 4 - \frac{16}{3}\cos(t/2) + \frac{4}{3}\cos t, 3t - 8\sin(t/2) + \sin t\right),$$

$t \in [0, 2\pi]$, whose wronskian matrix at $t = 0$

$$W(u_0, \dots, u_5)(0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 3/4 \end{pmatrix}$$

is lower triangular with positive diagonal entries.

Let us observe that $u_4(t)$ can be factorized as $u_4(t) = \frac{8}{3}\left(1 - \cos(t/2)\right)^2$.

Step 2. We evaluate the wronskian matrix at $t = 2\pi$

$$W(u_0,\dots,u_5)(2\pi) = \begin{pmatrix} 1 & 2\pi & 0 & 2\pi & 32/3 & 6\pi \\ 0 & 1 & 0 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & -8/3 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 5/3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 5/4 \end{pmatrix},$$

reverse the columns and compute the factorization, $W(u_5,\dots,u_0)(2\pi) = LU$ obtaining

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 4\pi^{-1}/3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3\pi/16 & 1 & 0 & 0 & 0 \\ -\pi^{-1}/3 & -1/4 & 2\pi^{-1} & 1 & 0 & 0 \\ 0 & -15\pi/128 & -5/8 & 3\pi/16 & 1 & 0 \\ 5\pi^{-1}/24 & 5/32 & -2\pi^{-1} & -1 & 4\pi^{-1}/3 & 1 \end{pmatrix},$$

$$U = \begin{pmatrix} 6\pi & 32/3 & 2\pi & 0 & 2\pi & 1 \\ 0 & -128\pi^{-1}/9 & -8/3 & 0 & -5/3 & -4\pi^{-1}/3 \\ 0 & 0 & \pi/2 & 1 & 5\pi/16 & 1/4 \\ 0 & 0 & 0 & -2\pi^{-1} & -3/8 & -\pi^{-1}/2 \\ 0 & 0 & 0 & 0 & 9\pi/128 & 3/32 \\ 0 & 0 & 0 & 0 & 0 & -\pi^{-1}/8 \end{pmatrix}.$$

Step 3. In order to construct the basis $(b_0,\dots,b_5)$ defined by

$$(b_5, -b_4, b_3, -b_2, b_1, -b_0) = (u_5, u_4, u_3, u_2, u_1, u_0)U^{-1},$$

we compute

$$U^{-1} = \begin{pmatrix} \pi^{-1}/6 & 1/8 & 0 & 0 & -16\pi^{-1}/9 & -4/3 \\ 0 & -9\pi/128 & -3/8 & -3\pi/16 & -1 & 0 \\ 0 & 0 & 2\pi^{-1} & 1 & -32\pi^{-1}/9 & -8/3 \\ 0 & 0 & 0 & -\pi/2 & -8/3 & 0 \\ 0 & 0 & 0 & 0 & 128\pi^{-1}/9 & 32/3 \\ 0 & 0 & 0 & 0 & 0 & -8\pi \end{pmatrix}.$$

Then we have

$$\begin{aligned} b_5(t) &:= \frac{1}{6\pi}(3t - 8\sin(t/2) + \sin t), \\ b_4(t) &:= \frac{1}{8}(3t - 8\sin(t/2) + \sin t) - \frac{3\pi}{16}(1 - \cos(t/2))^2, \\ b_3(t) &:= \frac{2}{\pi}(t - \sin t) - (1 - \cos(t/2))^2. \end{aligned}$$

Step 4. Solving the system

$$L(c_5, c_4, c_3, c_2, c_1, c_0)^T = (1, 0, 0, 0, 0, 0)^T,$$

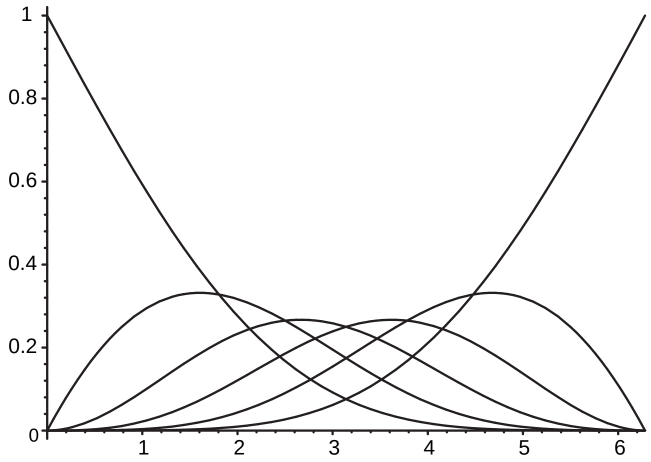

Figure 1: Normalized B-basis of $\bar{T}_{1/2}$ on $[0, 2\pi]$.

we obtain $c_5 = 1$, $c_4 = -4\pi^{-1}/3$, $c_3 = 1/4$ and then the normalized B-basis $(B_0, \ldots, B_5)$ is given by

$$B_5(t) := \frac{1}{6\pi}(3t - 8\sin(t/2) + \sin t),$$
$$B_4(t) := \frac{-1}{6\pi}(3t - 8\sin(t/2) + \sin t) + \frac{1}{4}(1 - \cos(t/2))^2,$$
$$B_3(t) := \frac{1}{2\pi}(t - \sin t) - \frac{1}{4}(1 - \cos(t/2))^2,$$

and by Remark 1, we obtain the remaining basis functions

$$B_2(t) := B_4(2\pi - t),\ B_1(t) := B_4(2\pi - t),\ B_0(t) := B_5(2\pi - t).$$

Figure 1 shows the graphs of the functions of the normalized B-basis of $\bar{T}_{1/2}$ on $[0, 2\pi]$.

Now we are going to obtain control polygons of different curves. For this purpose, we need the coefficients of some functions with respect to the normalized B-basis. The coefficients of the function t, shown in Table 1, are called the Greville abscissae and are used for obtaining the control polygon

$$\begin{pmatrix}0\\c_0\end{pmatrix}\begin{pmatrix}3\pi/4\\c_1\end{pmatrix}\begin{pmatrix}3\pi/4\\c_2\end{pmatrix}\begin{pmatrix}5\pi/4\\c_3\end{pmatrix}\begin{pmatrix}5\pi/4\\c_4\end{pmatrix}\begin{pmatrix}2\pi\\c_5\end{pmatrix}$$

of the graph of $f(t) = \sum_{i=0}^{5} c_i B_i(t)$.

Table 1 contains the coefficients of the usual trigonometric functions $\cos t, \sin t$, the cycloidal cosine, $1-\cos t$, and the cycloidal sine, $t-\sin t$, with respect to the normalized B-basis.

Figure 2 (left) shows a circle $(\sin t, 1 - \cos t)$, $t \in [0, 2\pi]$, and its control polygon

$$\begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}3\pi/4\\0\end{pmatrix}\begin{pmatrix}3\pi/4\\4\end{pmatrix}\begin{pmatrix}-3\pi/4\\4\end{pmatrix}\begin{pmatrix}-3\pi/4\\0\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}.$$

Function	c_0	c_1	c_2	c_3	c_4	c_5
1	1	1	1	1	1	1
t	0	$3\pi/4$	$3\pi/4$	$5\pi/4$	$5\pi/4$	2π
$\cos t$	1	1	-3	-3	1	1
$\sin t$	0	$3\pi/4$	$3\pi/4$	$-3\pi/4$	$-3\pi/4$	0
$1-\cos t$	0	0	4	4	0	0
$t-\sin t$	0	0	0	2π	2π	2π
$\sin(t/2)$	0	$3\pi/8$	$3\pi/8$	$3\pi/8$	$3\pi/8$	0
$\cos(t/2)$	1	1	0	0	-1	-1
$(1+\cos t)/2$	1	1	-1	-1	1	1

Table 1: Coefficients of relevant functions in $\bar{T}_{1/2}$

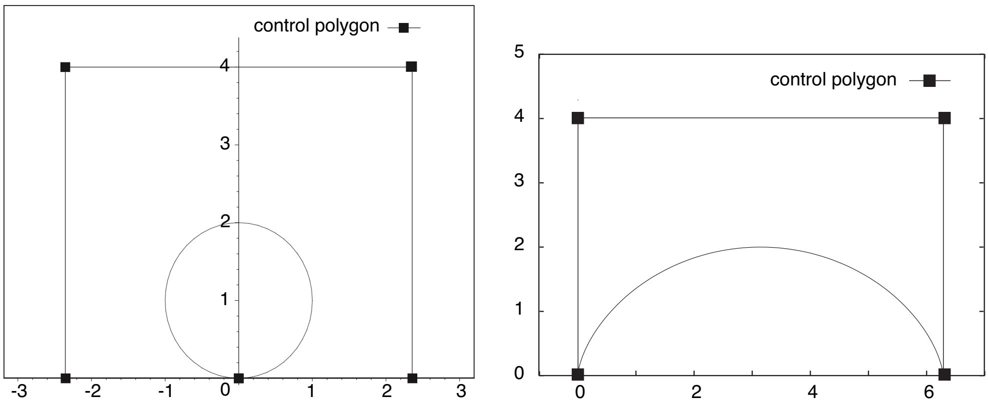

Figure 2: Control polygon of a circle (left) and a complete cycloid arc (right) in $\bar{T}_{1/2}$

Figure 2 (right) shows the cycloid $(t-\sin t, 1-\cos t)$, $t \in [0, 2\pi]$, and its control polygon

$$\begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}0\\4\end{pmatrix}\begin{pmatrix}2\pi\\4\end{pmatrix}\begin{pmatrix}2\pi\\0\end{pmatrix}\begin{pmatrix}2\pi\\0\end{pmatrix}.$$

Quadratic curves can also be represented in this space. In fact, the parabola $y = x^2$, $x \in [-1, 1]$, can be represented in this space by the parametric curve $(\cos(t/2), (1+\cos t)/2)$, $t \in [0, 2\pi]$, which, in view of Table 1, has the following control polygon

$$\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}0\\-1\end{pmatrix}\begin{pmatrix}0\\-1\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix}.$$

Other remarkable parametric curves which can be represented in this space are

Cardioid: $(a(2\cos(t/2)+1+\cos t), a(2\sin(t/2)+\sin t))$,

Deltoid: $(a(2\cos(t/2)+\cos t), a(2\sin(t/2)-\sin t))$,

Trifolium: $(a\cos(3t/4)\cos(t/4), a\cos(3t/4)\sin(t/4))$.

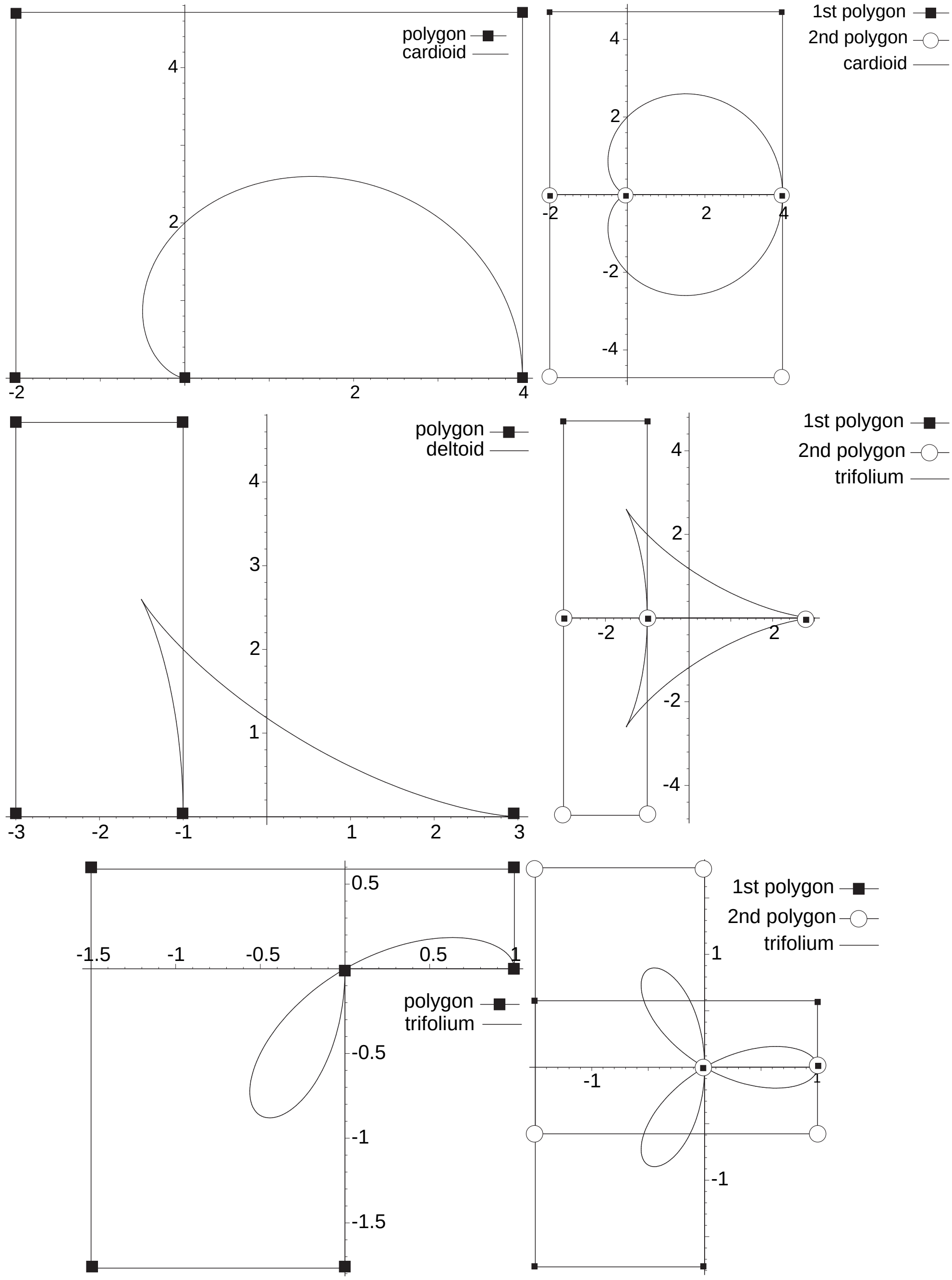

Figure 3: Cardioid, Deltoid and Trifolium

Let us observe that these curves are defined on the parameter interval $[0,4\pi]$. The representation of $(x(t), y(t))$, $t \in [0,4\pi]$, $x, y \in \bar{T}_{1/2}$, requires two control polygons. The first one is needed to represent the curve $(x(t), y(t))$, $t \in [0,2\pi]$, and the second one to represent $(x(t+2\pi), y(t+2\pi))$, $t \in [0,2\pi]$. Since the space $\bar{T}_{1/2}$ is invariant under translations, the functions $x(t+2\pi), y(t+2\pi)$, $t \in [0,2\pi]$, also belong to $\bar{T}_{1/2}$. In view of the symmetry of these curves, $x(4\pi - t) = x(t)$, $y(4\pi - t) = -y(t)$, we have that the control polygon $\tilde{P}_0 \cdots \tilde{P}_5$, $\tilde{P}_i = (\tilde{x}_i, \tilde{y}_i)$, $i = 0, \ldots, 5$, of the curve $x(t+2\pi), y(t+2\pi)$, $t \in [0,2\pi]$, can be expressed in terms of the control polygon $P_0 \cdots P_5$, $P_i = (x_i, y_i)$, $i = 0, \ldots, 5$, of $x(t), y(t)$, $t \in [0,2\pi]$, by

$$\tilde{x}_i = x_{5-i}, \quad \tilde{y}_i = -y_{5-i}, \quad i = 0, \ldots, 5.$$

Figure 3 shows the cardiod (top), deltoid (middle) and Descartes' trifolium (bottom), corresponding to the parameter value $a = 1$ and their corresponding couple of control polygons with respect to the normalized B-basis of $\bar{T}_{1/2}$.

Acknowledgements

Partially supported by the Spanish Research Grant MTM2006-03388 and by Gobierno de Aragón

References

[1] Carnicer, J. M., Mainar, E., and Peña, J. M. Shape preserving representations and optimality of the Bernstein basis. *Adv. Comput. Math. 1* (1993), 173–196.

[2] Carnicer, J. M., Mainar, E., and Peña, J. M. Totally positive bases for shape preserving curve design and optimality of B-splines. *Comput. Aided Geom. Design 11* (1994), 635–656.

[3] Carnicer, J. M., Mainar, E., and Peña, J. M. Critical length for design purposes and extended chebyshev spaces. *Constr. Approx. 20* (2004), 55–71.

[4] Carnicer, J. M., Mainar, E., and Peña, J. M. Shape preservation regions for six-dimensional spaces. *Adv. Comput. Math. 26* (2007), 121–136.

[5] Goodman, T. N. T. Shape preserving representations. In *Mathematical Methods in CAGD* (Boston, 1989), Academic Press, pp. 333–351.

[6] Peña, J. M. Shape preserving representations for trigonometric polynomial curves. *Comput. Aided Geom. Design 14* (1997), 5–11.

J. M. Carnicer and J. M. Peña
Departamento de Matemática Aplicada/IUMA
Universidad de Zaragoza
50009 Zaragoza, Spain
carnicer@unizar.es, jmpena@unizar.es

E. Mainar
Departamento de Matemáticas, Estadística y Computación
Universidad de Cantabria
39005 Santander, Spain
mainare@unican.es