Monografías Matemáticas *García de Galdeano* **35**, 11–18 (2010)

A LEAST SQUARES APPROACH FOR AN INVERSE TRANSMISSION PROBLEM

Lekbir Afraites, Marc Dambrine and Djalil Kateb

Abstract. We consider the question of recovering the shape of an unknown inclusion ω inside a body Ω from a single boundary measurement. This inverse problem —known as electrical impedance tomography— is seen through the minimization of some Least Squares criteria. We provide the first and second order derivatives with respect of perturbations of the shape of the interface $\partial\omega$ of the state functions and of the objectives. We study the stability of the optimization and prove that the shape Hessian at an optimal inclusion is not coercive but compact explaining the ill-posedness of the proposed approach.

Keywords: Inverse conductivity problem, shape optimization, second order method.

AMS classification: 49Q10, 49Q12, 65N21.

§1. Introduction

Consider a body constant conductivity σ_1 occupying a bounded domain Ω in $\mathbb{R}^N$ with $N \geq 3$. Inside Ω, there is an unknown inclusion ω whose conductivity σ_2 differs from the background conductivity σ_1 ($\sigma_1, \sigma_2 > 0$). The electrical potential u solves the partial differential equation

$$-\mathrm{div}\,(\sigma_\omega(x)\nabla u) = 0 \text{ in } \Omega, \tag{1}$$

with $\sigma_\omega = \sigma_1\chi_{\Omega\setminus\overline{\omega}} + \sigma_2\chi_\omega$. The notation χ_E denotes the characteristic function of a measurable subset E of Ω. By measuring the input voltage and the corresponding output current on $\partial\Omega$, we gain access to a Cauchy pair (f, g) for (1). In others words, both Dirichlet boundary condition $u = f$ and Neumann boundary condition $\sigma_1\partial_{\mathbf{n}}u = g$ are known on $\partial\Omega$. We consider the question of a practical reconstruction of ω by these redundant informations on $\partial\Omega$.

This problem is a particular case of the inverse conductivity problem of Calderón that concerns the determination of the conductivity distribution σ from boundary measurements ([11, 9, 4]). The identification problem of an inclusion by boundary measurements is usually written from a numerical point a view as the minimization of a cost function: typically a Least Squares matching criterion. Many authors have investigate the steepest descent method for this problem [7, 6, 2] with the methods of shape optimization.

We address in this manuscript the stability of the optimization problems obtained with different Least Square cost function. By introducing second order methods, we analyze the wellposedness of the optimization method. We explain the instability in the continuous settings in terms of shape optimization: the shape Hessian is not coercive —in fact its Riesz operator turns out to be compact— and hence the criterion to minimize does not have necessarily a local strict minimum. A Kohn-Vogelius type objective is studied in [3] and simplified models can be found in [5, 1]. In this note, we present a Least Squares approach for this inverse problem and obtain similar results. This fact is surprising since a Kohn-Vogelius criteria is expected to lead to more stable optimization schemes.

The present manuscript is organized as follows. In Section 2, we reformulate the identification problem as shape optimization problems, tracking with a Least Squares formulation the Dirichlet and Neumann boundary conditions. We precise the first and second derivative of the state and the corresponding expressions for the criteria by introducing an adjoint state. Finally, we present our main result: a compactness result for the shape hessian at a critical point. In Section 3, we justify some shape derivatives and explain the main steps of the proof for the compactness theorem that explains the ill-posedness of the underlying identification problem.

§2. The results

Let us fix the geometrical setting under consideration and the notations. We consider a bounded domain $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) with a C^2 boundary. It is fulfilled with a material whose conductivity is σ_1, an unknown inclusion ω in Ω of conductivity $\sigma_2 \neq \sigma_1$. In the sequel, we fix $d_0 > 0$ and consider inclusions ω such that $\omega \subset\subset \Omega_{d_0} = \{x \in \Omega,\ d(x, \partial\Omega) > d_0\}$. We also assume that the boundary $\partial\omega$ is of class $C^{4,\alpha}$.

In the sequel, a bold character denotes a vector. If $\mathbf{h}$ denotes a deformation field, it can be written as $\mathbf{h} = \mathbf{h}_\tau + h_n\mathbf{n}$ on $\partial\omega$. Note also that in the following lines, $\mathbf{n}$ denotes the outer normal field to $\partial\omega$ pointing into $\Omega \setminus \overline{\omega}$. Hence, for $x \in \partial\omega$, we define, when the limit exists, $u^\pm(x)$ (resp. $(\partial_n u)^\pm(x)$) as the limit of $u(x \pm t\mathbf{n}(x))$ (resp. $\langle \nabla u(x \pm t\mathbf{n}(x), \mathbf{n}(x))\rangle$) when $t > 0$ tends to 0. Note that $\mathbf{h}_\tau$ is a vector while h_n is a scalar quantity. Admissible deformation fields have to preserve $\partial\Omega$ and the regularity of the boundaries. Therefore, we consider the space of admissible fields

$$\mathcal{H} = \left\{ \mathbf{h} \in C^{4,\alpha}(\mathbb{R}^N, \mathbb{R}^N), \mathrm{supp}(\mathbf{h}) \subset \Omega_{d_0} \right\}.$$

2.1. The shape optimization problem

In order to recover the shape of the inclusion ω, an possible strategy is to minimize a cost function. Many choices are possible, in particular a Least Squares type objective. In this paper, we study two different Least Square cost functions. We now define these criteria. Fixing the Neumann boundary data, we can track Dirichlet boundary conditions:

$$J_{LS}(\omega) = \frac{1}{2}\int_{\partial\Omega} |u_n - f|^2,$$

where f is the disturbed boundary measurements and the u_n is solution of the Neumann boundary value problem:

$$\begin{cases} -\mathrm{div}\,(\sigma_\omega \nabla u_n) = 0 & \text{in } \Omega, \\ \sigma_1 \partial_n u_n = g & \text{on } \partial\Omega. \end{cases} \tag{2}$$

To obtain uniqueness of the solution of (2), we add the normalization condition

$$\int_{\partial\Omega} u_n = \int_{\partial\Omega} f. \tag{3}$$

Another possible choice is to fix Dirichlet boundary condition and track the outgoing flux:

$$J_{DLS}(\omega) = \frac{1}{2}\int_{\partial\Omega} |\sigma_1 \partial_n u_d - g|^2,$$

where u_d is solution of the Dirichlet boundary value problem:

$$\begin{cases} -\mathrm{div}\,(\sigma_\omega \nabla u_d) = 0 & \text{in } \Omega, \\ u_d = f & \text{on } \partial\Omega. \end{cases} \tag{4}$$

To ensure that the cost function J_{DLS} is well defined, we assume that the Dirichlet data $f \in H^{3/2}(\partial\Omega)$. To avoid this assumption, one usually prefers to consider J_{LS} than J_{DLS}.

2.2. Differentiability results for the state u_n and u_d

We quote from [6, 10, 2] the first order derivative of the state u_n and u_d.

Theorem 1. *Let Ω be a open subset of $\mathbb{R}^N$ with a C^2 boundary and ω a subdomain in Ω_{d_0} with a $C^{4,\alpha}$ boundary. The state functions u_n and u_d are shape differentiable and their shape derivative u_n' and u_d' belong to $\mathrm{H}^1(\Omega \setminus \overline{\omega}) \cup \mathrm{H}^1(\omega)$ and satisfy*

$$\begin{cases} \Delta u_n' = 0 \ \ \textit{in } \Omega \setminus \overline{\omega} \textit{ and in } \omega, \\ [u_n'] = h_n \dfrac{[\sigma]}{\sigma_1}\, \partial_{\mathbf{n}} u_n^- \ \ \textit{on } \partial\omega, \\ [\sigma \partial_n u_n'] = [\sigma]\mathrm{div}_\tau\,(h_n \nabla_\tau u_n) \ \ \textit{on } \partial\omega, \\ \sigma_1 \partial_n u_n' = 0 \ \ \textit{on } \partial\Omega, \end{cases} \quad \textit{and} \quad \begin{cases} \Delta u_d' = 0 \ \ \textit{in } \Omega \setminus \overline{\omega} \textit{ and in } \omega, \\ \left[u_d'\right] = h_n \dfrac{[\sigma]}{\sigma_1} \partial_{\mathbf{n}} u_d^- \ \ \textit{on } \partial\omega, \\ \left[\sigma \partial_n u_d'\right] = [\sigma]\mathrm{div}_\tau\,(h_n \nabla_\tau u_d) \ \ \textit{on } \partial\omega, \\ u_d' = 0 \ \ \textit{on } \partial\Omega. \end{cases} \tag{5}$$

The second order derivative of the state functions u_n is computed in [3].

Theorem 2. *Let Ω be a open subset of $\mathbb{R}^N$ with a C^2 boundary and ω a element of Ω_{d_0} with a $C^{4,\alpha}$ boundary. Let $\mathbf{h}_1$ and $\mathbf{h}_2$ be two deformation fields in $\mathcal{H}$. The state u_n is has a second order shape derivative $u_n'' \in \mathrm{H}^1(\Omega \setminus \overline{\omega}) \cup \mathrm{H}^1(\omega)$ that solves*

$$\begin{cases} \Delta u_n'' = 0 \textit{ in } \Omega \setminus \overline{\omega} \textit{ and in } \omega, \\ [u_n''] = (h_{1,n} h_{2,n} H - \mathbf{h_1}_\tau.(D\mathbf{n}\, \mathbf{h_2}_\tau))\,[\partial_{\mathbf{n}} u_n] - (h_{1,n}[\partial_{\mathbf{n}}(u_n)_2'] + h_{2,n}[\partial_{\mathbf{n}}(u_n)_1']) \\ \qquad\qquad + (\mathbf{h_1}_\tau.\nabla h_{2,n} + \mathbf{h_2}_\tau.\nabla h_{1,n})\,[\partial_{\mathbf{n}} u_n] \textit{ on } \partial\omega, \\ [\sigma \partial_n u_n''] = \mathrm{div}_\tau\,(h_{2,n}\,[\sigma \nabla_\tau (u_n)_1'] + h_{1,n}\,[\sigma \nabla_\tau (u_n)_2'] + \mathbf{h_1}_\tau.(D\mathbf{n}\,\mathbf{h_2}_\tau)[\sigma \nabla_\tau u_n]) \\ \qquad\qquad - \mathrm{div}_\tau\,((\mathbf{h_1}_\tau.\nabla_\tau h_{2,n} + \nabla_\tau h_{1,n}.\mathbf{h_2}_\tau)\,[\sigma \nabla_\tau u_n]) \\ \qquad\qquad + \mathrm{div}_\tau\,(h_{2,n} h_{1,n}(2D\mathbf{n} - HI)\,[\sigma \nabla_\tau u_n]) \ \textit{ on } \partial\omega, \\ \sigma_1 \partial u_n'' = 0 \textit{ on } \partial\Omega. \end{cases} \tag{6}$$

Here, $(u_n)_i'$ denotes the first order derivative of u in the direction of h_i as given in (5)*, $D\mathbf{n}$ stands for the second fundamental form of the manifold $\partial\omega$ and H stands for the mean curvature of $\partial\omega$. Note that H is then the sum of the main curvatures and not the scaled version (divided by $n-1$) in dimension n.*

The result concerning u_d is an easy adaption of Theorem 2. Once the differentiability of the state function has been established, the chain rule provides the differentiability with respect to the shape of criterion.

2.3. Differentiability of the objective

As usual for Least Squares objective, this derivative can be simplified thanks to an adjoint state denoted by w_{LS} for J_{LS} and w_{DLS} for J_{DLS}.

Theorem 3. *Let Ω be a open subset of $\mathbb{R}^N$ ($N \geq 3$) with a C^2 boundary and ω a element of Ω_{d_0} with a $C^{4,\alpha}$ boundary. The Least-Square objective J_{LS} and J_{DLS} are differentiable with respect to the shape and their derivatives in the direction of a deformation field* **h** *in $\mathcal{H}$ are given by*

$$DJ_{LS}(\omega).h = \frac{\sigma_1 - \sigma_2}{\sigma_2} \int_{\partial\omega} (\sigma_1 \partial_n w_{LS}^+ \partial_n u_n^+ + \nabla_\tau u_n . \nabla_\tau w_{LS})\, h_n,$$
$$DJ_{DLS}(\omega).h = -\frac{\sigma_1 - \sigma_2}{\sigma_2} \int_{\partial\omega} \left(\sigma_1 \partial_n w_{DLS}^+ \partial_n u_d^+ + \nabla_\tau u_d . \nabla_\tau w_{DLS}\right) h_n,$$

where the adjoint functions w_{LS} and w_{DLS} solve the boundary value problem

$$\begin{cases} -\mathrm{div}\,(\sigma \nabla w_{LS}) = 0 & \text{in } \Omega, \\ \sigma_1 \partial_n w_{LS} = u_n - f & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} -\mathrm{div}\,(\sigma \nabla w_{DLS}) = 0 & \text{in } \Omega, \\ w_{DLS} = \sigma_1 \partial u_d - g & \text{on } \partial\Omega, \end{cases} \tag{7}$$

The compatibility condition is satisfied thanks to the normalization (3). The adjoint has to be normalized for example as in (3).

In this work, we are interested by the second order shape derivative of the cost functions objectives and the study of the stability of these criteria. For this, will need the shape derivatives of the adjoint states w_{LS} and w_{DLS} obtained as a consequence of Theorem 1. The state functions w_{LS} and w_{DLS} are shape differentiable and their shape derivatives w'_{LS} and w'_{DLS} belong to $\mathrm{H}^1(\Omega \setminus \overline{\omega}) \cup \mathrm{H}^1(\omega)$ and satisfy

$$\begin{cases} \Delta w'_{LS} = 0 & \text{in } \Omega \setminus \overline{\omega} \text{ and in } \omega, \\ [w'_{LS}] = h_n \dfrac{[\sigma]}{\sigma_1} \partial_{\mathbf{n}} w_{LS}^- & \text{on } \partial\omega, \\ [\sigma \partial_n w'_{LS}] = [\sigma] \mathrm{div}_\tau (h_n \nabla_\tau w_{LS}) & \text{on } \partial\omega, \\ \sigma_1 \partial_n w'_{LS} = u'_n & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} \Delta w'_{DLS} = 0 & \text{in } \Omega \setminus \overline{\omega} \text{ and in } \omega, \\ [w'_{DLS}] = h_n \dfrac{[\sigma]}{\sigma_1} \partial_{\mathbf{n}} w_{DLS}^- & \text{on } \partial\omega, \\ [\sigma \partial_n w'_{DLS}] = [\sigma] \mathrm{div}_\tau (h_n \nabla_\tau w_{DLS}) & \text{on } \partial\omega, \\ w'_{DLS} = \sigma_1 \partial_n u'_d & \text{on } \partial\Omega. \end{cases}$$

We now give the second order derivatives of the Least Square criterions J_{LS} and J_{DLS}:

Theorem 4. *Let Ω be a open subset of $\mathbb{R}^N$ with a C^2 boundary and ω a element of Ω_{d_0} with a $C^{4,\alpha}$ boundary. Let $\mathbf{h}_1$ and $\mathbf{h}_2$ be two deformation fields in $\mathcal{H}$. The Least Square objectives J_{LS} and J_{DLS} are twice differentiable with respect to the shape and their second derivatives in the direction $\mathbf{h}$ are given by*

$$\begin{aligned} D^2 J_{LS}(\omega)(\mathbf{h},\mathbf{h}) = &\int_{\partial\omega} \sigma_1 \partial_n w_{LS}'^{+} \left[(u_n')\right] + \left[\sigma\partial_n w_{LS}'\right](u_n')^- - \left[\sigma\partial_n(u_n')\right] w_{LS}'^{-} \\ &+ \int_{\partial\omega} \sigma_2 \partial_{\mathbf{n}} w_{LS}^{-} \left[(u_n)''\right] - \sigma_1\partial_n(u_n')^+ \left[w_{LS}'\right] - w_{LS}\left[\sigma\partial_{\mathbf{n}}(u_n)''\right], \\ D^2 J_{DLS}(\omega)(\mathbf{h},\mathbf{h}) = &\int_{\partial\omega} \left[\sigma\partial_n(u_d')\right] w_{DLS}'^{-} + \sigma_1\partial_n(u_d')_1^+ \left[w_{DLS}'\right] - \sigma_1\partial_n w_{DLS}'^{-}\left[(u_d')\right] \\ &+ \int_{\partial\omega} \left[(u_d)''\right] + \left[\sigma\partial_n w_{DLS}'\right](u_d')^- - w_{DLS}\left[\sigma\partial_{\mathbf{n}}(u_d)''\right] - \sigma_2\partial_{\mathbf{n}} w_{DLS}^{-}. \end{aligned} \tag{8}$$

Let us investigate the properties of stability of this cost functions. We focus the study J_{LS} cost function but we can use the same techniques for J_{DLS}. We *assume* that there exists an admissible inclusion ω^* such that $J_{LS}(\omega^*) = 0$. It realizes the absolute minimum of the criterion J_{LS}. This is satisfied by solution of the inverse problem.Then, Euler's equation $DJ_{LS}(\omega^*)(\mathbf{h}) = 0$ holds and that we prove that

$$D^2 J_{LS}(\omega^*)(\mathbf{h},\mathbf{h}) = \int_\Omega (u_n')^2. \tag{9}$$

Moreover, if $h_n \neq 0$, then $D^2 J_{LS}(\omega^*)(\mathbf{h},\mathbf{h}) > 0$ holds. Nevertheless, (9) does not means that the minimization problem is well posed. In fact, the following theorem explains the instability of standard minimization algorithms.

Theorem 5. *Assume that ω^* is a critical shape of J_{LS} for which the additional condition $u_n = f$ holds, then the Riesz operator corresponding to $D^2 J_{LS}(\omega^*)$ defined from $\mathrm{H}^{1/2}(\partial\omega^*)$ with values in $\mathrm{H}^{-1/2}(\partial\omega^*)$ is compact.*

Theorem 5 has two main consequences. First, the shape Hessian at the global minimizer is not coercive. This means that this minimizer may be no local strict minimum of the criterion. Moreover, J_{LS} is not locally convex (at least uniformly in the directions of deformations) around the minimizer ω^*: the criterion provide no control of the distance between the parameter ω and the target ω^*. The second consequence concerns any numerical scheme used to obtain this optimal domain ω^*. One has to face this difficulty. This explains why frozen Newton schemes or Levenberg-Marquard schemes are used to numerically solve this problem [6, 2].

§3. Ideas of the proofs

3.1. Proof of Theorem 4

The differentiability of the objective is a direct application of Theorem 2. The computation we make here is based on the relation

$$D^2 J_{LS}(\omega)(\mathbf{h}_1,\mathbf{h}_2) = D\left(DJ_{LS}(\omega)\mathbf{h}_1\right)(\omega)\mathbf{h}_2 - DJ_{LS}(\omega)D\mathbf{h}_1\mathbf{h}_2). \tag{10}$$

To obtain (8), we first compute the shape gradient in the direction $\mathbf{h}_1$, then differentiate it in the direction of $\mathbf{h}_2$ to get

$$DJ_{LS}(\omega)\mathbf{h}_1 = \int_{\partial\Omega} (u_n - f)\,(u_n)'_1.$$

Then,

$$D\,(DJ_{LS}(\omega)\mathbf{h}_1)\,\mathbf{h}_2 = \int_{\partial\Omega} (u_n)'_1(u_n)'_2 + ((u_n)'_1)'_2(u_n - f).$$

Thanks to formula (10), we obtain

$$D^2 J_{LS}(\omega)(\mathbf{h}_1, \mathbf{h}_2) = \int_{\partial\Omega} (u_n)'_1(u_n)'_2 + (u_n)''_{1,2}(u_n - f). \tag{11}$$

Introducing the adjoint state function w_{LS} and the first derivative adjoint state w'_{LS}, we transform the integral on $\partial\Omega$ at integral on $\partial\omega$ thanks to Green's formulas:

$$\begin{aligned}
\int_{\partial\Omega} (u_n)'_1(u_n)'_2 &= \int_{\partial\Omega} \sigma_1\partial_n w'_{LS}\,(u_n)'_1 \\
&= \int_{\partial\omega} \sigma_1\partial_n(w'_{LS})^+\,[(u_n)'_1] + (u_n^-)'_1\,[\sigma_1\partial_n w'_{LS}] - [\sigma\partial_n(u_n)'_1]\,(w'_{LS})^- - [w'_{LS}]\,\sigma_1\partial_n(u_n^+)'_1, \\
\int_{\partial\Omega} (u_n)''_{1,2}(u_n - f) &= \int_{\partial\Omega} \sigma_1\partial_n w_{LS}\,(u_n)''_{1,2} = \int_{\partial\omega} \sigma_2\partial_n w_{LS}^-\left[(u_n)''_{1,2}\right] - w_{LS}\left[\sigma\partial_n(u_n)''_{1,2}\right].
\end{aligned}$$

We gather these formulae to obtain the result (8).

3.2. Sketch of proof of Theorem 5

We follow the strategy of analysis of [5, 3]. We specify the domain ω that is assumed to be a critical shape for J_{LS}. Moreover, we assume that the additional condition $u_n = f$ on $\partial\Omega$ holds, then the adjoint state $w_{LS} = 0$ in the Ω and the first derivative adjoint state w'_{LS} becomes :

$$\begin{cases} -\mathrm{div}\,(\sigma_\omega \nabla w'_{LS}) = 0 \text{ in } \Omega, \\ \qquad \sigma_1\partial_n w'_{LS} = u'_n \text{ on } \partial\Omega. \end{cases}$$

To emphasize that we deal with such a special domain, we will denote it by ω^*. The assumptions mean that the measurements are compatible and that ω^* is a global minimum of the criterion. From the necessary condition of order two at a minimum, the shape Hessian is positive at such a point.

Let us notice that only the normal component of $\mathbf{h}$ appears. Let us also emphasize that there is no hope to get $\mathbf{h} = 0$ from the structure theorem for second order shape derivative. The deformation field $\mathbf{h}$ appears in $D^2 J_{LS}(\omega^*)(\mathbf{h}, \mathbf{h})$ only thought its normal component h_n since ω^* is a critical point for J_{LS}. This remark explains why we consider in the statement of Theorem 5 the scalar Sobolev space corresponding to the normal components of the deformation field.

We now prove Theorem 5. From (8), we deduce

$$D^2 J_{LS}(\omega^*)(\mathbf{h}, \mathbf{h}) = \int_{\partial\omega^*} \sigma_1\partial_n w'^{+}_{LS}\,[(u'_n)] - \int_{\partial\omega^*} [\sigma\partial_n(u'_n)]\,w'^{-}_{LS}$$

Substituting their values to the quantities $[(u'_n)]$ and $[\sigma\partial_n(u'_n)]$, we get

$$D^2 J_{LS}(\omega^*)(\mathbf{h},\mathbf{h}) = [\sigma]\left(\left\langle \sigma_1 h_n \partial_n u_n^-, \partial_n w_{LS}'^+ \right\rangle - \left\langle \mathrm{div}_\tau\left(h_n \nabla_\tau u_n\right), w_{LS}'^- \right\rangle\right),$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between $\mathrm{H}^{1/2}(\partial\omega^*)\times\mathrm{H}^{-1/2}(\partial\omega^*)$. Let us introduce the operators

$$\begin{aligned} T_1 : \mathrm{H}^{1/2}(\partial\omega^*) &\to \mathrm{H}^{-1/2}(\partial\omega^*) & M_1 : \mathrm{H}^{1/2}(\partial\omega^*) &\to \mathrm{H}^{1/2}(\partial\omega^*) \\ \mathbf{h} &\mapsto \mathrm{div}_\tau\left(h_n \nabla_\tau u_n\right) & \mathbf{h} &\mapsto w_{LS}'^- \\ T_2 : \mathrm{H}^{1/2}(\partial\omega^*) &\to \mathrm{H}^{1/2}(\partial\omega^*) & M_2 : \mathrm{H}^{1/2}(\partial\omega^*) &\to \mathrm{H}^{-1/2}(\partial\omega^*) \\ \mathbf{h} &\mapsto h_n \partial_{\mathbf{n}} u_n^- & \mathbf{h} &\mapsto \partial_{\mathbf{n}} w_{LS}'^+ \end{aligned}$$

The Hessian can then be written under the form

$$D^2 J_{LS}(\omega^*)(\mathbf{h},\mathbf{h}) = [\sigma]\left(\left\langle M_2(\mathbf{h}), T_2(\mathbf{h})\right\rangle - \sigma_1\left\langle T_1(\mathbf{h}), M_1(\mathbf{h})\right\rangle\right).$$

From the classical results of Maz'ya and Shaposhnikova on multipliers ([8]), we get easily that T_1 and T_2 are continuous operators. Operator M_1 is the composition of the operators

$$\begin{aligned} R_1 : \mathrm{H}^{1/2}(\partial\omega^*) &\to \mathrm{H}^{1/2}_\diamond(\partial\Omega) & \text{and} \qquad R_2 : \mathrm{H}^{1/2}_\diamond(\partial\Omega) &\to \mathrm{H}^{1/2}(\partial\omega^*) \\ \mathbf{h} &\mapsto u'_n & \phi &\mapsto \psi \end{aligned}$$

where ψ is the trace on $\partial\omega^*$ of Ψ solution of

$$\begin{cases} -\mathrm{div}\left(\sigma_{\omega^*}\nabla\Psi\right) = 0 & \text{in } \Omega, \\ \sigma_1 \partial_n \Psi = \phi & \text{on } \partial\Omega, \end{cases} \tag{12}$$

and $\mathrm{H}^{1/2}_\diamond(\partial\Omega)$ is the Sobolev space

$$\mathrm{H}^{1/2}_\diamond(\partial\Omega) = \left\{\phi \in \mathrm{H}^{1/2}(\partial\Omega) \ : \ \int_{\partial\Omega}\phi = 0\right\}.$$

While R_1 is a continuous operator, R_2 is compact. To prove this claim, let us express $u_{|\partial\omega^*} = \psi$. We use the integral representation formula and classical notation for the layers operators: we use the convention that the letter S is used for single layer potentials while K is used for double layer potentials. All the justifications of next claims are standart in the theory of integral equations. If u solves the boundary value problem (12), then it also solves the following system of integral equation

$$\begin{bmatrix} \frac{1}{2}I + \mu K_{\omega^*} & \kappa K_{\partial\Omega\partial\omega^*} \\ \mu K_{\partial\omega^*\partial\Omega} & \kappa\left(-\frac{1}{2}I + K_\Omega\right) \end{bmatrix} \begin{bmatrix} (u)_{|\partial\omega^*} \\ (u)_{|\partial\Omega} \end{bmatrix} = \kappa \begin{bmatrix} S_{\partial\Omega\partial\omega^*}\phi \\ S_\Omega \phi \end{bmatrix},$$

where $\kappa = -\sigma_1/(\sigma_1+\sigma_2)$ and $\mu = [\sigma]/(\sigma_1+\sigma 2)$. The matricial operator arising in this equation has a continuous inverse. A straightforward computation gives that $u|_{\partial\omega^*} = \psi$ solves

$$\left[\left(\tfrac{1}{2}I + \mu K_{\omega^*}\right) + \mu K_{\partial\Omega\partial\omega^*}\left(-\tfrac{1}{2}I + K_\Omega\right)^{-1} K_{\partial\omega^*\partial\Omega}\right]\psi = \kappa\left[S_{\partial\Omega\partial\omega^*} - K_{\partial\Omega\partial\omega^*}\left(-\tfrac{1}{2}I + K_\Omega\right)^{-1} S_\Omega\right]\phi.$$

Since the operators $K_{\partial\Omega\partial\omega^*}$ and $S_{\partial\Omega\partial\omega^*}$ are compact, the operator R_2 is compact, hence M_1 is compact. The proof of compactness of M_2 is similar and therefore the Hessian is compact.

References

[1] Afraites, L., Dambrine, M., Eppler, K., and Kateb, D. Detecting perfectly insulated obstacles by shape optimization techniques of order two. *Discrete Contin. Dyn. Syst. Ser. B 8*, 2 (2007), 389–416 (electronic).

[2] Afraites, L., Dambrine, M., and Kateb, D. Shape methods for the transmission problem with a single measurement. *Numer. Funct. Anal. Optim. 28*, 5-6 (2007), 519–551.

[3] Afraites, L., Dambrine, M., and Kateb, D. On second order shape optimization methods for electrical impedance tomography. *SIAM J. Control Optim. 47*, 3 (2008), 1556–1590.

[4] Astala, K., and Päivärinta, L. Calderón's inverse conductivity problem in the plane. *Ann. of Math. (2) 163*, 1 (2006), 265–299.

[5] Eppler, K., and Harbrecht, H. A regularized Newton method in electrical impedance tomography using shape Hessian information. *Control Cybernet. 34*, 1 (2005), 203–225.

[6] Hettlich, F., and Rundell, W. The determination of a discontinuity in a conductivity from a single boundary measurement. *Inverse Problems 14*, 1 (1998), 67–82.

[7] Kirsch, A. The domain derivative and two applications in inverse scattering theory. *Inverse Problems 9*, 1 (1993), 81–96.

[8] Maz′ya, V. G., and Shaposhnikova, T. O. *Theory of multipliers in spaces of differentiable functions*, vol. 23 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[9] Nachman, A. I. Reconstructions from boundary measurements. *Ann. of Math. (2) 128*, 3 (1988), 531–576.

[10] Pantz, O. Sensibilité de l'équation de la chaleur aux sauts de conductivité. *C. R. Math. Acad. Sci. Paris 341*, 5 (2005), 333–337.

[11] Sylvester, J., and Uhlmann, G. A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math. (2) 125*, 1 (1987), 153–169.

Lekbir Afraites
École Nationale des Sciences Appliquées, Safi
Université Cadi Ayyad, Maroc
`Lekbir.Afraites@cea.fr`

Marc Dambrine
Laboratoire de Mathématiques et de leurs Applications
Université de Pau et des Pays de l'Adour
`Marc.Dambrine@univ-pau.fr`

Djalil Kateb
Laboratoire de Mathématiques Appliquées de Compiègne
Université de Technologie de Compiègne
`Djalil.Kateb@utc.fr`