Monografías Matemáticas *García de Galdeano* **35**, 83–90 (2010)

ERROR GROWTH IN THE NUMERICAL INTEGRATION OF PERIODIC ORBITS WITH PROJECTION METHODS

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Abstract. The aim of this work is to show that, when projection techniques are used in connection with Runge–Kutta (RK) methods to preserve first integrals of some periodic differential systems, the global error of the numerical solution presents a linear growth, even though the integration advances with a variable stepsize strategy.

Keywords: initial value problems, periodic solutions, Runge–Kutta methods, projections methods, error growth, preservation of first integrals.
AMS classification: 65L05, 65L06.

§1. Introduction

In this paper, autonomous initial value problems of the form

$$x'(t) = f(x(t)), \tag{1}$$
$$x(t_0) = x_0 \in \mathbb{R}^m, \tag{2}$$

which possess a unique periodic solution $x = \varphi(t; x_0)$ with period $T_0 > 0$, are considered. The function f is supposed as smooth as necessary.

Let φ_t be the t-flow map of (1), and ψ_h the function that defines a smooth one-step method to solve numerically (1)–(2). Thus, we obtain approximations x_n to the exact solution of this initial value problem at the gridpoints $t_n = t_{n-1} + h_{n-1}$:

$$x_n = \psi_{h_{n-1}}(x_{n-1}) = \psi_{h_{n-1}} \circ \cdots \circ \psi_{h_0}(x_0) \simeq \varphi_{t_n}(x_0), \quad n = 1, 2, 3, \dots,$$

where $h_0, h_1, h_2, \dots$ is a sequence of positive stepsizes.

B. Cano and J. M. Sanz-Serna have considered in [5] one-step methods ψ_h for the numerical integration of (1)–(2) satisfying the following conditions:

(i) ψ_h is defined in $\mathbb{R}^m$ for $|h| \le |h_0|$ for some $h_0 > 0$.

(ii) $\psi_h(x)$ depends smoothly on h and x.

(iii) The method ψ_h is consistent of order $r \ge 1$, r integer:

$$\psi_h(x) - \varphi_h(x) = O(h^{r+1}), \ h \to 0.$$

(iv) The Jacobian matrices satisfy

$$\psi'_h(x) - \varphi'_h(x) = O(h^{r+1}), \ h \to 0.$$

(v) The stepsizes are determined by

$$h_n = h\, s(x_n, h), \quad h > 0, \quad n = 0, 1, 2, \ldots,$$

where $s(x, h)$ is a smooth real-valued function such that

$$s_{\min} \leq s(x, h) \leq s_{\max},$$

for suitable positive constants $s_{\min}$ and $s_{\max}$.

Under these five assumptions, the authors prove in [5] the following asymptotic expansion in powers of h for the global error:

$$x_n - \varphi(t_n; x_0) = h^r e_r(t_n) + \cdots + h^{2r-1} e_{2r-1}(t_n) + h^{2r-1} R(t_n, h),\ h \to 0,$$

where the error functions $e_k(t)$ satisfy non-homogeneous variational equations of (1) with respect to $\varphi(t; x_0)$, and $R(t, h) \to 0$ as $h \to 0$ in bounded time intervals. The authors also prove in [5] that these error functions at integer multiples of the period, $e_k^{(N)} = e_k(NT_0)$, $r \leq k \leq 2r - 1$, satisfy:

$$e_k^{(N)} = \left(\sum_{i=0}^{N-1} M_{t_0}^i \right) e_k^{(1)}, \quad N = 1, 2, \ldots,$$

where $M_{t_0} = M(t_0 + T_0, t_0)$ is the monodromy matrix associated to the T_0-periodic solution $\varphi(t; x_0)$.

In particular, M. P. Calvo and J. M. Sanz-Serna had shown in [4] that integrating elliptic orbits in the two-body problem with a symplectic method using a constant stepsize policy, the global error grows linearly with the number of periods. They point out that such study is extensible to periodic Hamiltonian problems whose period depends only on the energy.

In this article we present a study of the growth of the global error integrating periodic differential systems (not necessarily Hamiltonian) so that the periodic orbit is embedded into a family of periodic orbits. We present some numerical experiments over this kind of problems with projection Runge–Kutta (RK) methods.

§2. Error behaviour

We assume the following hypothesis (H) for the differential system (1):

For all $\widetilde{x}_0$ in some neighbourhood of x_0, the solution of (1) with initial value $\widetilde{x}_0$ at time t_0, $\varphi(t; \widetilde{x}_0)$, is periodic with period $T = T(\widetilde{x}_0)$ where the function T is as smooth as required (H)

To integrate this kind of differential problems we consider one step methods satisfying the above assumptions (i)–(v). In [3], we show that the matrix M_0 can be written as

$$M_0 = I - f(x_0) \nabla T(x_0)^T.$$

This expression for M_0 allows to simplify the powers of this matrix, and we obtain that the global error coefficients after N periods satisfy

$$e_k^{(N)} = N e_k^{(1)} - \frac{N(N-1)}{2} f(x_0) \nabla T(x_0)^T e_k^{(1)}, \quad r \leq k \leq 2r - 1$$

Furthermore, in [3] we show that the global error of the numerical method after N periods can be written as:

$$\begin{aligned} ge^{(N)} &= Nge^{(1)} - \frac{N(N-1)}{2} f(x_0)\nabla T(x_0)^T ge^{(1)} - h^{2r-1} NR(T_0, h) \\ &\quad + h^{2r-1}\frac{N(N-1)}{2} f(x_0)\nabla T(x_0)^T R(T_0, h) + h^{2r-1} R(NT_0, h). \end{aligned} \tag{3}$$

As a consequence, we prove the following result:

Theorem 1. *Let us consider a differential system (1), (2) satisfying the hypothesis* (H)*, and an one-step method satisfying the conditions (i)–(v). Then, if the method preserves the period T up to order $O(h^{2r})$, the global error grows linearly on t, provided that Nh^r is small.*

Let us see now how the error of first integrals of the differential system behaves. Let $G(x)$ be an scalar first integral of (1). We denote by

$$\Delta^{(N)} G = G(x_0 + ge^{(N)}) - G(x_0), \quad N = 1, 2, \ldots,$$

the error in the invariant G for the method ψ_h after N periods. Since

$$\Delta^{(N)} G = \nabla G(x_0)^T ge^{(N)} + O\left(\|ge^{(N)}\|^2\right),$$

taking into account (3) we obtain:

Theorem 2. $\Delta^{(N)} G = N\, \Delta^{(1)} G + O(Nh^{2r-1}) + h^{2r-1}\nabla G(x_0)^T R(NT_0, h) + O\left(\|ge^{(N)}\|^2\right).$

This asymptotic relation implies a linear error growth in the invariant with the number of periods provided that $\|ge^{(N)}\|$ is not too large.

A Runge–Kutta method with a projection technique applied to (1) with $x(t_0) = u$, provides a numerical approximation $\widehat{\psi}_h(u)$ given by

$$\widehat{\psi}_h(u) = \psi_h(u) + \lambda(u, h) w(u, h),$$

where:

- ψ_h is an s-stage RK method of order r:

$$\begin{cases} \psi_h(u) = u + h \displaystyle\sum_{j=1}^{s} b_j\, f(U_j), \\ U_j = u + h \displaystyle\sum_{k=1}^{s} a_{jk} f(U_k), \quad (j = 1, \ldots, s), \end{cases}$$

- The coefficient $\lambda(u, h) \in \mathbb{R}$ is computed so that $\widehat{\psi}_h(u)$ preserves the invariant G of (1): $G(\widehat{\psi}_h(u)) = G(u)$.
- The vector $w(u, h) \in \mathbb{R}^m$ depends on the type of projection used. In this article, we will use directional projection [2], which takes $w = \psi_h - \widetilde{\psi}_h$, where $\widetilde{\psi}_h$ is a RK method of order $q < r$ embedded in ψ_h.

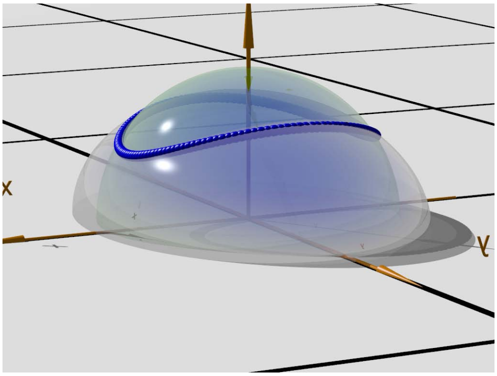

Figure 1: Solution of Euler's equations obtained with the projection dopri54 method

The results of the previous section on the growth of the errors are stated on the assumption that the one-step method ψ_h commutes with respect to differentiations with respect to the initial conditions, i.e. assumption (*iv*) is satisfied. It is known [1] that this relation holds for all the RK methods. We have proved in [3] that it is also satisfied for projection methods.

Next, we present some numerical experiments to corroborate the theoretical results presented in this article. The numerical method considered is the Runge–Kutta embedded pair of order 5(4) constructed by Dormand and Prince (see e.g. [7, p. 178]). We denote this pair "standard dopri54", whereas "projection dopri54" refers to that pair combined with the directional projection technique (see [2]) which makes the resulting method to preserve certain first integrals of the problem. Integrations have been carried out with a local error tolerance of 10^{-6}.

Our first test problem describes the motion of a free rigid body represented by the Euler's equations (see e.g. [6, p. 95]):

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 0 & c_3 y_3 & -c_2 y_2 \\ -c_3 y_3 & 0 & c_1 y_1 \\ c_2 y_2 & -c_1 y_1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

The vector $y = (y_1, y_2, y_3)^T$ is the angular momentum, and $c_j^{-1} > 0$, $j = 1, 2, 3$, are the principal momenta of inertia. This Poisson differential system has the two first integrals:

$$2H = c_1 y_1^2 + c_2 y_2^2 + c_3 y_3^2,$$
$$L^2 = y_1^2 + y_2^2 + y_3^2,$$

where H and L represent the kinetic energy and the modulus of the angular momentum, respectively.

We have taken $c_1 = 1$, $c_2 = 1 - 0.51/\sqrt{1.51}$, $c_3 = 1 + 1/\sqrt{1.51}$, with initial conditions:

$$y_1(0) = 0, \; y_2(0) = y_3(0) = 1.$$

The solution of this initial value problem is periodic, and its period $T = T(H, L)$ depends only on those two quadratic invariants. In Figure 1 we have plotted the numerical solution of this problem obtained with the projection dopri54 method. It lies on the intersection of the sphere $L^2(y_1, y_2, y_3) = 2$ with the ellipsoid $2H(y_1, y_2, y_3) = c_2 + c_3$.

In Figure 2, the Euclidean norm of the global error against the number of periods is shown in a log-log scale for the Euler's equations. The integration has been carried out up to 8000 periods. The projection dopri54 method has been designed so that it preserves the two invariants of the problem (see [2]) and, in consequence, it preserves its period. As it can be seen, the global error grows linearly with the number of periods for this projection method which is in agreement with Theorem 1. As expected, this growth is quadratic for the standard dopri54. Dashed straight lines with slopes $m = 1$ and $m = 2$ have been drawed in order to show up clearly the type of growth. In Figure 3, the preservation of the invariants is clear for the projected RK method, whereas the error of the invariant grows linearly for the standard one, which agrees with Theorem 2. Here the dashed reference line has slope 1.

The second test problem is the well known planar two body problem, also called Kepler's problem, given by the equations:

$$p_i' = -\frac{q_i}{(q_1^2 + q_2^2)^{3/2}}, \; q_i' = p_i, \; i = 1, 2.$$

This is a Hamiltonian system with Hamiltonian function given by

$$H = \frac{1}{2}(p_1^2 + p_2^2) - \frac{1}{\sqrt{q_1^2 + q_2^2}}.$$

We have considered the initial conditions

$$p_1 = 0, \; p_2 = \sqrt{\frac{1+e}{1-e}}, \; q_1 = 1 - e, \; q_2 = 0,$$

which correspond to a 2π-periodic elliptic orbit with eccentricity e, $0 \leq e < 1$. For these numerical experiments we have taken $e = 0.3$, and we have integrated along 8000 periods.

In Figure 4, the evolution of the global error against the number of periods is shown. In this case, the projection has been made so that the resulting projection method preserves the Hamiltonian H. Therefore, it also preserves the period since the period only depends on the energy H. According to Theorem 1, the growth of the global error must be linear in this case, and it is just what happens for the projection dopri54. Once again, the global error grows quadratically for the standard method. Figure 5 shows up the preservation of the first integral H for the projection method and the linear growth of H with the number of periods for the standard one.

Acknowledgements

This work was supported by D.G.I. project MTM2007-67530-C02-01.

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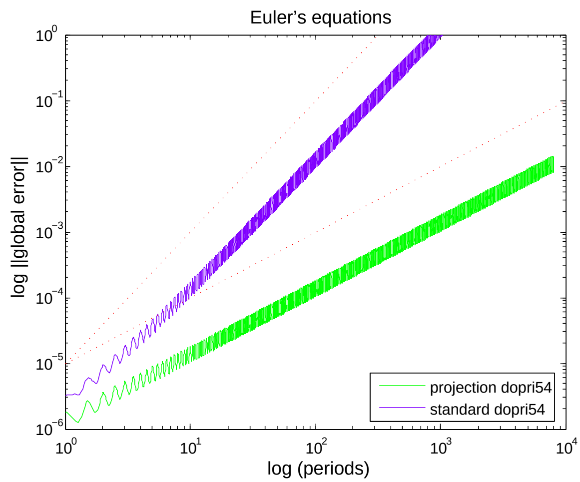

Figure 2: Euler's equations: global error vs. periods, tol=10^{-6}

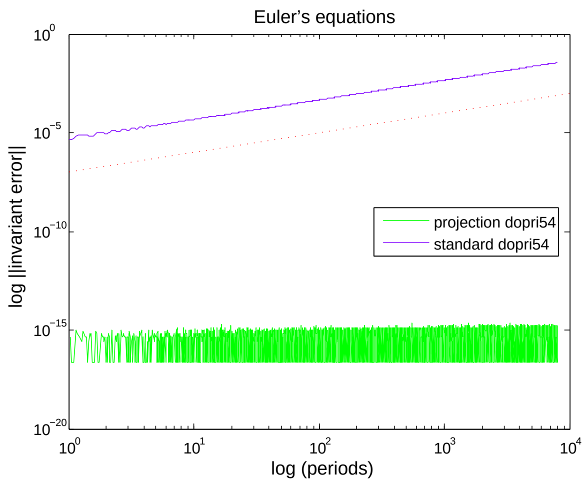

Figure 3: Euler's equations: invariants' error vs. periods, tol=10^{-6}

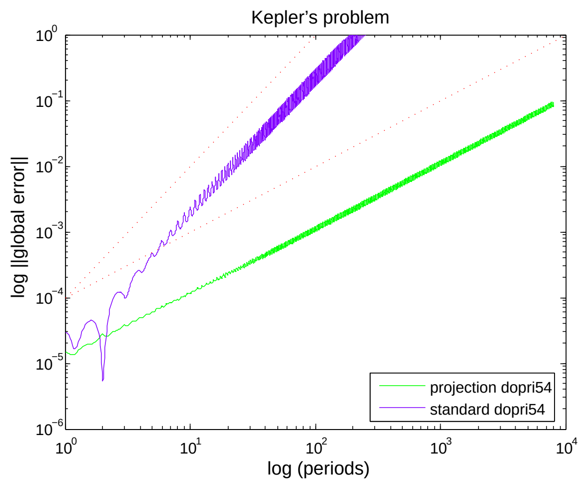

Figure 4: Kepler's problem: global error vs. periods, $e = 0.3$, tol=10^{-6}

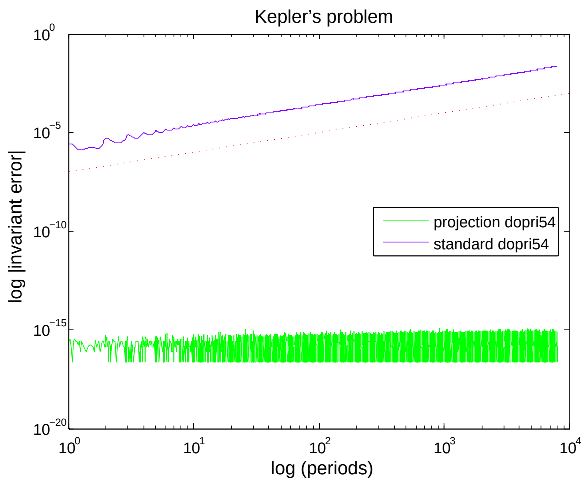

Figure 5: Kepler's problem: error in H vs. periods, $e = 0.3$, tol=10^{-6}

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