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ASYMPTOTIC KINETIC ENERGY CONSERVATION FOR LOW-MACH NUMBER FLOW COMPUTATIONS

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Abstract. Numerical dissipation, often used in collocated mesh schemes to enforce stability or to avoid odd-even decoupling problem, may be undesirable, for example to compute turbulent fluid flows in *DNS* or *LES*. Unfortunately, on the other hand, central discretization suffers from loss of stability problems, in particular when the Reynolds number increases. Therefore, an important attention has been devoted to find criteria that could ensure the stability without any addition of non-physical dissipation into the numerical schemes.

It appears experimentally that, for incompressible flow, the discrete kinetic energy conservation is one of these criteria. The present study deals with *(1)* the asymptotic signification of this conservation property in the incompressible limit of the compressible flow model; *(2)* the conditions under which the discrete kinetic energy is conserved in the sense previously evidenced; *(3)* the benefits that can be expected from this conservation property in the computations and the numerical problems that it does not prevent.

Keywords: Low-Mach number, kinetic energy conservation, Mach-uniformity, pressure correction, numerical dissipation.

AMS classification: 65N22, 76G25, 76M45, 76N15.

§1. Introduction

In the context of finite volume method, it has been proved recently in Georges *et al.* [1] that central interpolations ensure the discrete kinetic energy conservation in the "incompressible limit", which is in fact reduced to the condition $\mathrm{div}(v) = 0$ in this reference. Here, we are aiming to extend this result in the more general case of the incompressible limit in the *asymptotic* sense of this expression, that is, the limit of the compressible flow model when the reference Mach number of the flow goes to zero. Physically, this can be interpreted as the non-convertion of the kinetic energy into the elastic one allowing the propagation of sound waves.

First, following Nicoud [4], a single scale continuous asymptotic analysis is employed to precise the conditions under which the kinetic energy is conserved when the reference Mach number of the flow goes to zero. Secondly, this conservation property is investigated at the discretized level. Finally, few numerical experiments based on an "all-Mach" algorithm described in Ref. [3] demonstrate that the check board decoupling problem is unfortunately not avoided when the Mach number becomes sufficiently small before unity.

§2. Continuous asymptotics

We claim that the "incompressible limit" of the compressible flow model should be understood rather as the model obtained when the characteristic Mach number of the flow goes to zero in the general compressible flow model, than the divergence-free velocity flow model. In fact this latter is only a particular case of the asymptotic model. In this section, we recall shortly the derivation of the convective space and time scale continuous asymptotics, and few basic properties of it (see *e.g.* Ref. [2] for further details).

The Euler equations are written in dimensional form as

$$\begin{aligned}
\partial_{\hat{t}}\hat{\varrho} + \widehat{\mathrm{div}}(\hat{\varrho}\hat{v}) &= 0,\\
\partial_{\hat{t}}(\hat{\varrho}\hat{v}) + \widehat{\mathrm{div}}(\hat{\varrho}\hat{v}\otimes\hat{v}) + \hat{\nabla}\hat{p} &= 0,\\
\partial_{\hat{t}}(\hat{\varrho}\hat{E}) + \widehat{\mathrm{div}}(\hat{\varrho}\hat{v}\hat{H}) &= 0,\\
\hat{E} &= \hat{e} + \hat{K},\\
\hat{\varrho}\hat{H} &= \hat{\varrho}\hat{E} + \hat{p},\\
\hat{\varrho}\hat{e} &= \frac{\hat{p}}{\gamma-1},
\end{aligned}$$

where $\hat{\varrho}$, $\hat{v}$, $\hat{p}$, $\hat{e}$, $\hat{E}$, $\hat{H}$ and γ denote the density, velocity, pressure, internal energy, total energy, total enthalpy and the ratio of the specific heats at constant pressure and constant volume, respectively. The kinetic energy is $\hat{\varrho}\hat{K}$ with

$$\hat{K} = \frac{\hat{v}\cdot\hat{v}}{2}.$$

Let us suppose that the following reference quantities are given: length $\hat{l}_r$, density $\hat{\varrho}_r$, pressure $\hat{p}_r$ and norm velocity $\hat{v}_r$. Then, non-dimensional quantities are defined, $x = \hat{x}/\hat{l}_r$, $v = \hat{v}/\hat{v}_r$, $p = \hat{p}/\hat{p}_r$, $\varrho = \hat{\varrho}/\hat{\varrho}_r$, $t = \hat{t}/(\hat{l}_r/\hat{v}_r)$, $E = \hat{E}/(\hat{p}_r/\hat{\varrho}_r)$, $e = \hat{e}/(\hat{p}_r/\hat{\varrho}_r)$ and $H = \hat{H}/(\hat{p}_r/\hat{\varrho}_r)$. In the following, ∇ and div denote the gradient and the divergence operators with respect to the non-dimensionalized spatial variable x. The non-dimensional Euler equations read:

$$\begin{aligned}
\partial_t\varrho + \mathrm{div}(\varrho v) &= 0,\\
\partial_t(\varrho v) + \mathrm{div}(\varrho v\otimes v) + \frac{1}{\mathrm{M}^2}\,\nabla p &= 0, &(1)\\
\partial_t(\varrho E) + \mathrm{div}(\varrho v H) &= 0, &(2)\\
E &= e + \mathrm{M}^2 K, &(3)\\
\varrho H &= \varrho E + p, &(4)\\
\varrho e &= \frac{p}{\gamma-1}, &(5)
\end{aligned}$$

where we set

$$\mathrm{M} = \sqrt{\gamma}\,\frac{\hat{v}_r}{\sqrt{\gamma\hat{p}_r/\hat{\varrho}_r}}, \quad K = \frac{v\cdot v}{2}.$$

By taking the scalar product of the velocity with the momentum equation (1), one obtains the transport equation of the kinetic energy

$$\partial_t(\varrho K) + \mathrm{div}(v\varrho K) = \frac{1}{\mathrm{M}^2}(p\,\mathrm{div}(v) - \mathrm{div}(pv)). \tag{6}$$

Next, let us suppose that

$$p(x,t,\mathrm{M}) = \sum_{n=0}^{N} \mathrm{M}^n p^{(n)}(x,t) + o(\mathrm{M}^N), \quad N = 0,1,2, \quad \mathrm{M} \to 0,$$

with similar expansions for ϱ and v. Then, these expansions are substituted into the non-dimensional Euler equations. From the momentum equation, collecting coefficients of powers -2 and -1 in the characteristic Mach number M,

$$p^{(0)} = p^{(0)}(t), \quad p^{(1)} = p^{(1)}(t). \tag{7}$$

This means that, at convective space and time scale, the spatial pressure variations are taken into account by the second-order pressure $p^{(2)}$, called the hydrodynamic pressure. Now, from Eqs. (3), (5) and (7),

$$\partial_t(\varrho E) = \frac{1}{\gamma - 1}\mathrm{d}_t p^{(0)} + o(\mathrm{M}), \quad \mathrm{M} \to 0. \tag{8}$$

Consequently, energy equation (2) leads to

$$\mathrm{d}_t p^{(0)} + \gamma p^{(0)}\,\mathrm{div}(v^{(0)}) = 0. \tag{9}$$

On the other hand, from Eq. (6), the zeroth-order transport equation of the kinetic energy reads, after integration over the computational domain Ω,

$$-\int_{\partial\Omega} p^{(2)} v^{(0)} \cdot n = \partial_t \int_{\Omega} \varrho^{(0)} K^{(0)} + \int_{\partial\Omega} \varrho^{(0)} K^{(0)} v^{(0)} \cdot n - \int_{\Omega} p^{(2)}\,\mathrm{div}(v^{(0)}). \tag{10}$$

Consequently, a sufficient condition for the kinetic energy conservation in the incompressible limit (asymptotically) is that the zeroth-order velocity field is divergence-free. Let us notice that, from Eqs. (8) and (9), this condition is equivalent to

$$\mathrm{d}_t(\varrho E)^{(0)} = 0 \quad \text{or} \quad \mathrm{d}_t p^{(0)} = 0.$$

In this case the zeroth-order pressure of the flow is constant in time and space. In fact, $p^{(0)}$ (called the thermodynamical pressure) is related to the adiabatic compression of the gas flow, because it verifies:

$$\mathrm{D}_t \log \varrho^{(0)} = \mathrm{d}_t \log(p^{(0)})^{1/\gamma}, \quad \mathrm{D}_t \equiv \partial_t + v^{(0)} \cdot \nabla.$$

If $p^{(0)}$ is constant, then the power of the hydrodynamic pressure forces on the boundary of the computational domain equals the time variation of the the zeroth-order kinetic energy, plus its injection or evacuation by the boundary. This is the significance of the kinetic energy conservation in the incompressible limit, given by Eq. (10) through the analysis of the continuous flow model.

§3. Semi-discrete asymptotics

Let us now consider the discretized counterpart of the continuous property of kinetic energy conservation in the incompressible limit. We are aiming to conserve this property after applying the discretization procedure, when a first-order cell centered finite-volume method is used. We follow the presentation of Georges *et al.* [1], but here we adopt an asymptotic point of view. As in the continuous case, the convective time and space scale is considered.

Let $V^h \subset \Omega$ a polygonal bounded domain in $\mathbb{R}^d$ ($d = 1, 2$ or 3), which consists of cells V_i such that

$$V^h = \bigcup_i V_i\,; \quad \left|V_i \cap V_j\right| = 0,\ i \neq j.$$

S_{ij} is the surface (if $d = 3$), the edge (if $d = 2$) or the point (if $d = 1$) between two adjacent cells V_i and V_j.

Let us first introduce the asymptotic semi-discrete continuity operator. On each cell V_i, the semi-discrete zeroth-order mass equation reads:

$$|V_i|\, \mathrm{d}_t \varrho^{(0)} + \sum_{S_{ij}} (\varrho v)^{(0)}_{ij} \cdot n_{ij} \left|S_{ij}\right| = 0,$$

where n_{ij} denotes the V_i unit outer normal on S_{ij}. Here, mass fluxes, pressures and velocities are centrally interpolated. Thus, the asymptotic semi-discrete Continuity operator is defined on the cell V_i by

$$\mathrm{C}_i = |V_i|\, \mathrm{d}_t \varrho^{(0)}_i + \sum_{S_{ij}} F^{\mathrm{C}}_{ij}, \quad F^{\mathrm{C}}_{ij} = \frac{(\varrho v)^{(0)}_i + (\varrho v)^{(0)}_j}{2} \cdot n_{ij} \left|S_{ij}\right|.$$

Let us now consider the zeroth-order semi-discrete momentum equation, the k^{th} component of which reads

$$|V_i|\, \mathrm{d}_t(\varrho^{(0)}_i u^{(0)}_i) + \sum_{S_{ij}} (\varrho v)^{(0)}_{ij} \cdot n_{ij} u^{(0)}_{ij} \left|S_{ij}\right| + \sum_{S_{ij}} p^{(2)}_{ij} n^k_{ij} \left|S_{ij}\right| = 0, \tag{11}$$

where we note $u \equiv v^k$ for convenience. As a generalization of the two first terms of the left-hand side of Eq. (11), we set

$$\mathrm{D}(\phi)_i = |V_i|\, \mathrm{d}_t(\varrho^{(0)}_i \phi^{(0)}_i) + \sum_{S_{ij}} F^{\mathrm{D}}_{ij}(\phi), \quad F^{\mathrm{D}}_{ij}(\phi) = \frac{(\varrho v)^{(0)}_i + (\varrho v)^{(0)}_j}{2} \cdot n_{ij} \frac{\phi^{(0)}_i + \phi^{(0)}_j}{2} \left|S_{ij}\right|,$$

where ϕ is a scalar field on V^h, constant on each cell V_i. It is called the Divergence operator. Let us also introduce the Advection operator,

$$\mathrm{A}(\phi)_i = |V_i|\, \varrho^{(0)}_i \mathrm{d}_t \phi^{(0)}_i + \sum_{S_{ij}} F^{\mathrm{A}}_{ij}(\phi), \quad F^{\mathrm{A}}_{ij}(\phi) = \frac{(\varrho v)^{(0)}_i + \varrho v)^{(0)}_j}{2} \cdot n_{ij} \frac{\phi^{(0)}_j - \phi^{(0)}_i}{2} |S_{ij}|,$$

and the Skew-symmetric operator,

$$\mathrm{S}(\phi)_i = \frac{1}{2} \left(\mathrm{D}(\phi)_i + \mathrm{A}(\phi)_i\right).$$

A simple calculus leads to

$$\mathrm{S}(\phi)_i = \frac{|V_i|}{2}\left(\mathrm{d}_t(\varrho_i^{(0)}\phi_i^{(0)}) + \varrho_i^{(0)}\mathrm{d}_t\phi_i^{(0)}\right) + \sum_{S_{ij}} F_{ij}^{\mathrm{S}}(\phi), \quad F_{ij}^{\mathrm{S}}(\phi) = \frac{(\varrho v)_i^{(0)} + (\varrho v)_j^{(0)}}{2} \cdot n_{ij}\frac{\phi_j^{(0)}}{2}\left|S_{ij}\right|.$$

Moreover, as

$$\mathrm{D}(\phi)_i = \mathrm{S}(\phi)_i + \frac{\phi_i^{(0)}}{2}\mathrm{C}_i,$$

Eq. (11) yields

$$\sum_i \left\{ u_i^{(0)}\mathrm{S}(u^{(0)})_i + \frac{u_i^{(0)}}{2}\mathrm{C}_i + u_i^{(0)}\sum_{S_{ij}} F_{ij}^{p} \right\} = 0, \tag{12}$$

where

$$F_{ij}^{p} = \frac{p_i^{(2)} + p_j^{(2)}}{2} n_{ij}^{k}\left|S_{ij}\right|.$$

It is worthwhile to notice that

$$\phi_i^{(0)}\mathrm{S}(\phi)_i = |V_i|\ \mathrm{d}_t\left(\frac{\varrho_i^{(0)}(\phi_i^{(0)})^2}{2}\right) + \sum_{S_{ij}} F_{ij}^{\mathrm{KS}}(\phi),$$

where

$$F_{ij}^{\mathrm{KS}}(\phi) = \frac{(\varrho v)_i^{(0)} + (\varrho v)_j^{(0)}}{2} \cdot n_{ij}\frac{\phi_i^{(0)}\phi_j^{(0)}}{2}\ \left|S_{ij}\right|.$$

Thus, Eq. (12) becomes

$$\sum_i \left\{ |V_i|\ \mathrm{d}_t\left(\frac{\varrho_i^{(0)}(u_i^{(0)})^2}{2}\right) + \sum_{S_{ij}} F_{ij}^{\mathrm{KS}}(u) + \frac{u_i^{(0)}}{2}\mathrm{C}_i + u_i^{(0)}\sum_{S_{ij}} F_{ij}^{p} \right\} = 0. \tag{13}$$

Focusing on the last terms in the brackets, let us mention that

$$u_i^{(0)}\sum_{S_{ij}} F_{ij}^{p} = \sum_{S_{ij}} F_{ij}^{\mathrm{K}p}(u) - p_i^{(2)}\sum_{S_{ij}}\frac{u_i^{(0)} + u_j^{(0)}}{2} n_{ij}^{k}\ |S_{ij}| + p_i^{(2)}u_i^{(0)}\sum_{S_{ij}} n_{ij}^{k}\ |S_{ij}|, \tag{14}$$

where

$$F_{ij}^{\mathrm{K}p}(u) = \frac{u_i^{(0)}p_j^{(2)} + u_j^{(0)}p_i^{(2)}}{2}\ n_{ij}^{k}\ \left|S_{ij}\right|.$$

As $F_{ji}^{\mathrm{K}p}(u) = -F_{ij}^{\mathrm{K}p}(u)$, Eq. (13) leads to

$$\sum_i \left\{ |V_i|\ \mathrm{d}_t\left(\frac{\varrho_i^{(0)}\|v_i^{(0)}\|^2}{2}\right) - p_i^{(2)}\sum_{S_{ij}}\frac{v_i^{(0)} + v_j^{(0)}}{2} \cdot n_{ij}\ \left|S_{ij}\right| \right\} = 0, \tag{15}$$

where $\|\cdot\|$ denotes the euclidean norm in $\mathbb{R}^d$. Finally, one obtains the following result:

Theorem 1. *Consider the Euler equations discretized using a first-order cell-centered finite-volume method. Let us assume that the thermodynamical pressure $p^{(0)}$ is constant in time, or equivalently, the divergence of the zeroth-order velocity is zero at any time. Then, the discrete kinetic energy is conserved on the whole computational domain when the Mach number goes to zero, provided that:*

1. *convective terms are spatially discretized in skew-symmetric form,*
2. *mass fluxes, velocities and pressures are centrally interpolated at the cell interfaces.*

One observes that Eq. (15) is in accordance with the original continuous form (10) of the kinetic energy conservation equation in the incompressible limit. The numerical benefits that can be expected from this property are carried out by the non-growth of the sum of the square of the velocities. This contributes to ensure the stability of the time scheme without any explicit numerical dissipation to introduce.

Unfortunately, a glance at Fig. 1 suffices to realize that check board decoupling is carried on by central discretizations as the Mach number goes to zero. For this computation, an algorithm suggested by Nerinckx *et al.* [3] is used. In predictor/corrector form, it is based on the energy equation at the correction step, and enables one the proper handling of the pressure field. This one plays a specific role in the progressive decoupling between the flow equations when the Mach number goes to zero (see *e.g.* [2]).

A one-dimensional inviscid steady flow of perfect gas is considered in a nozzle with a variable section. The throat Mach number is about 10^{-6}. In Fig. 1, the asymmetry in the check board distribution along the nozzle is due to the boundary conditions. The flow is oriented from the left to the right. At the inlet, density and velocity are prescribed and the pressure gradient is zero. At the outlet, the pressure is prescribed, while velocity and density are allowing to float.

Semi-discrete asymptotic analysis enables one to explain the origin of the pressure numerical oscillations. Returning to the momentum equation, one has at orders -2 ($l = 0$) and -1 ($l = 1$):

$$\sum_{S_{ij}} \frac{p_i^{(l)} + p_i^{(l)}}{2} n_{ij}^k \left|S_{ij}\right| = 0, \quad l = 0, 1.$$

In fact, at convective time and space scale, $p^{(1)}$ disappears from the flow model in the incompressible limit (see *e.g.* [2]). Since at this scale the check board effect is due only to the hydrodynamic pressure $p^{(2)}$, it can be removed by the addition of an explicit numerical dissipation in the following form:

$$(\varrho v)_{ij} = \frac{(\varrho v)_i + (\varrho v)_j}{2} + \alpha_{ij}(\mathrm{M})\,(p_i - p_j), \quad \alpha_{ij}(\mathrm{M}) = O(1/\mathrm{M}^2),\ \mathrm{M} \to 0. \tag{16}$$

The efficiency of this technique can be viewed in Fig. 2. Let us emphasize that, in this figure, a steady flow with constant boundary conditions is considered. In contrast, for example when acoustic pressure fluctuations are imposed at the outlet of the nozzle, the non-physical coupling between pressure and velocity influences the kinetic energy. This is due to the fact that $p^{(1)}$, which is identified as the acoustic pressure in the flow, is taken into account when the numerical dissipation (16) is applied.

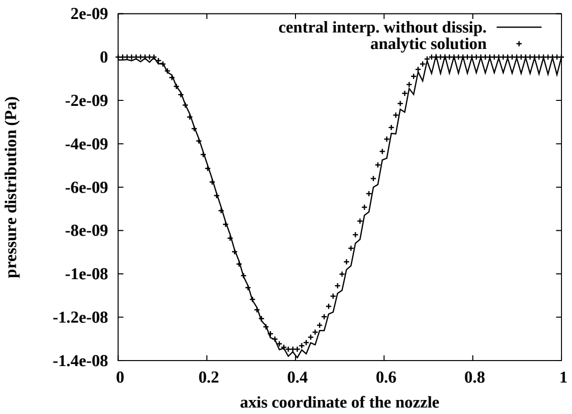

Figure 1: Pressure distribution (Pa) along the nozzle. Throat Mach number: 10^{-6}. Central discretization without numerical dissipation.

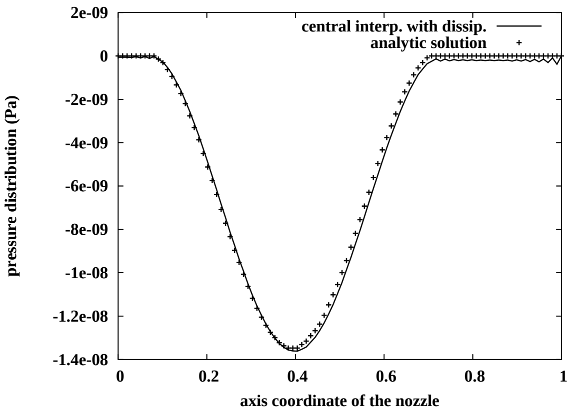

Figure 2: Pressure distribution (Pa) along the nozzle. Throat Mach number: 10^{-6}. Central discretizations with numerical dissipation.

§4. Conclusion

Kinetic energy conservation property in the incompressible limit was investigated at the continuous and discrete levels, through asymptotic analysis in convective time and space scale. Central interpolation of the pressures, mass fluxes and velocities allows one to conserve the continuous asymptotic property after applying the discretization procedure, provided that the skew-symmetric form is adopted for the convective terms. Unfortunately, check board effect is carried on by central interpolations. In the case of steady flows computations, a numerical dissipation enables one to avoid it.

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