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DEFORMATION AND DRAPE OF A PIECE OF FABRIC, SUBJECTED TO A FORCE FIELD

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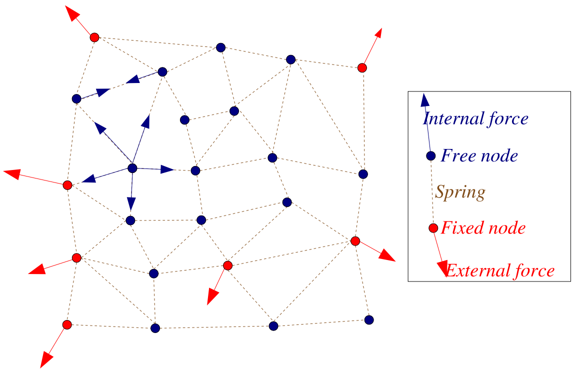
Part I
The problem, modelisation

§1. Hypothesis, notations



Let $\mathcal{N}$ be a set of n **nodes** or **masses** numbered in $\mathbb{R}^d$ (here $d = 2$ or 3). For simplicity, nodes are noted indifferently i or n_i, to compound a node and it's **number**. With no ambiguity, notations for the set $\mathcal{N}$ and its cardinal are identical.

- Each node i has a set $\mathcal{V}_i^* \subset \mathcal{N}$ of **neighbors** we shall note

$$\mathcal{V}_i = \mathcal{V}_i^* \cup \{i\}$$

- Each node i is bounded to its neighbors through **springs** R_{ij} with variable **stiffness** k_{ij}.
- Each node i is submitted to **an external force** $F_i, i \in \mathcal{N}$, (possibly zero).
- $\forall i \in \mathcal{N}$ let us denote by

$$X_i \quad = \quad (x_i, y_i, z_i) \in \mathbb{R}^3$$

  the i^{th} node **generic position** (say at one iteration or at generic time step).

- for all $(i,j) \in \mathcal{N}^2$, vector $\overrightarrow{X_i X_j}$ wil be denoted

$$\begin{aligned} X_{ij} &= X_j - X_j \\ X_{ij}^0 &= X_j^0 - X_j^0 \end{aligned}$$

- The notation

$$X_i^{00} = \left(x_i^{00}, y_i^{00}, z_i^{00}\right) \in \mathbb{R}^3$$

is set for let's position at equilibrium, without any external force, each springs being at its **reference length**.

- Let

$$X_i^0 = \left(x_i^0, y_i^0, z_i^0\right) \in \mathbb{R}^3$$

be i^{th} node **initial position** .

- Let us suppose

$$\forall i \in \mathcal{N}, \quad X_i^{00} = X_i^0$$

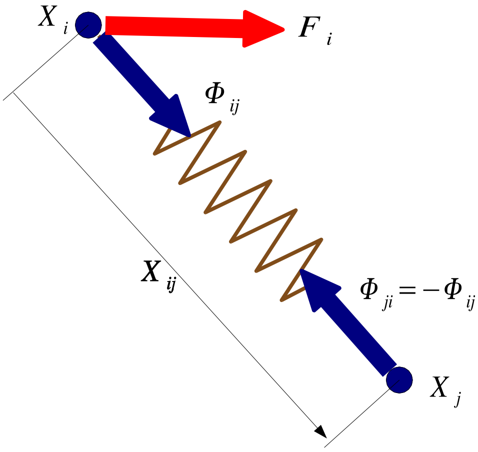
- spring reference length is given by distances

$$\left|X_{ij}^{00}\right| = \left|X_j^{00} - X_i^{00}\right|, 1 \leq i, j \leq \mathcal{N}$$

- For $1 \leq i \leq \mathcal{N}, j \in \mathcal{V}_i$, spring R_{ij} (stiffness k_{ij}) acts on node i through a force denoted Φ_{ij}, directed by normalized vector $u_{ij} = \frac{X_{ij}}{|X_{ij}|}$, from i to j . This force is proportional to elongation $|X_{ij}| - \left|X_{ij}^{00}\right|$ and verifies :

$$\Phi_{ij}(X) = k_{ij}\left(|X_{ij}| - \left|X_{ij}^{00}\right|\right)\frac{X_{ij}}{|X_{ij}|} \quad (1)$$

$$= k_{ij} X_{ij}\left(1 - \frac{\left|X_{ij}^{00}\right|}{|X_{ij}|}\right) \quad (2)$$



- Partition of $\mathcal{N}$: let us suppose some nodes have a fixed position,
  - $\mathcal{D}$ is the set of **fixed nodes** (or **Dirichlet nodes**) and
  - $\mathcal{L} = \mathcal{N} \setminus \mathcal{D}$ is the set of other nodes (**free nodes**).
  - so that, set $\mathcal{N}$ has a natural partition[1]:
$$\mathcal{N} = \mathcal{L} \cup \mathcal{D} \tag{3}$$
    variables are therefore partitioned the same way :
$$\begin{aligned} X &= (X_{\mathcal{L}}, X_{\mathcal{D}}) \\ F &= (F_{\mathcal{L}}, F_{\mathcal{D}}) \end{aligned}$$
- Then, for each node, i, we know :
  - even its position
$$X_i = X_i^0 \text{ if } i \in \mathcal{D} \tag{4}$$
  - or external force acting on it
$$F_i = F_i^0 \text{ if } i \in \mathcal{L} \tag{5}$$
- **Unknowns** are free nodes positions $X_i,\ i \in \mathcal{L}$ and external forces for fixed nodes $F_i\left(X\right),\ i \in \mathcal{D}$ (support reactions).

§2. Set of equations

2.1. for a node

$t \in \mathbb{R}_+$ is time, and $G \in \mathbb{R}^3$ is $\mathcal{N}$'s center of mass. Let us assume movement is submitted to viscous forces. More precisely, a particle with speed $\dot{X}\left(t\right)$ is submitted to the viscous force $-\beta \dot{X}\left(t\right),\ \beta > 0$.

The dynamic fundamental law applied to node i can be written :

$$\sum_{j \in \mathcal{V}_i^*} \Phi_{ij}\left(X\left(t\right)\right) + F_i\left(X\left(t\right)\right) - \beta \dot{X}_i\left(t\right) = m_i \ddot{X}_i\left(t\right) \tag{6}$$

Using Φ_{ij} expression [1], leads to, for each node i :

$$\begin{aligned} \forall i \in \mathcal{N},\ m_i \ddot{X}_i &= \sum_{j \in \mathcal{V}_i^*} k_{ij} X_{ij} \left(1 - \frac{\left|X_{ij}^{00}\right|}{\left|X_{ij}\right|}\right) + F_i - \beta \dot{X}_i \\ &= \sum_{j \in \mathcal{V}_i^*} k_{ij}\left(X_j - X_i\right) \left(1 - \frac{\left|X_{ij}^{00}\right|}{\left|X_{ij}\right|}\right) + F_i - \beta \dot{X}_i \end{aligned}$$

therefore

$$\forall i \in \mathcal{N},\ m_i \ddot{X}_i = \sum_{j \in \mathcal{V}_i^*} k_{ij} \left(1 - \frac{\left|X_{ij}^{00}\right|}{\left|X_{ij}\right|}\right) X_j - \left(\sum_{j \in \mathcal{V}_i^*} k_{ij} \left(1 - \frac{\left|X_{ij}^{00}\right|}{\left|X_{ij}\right|}\right)\right) X_i + F_i - \beta \dot{X}_i \tag{7}$$

[1] We could consider some node free for certain directionOx for instance, and fixed for others directions. In this case, we have three natural partitions of $\mathcal{N} = \mathcal{L}_x \cup \mathcal{D}_x = \mathcal{L}_y \cup \mathcal{D}_y = \mathcal{L}_z \cup \mathcal{D}_z$

2.2. Matrix formulation

Let us denote $A^0 = A\left(X^{00}\right)$ where $A\left(X\right) \in \mathbb{R}^{\mathcal{N},\mathcal{N}}$ is given by :

$$\begin{aligned} A_{ij}\left(X\right) &= -k_{ij}\frac{\left|X_{ij}^{00}\right|}{\left|X_{ij}\right|} \text{ if } i \neq j \\ &= \sum_{l \in \mathcal{V}_i^*} k_{il}\frac{\left|X_{il}^{00}\right|}{\left|X_{il}\right|} \text{ if } i = j \end{aligned}$$

and

$$\mathbb{A}\left(X\right) = \begin{pmatrix} A\left(X\right) & 0 & 0 \\ 0 & A\left(X\right) & 0 \\ 0 & 0 & A\left(X\right) \end{pmatrix} \in \mathbb{R}^{3\mathcal{N},3\mathcal{N}}$$

Fundamental equation [7] becomes :

$$\mathbb{A}\left(X\right)X - \mathbb{A}^0 X + F\left(X\right) = M\ddot{X} + \beta\dot{X} \tag{8}$$

where

$$\mathbb{M} = \begin{pmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{pmatrix} \in \mathbb{R}^{3\mathcal{N},3\mathcal{N}}$$

with $M = diag\left(m_i, i \in \mathcal{N}\right) \in \mathbb{R}^{\mathcal{N},\mathcal{N}}$ is the elementary masses diagonal matrix.

This non linear system [8] can be split into three equations uncoupled : we denote

$$\begin{aligned} X &= \left(x, y, z\right)^t \in \left(\mathbb{R}^{\mathcal{N}}\right)^3 \\ F &= \left(f, g, h\right)^t \in \left(\mathbb{R}^{\mathcal{N}}\right)^3 \end{aligned}$$

so we obtain :

$$\begin{aligned} A\left(X\right)x - A^0 x + f\left(X\right) &= M\ddot{x} + \beta\dot{x} \\ A\left(X\right)y - A^0 y + g\left(X\right) &= M\ddot{y} + \beta\dot{y} \\ A\left(X\right)z - A^0 z + h\left(X\right) &= M\ddot{z} + \beta\dot{z} \end{aligned} \tag{9}$$

2.3. $A\left(X\right)$ matrices properties

Node's partition (3) gives matrices A^0 et $A\left(X\right)$ a bloc expression, following template :

$$A = \begin{pmatrix} A_{\mathcal{L}} & A_{\mathcal{LD}} \\ A_{\mathcal{DL}} & A_{\mathcal{D}} \end{pmatrix} \tag{10}$$

Notice that $A_{\mathcal{LD}} = A_{\mathcal{DL}}^t$ because A^0 and $A\left(X\right)$ are symmetrical. Moreover, some properties hold :

properties :

- $A(X)$ is non invertible matrix, and $(1,1,....,1)^t \in Ker(A(X))$
- $A(X)$ is positive : $\forall u \in \mathbb{R}^{\mathcal{N}}, \langle A(X).u, u\rangle \geq 0$
- $0 \in Sp(A(X)) \subset \mathbb{R}_+$
- $A^0_{\mathcal{L}}$ is invertible because it is strongly diagonal dominant, if $\mathcal{D} \neq \emptyset$.
- $A^0_{\mathcal{L}}$ is symmetric positive definite.

Proof. $A(X)$ is symmetric, so spectrum of $A(X)$ is real. Relation $A_{ii}(X) = -\sum_{j\in\mathcal{V}_i^*} A_{ij}(X)$ shows that $(1,1,....,1)^t$ is an eigenvector associated to eigenvalue 0. So A is not invertible.

Moreover $\forall j \in \mathcal{V}_i^*,\ A_{ij}(X) < 0$ and $A_{ii}(X) > 0$ so $r_i = |A_{ii}(X)| = \sum_{j\in\mathcal{V}_i^*} |A_{ij}(X)|$
i^{th} Gershgorin disk is

$$\begin{aligned} D_i &= \{z \in \mathbb{C}, |z - A_{ii}(X)| \leq r_i\} \\ &= \{z \in \mathbb{C}, |z - r_i| \leq r_i\} \end{aligned}$$

Spectrum of A is included (Gershgorin's theorem) in $D = \bigcup_i D_i$, so $Sp(A(X)) \subset \mathbb{R}_+$.

If $\mathcal{D} \neq \emptyset$ matrix $A^0_{\mathcal{L}}$ is not equal to A^0 and when suppressing rows and columns from A^0, we suppress coefficients $A^0_{ij} < 0$, so that equality $A^0_{ii} = -\sum_{j\in\mathcal{V}_i^*} A^0_{ij}$ becomes strict inequality $A^0_{\mathcal{L}}$: $|(A^0_{\mathcal{L}})_{ii}| > \sum_{j\in\mathcal{V}_i^*, j\notin\mathcal{D}} \left|(A^0_{\mathcal{L}})_{ij}\right|$. So, matrix $A^0_{\mathcal{L}}$ is strongly diagonal dominant, consequently, it is positive definite. □

Expression of (9) by blocs Let us now examine each term in equations (9) :

Rewriting first equation (9) by bloc, following partition $\mathcal{N} = \mathcal{L} \cup \mathcal{D}$, we obtain :

$$\begin{aligned} [A(X)x]_{\mathcal{L}} - \left[A^0 x\right]_{\mathcal{L}} + \quad f_{\mathcal{L}}(X) &= [M\ddot{x} + \beta\dot{x}]_{\mathcal{L}} \\ [A(X)x]_{\mathcal{D}} - \left[A^0 x\right]_{\mathcal{D}} + \quad f_{\mathcal{D}}(X) &= [M\ddot{x} + \beta\dot{x}]_{\mathcal{D}} \end{aligned}$$

Now, A has a bloc expression we can use here and, using conditions (4 et 5), we can write :

$$[A(X)x]_{\mathcal{L}} - A^0_{\mathcal{L}} x_{\mathcal{L}} - A^0_{\mathcal{L}\mathcal{D}} x^{00}_{\mathcal{D}} + f^0_{\mathcal{L}} = M_{\mathcal{L}}\ddot{x}_{\mathcal{L}} + \beta\dot{x}_{\mathcal{L}} \tag{11}$$

$$[A(X)x]_{\mathcal{D}} - \left[A^0 x\right]_{\mathcal{D}} + f_{\mathcal{D}}(X) = 0 \tag{12}$$

Part II
Resolution

§3. Solving equilibrium equations

On equilibrium, $\dot{X} = (\dot{x}, \dot{y}, \dot{z}) = 0$ and $\ddot{X} = (\ddot{x}, \ddot{y}, \ddot{z}) = 0$. Moreover equation (8) :

$$\mathbb{A}(X)X - \mathbb{A}^0 X + F(X) = 0$$

suggests a fix point algorithm looking like :

$$\mathbb{A}^0 X^{n+1} = \mathbb{A}^n X^n + F^n$$

Of course, $\mathbb{A}^0$ isn't invertible, so we have to use bloc equations (11,12), which allows us to compute unknowns $x_{\mathcal{L}}$ et $f_{\mathcal{D}}$

$$[A(X)x]_{\mathcal{L}} - A^0_{\mathcal{L}} x_{\mathcal{L}} - A^0_{\mathcal{LD}} x^{00}_{\mathcal{D}} + f^0_{\mathcal{L}} = 0 \tag{13}$$

$$[A(X)x]_{\mathcal{D}} - \left[A^0 x\right]_{\mathcal{D}} + f_{\mathcal{D}}(X) = 0 \tag{14}$$

and we use suggested fix point algorithm only on first equation, the second one directly gives a value for $f_{\mathcal{D}}$. By this way, equations rewrites :

$$\begin{aligned} x^{n+1}_{\mathcal{L}} &= \left(A^0_{\mathcal{L}}\right)^{-1} \left([A^n x^n]_{\mathcal{L}} - A^0_{\mathcal{LD}} x^{00}_{\mathcal{D}} + f^0_{\mathcal{L}}\right) \\ f^{n+1}_{\mathcal{D}} &= \left[A^{n+1} x^{n+1}\right]_{\mathcal{D}} - \left[A^0 x^{n+1}\right]_{\mathcal{D}} \end{aligned}$$

3.1. Algorithm

Equilibrium algorithm

`given reference positions` $(x, y, z) = \left(x^{00}, y^{00}, z^{00}\right)$
`initial positions` $(x, y, z) = \left(x^0, y^0, z^0\right) = \left(x^{00}, y^{00}, z^{00}\right)$
`assembly` A^0`,` $A = A^0$
`given` $(f, g, h) = \left(f^0, g^0, h^0\right)$
`internal forces` $(\varphi, \psi, \theta) = (0, 0, 0)$
`residualforces` $(R_x, R_y, R_z) = (f, g, h) + (\varphi, \psi, \theta)$
$r_0 = \|R_x\|_2 + \|R_y\|_2 + \|R_z\|_2$
`while` $(r/r_0 > \epsilon)$
    `solve`
    $A^0_{\mathcal{L}} x = (R_x)_{\mathcal{L}}$
    $A^0_{\mathcal{L}} y = (R_y)_{\mathcal{L}}$
    $A^0_{\mathcal{L}} z = (R_z)_{\mathcal{L}}$
    `assemble` A
    `internal forces` $(\varphi, \psi, \theta) = (Ax, Ay, Az) - \left(A^0 x, A^0 y, A^0 z\right)$
    `external forces` $(f, g, h)_D = -(\varphi, \psi, \theta)_D$
    `residual forces` $(R_x, R_y, R_z) = (f, g, h) + (\varphi, \psi, \theta)$
    `residual` $r = \|R_x\|_2 + \|R_y\|_2 + \|R_z\|_2$
`end while`

This algorithm (see 4.1) reveals a slow but very stable behavior.

§4. Solving dynamic equations

In this part, we are interested on the same problem, from a dynamical point of view.

Dynamic equations are known, see [8] et [9]. In this case, of course $\dot{X} = (\dot{x}, \dot{y}, \dot{z}) \neq 0$ et $\ddot{X} = (\ddot{x}, \ddot{y}, \ddot{z}) \neq 0$

Fix point algorithm used for computing equilibrium, show to be very stable, therefore, we try slight modifications to take into account inertial and viscosity terms. A time step dt and an integer n are given, we approach $\dot{X}(n.dt) := \dot{X}^n$ and $\ddot{X}(n.dt) := \ddot{X}^n$ with order 2 central finite differences :

$$\begin{aligned}\dot{X}^n &= \frac{X^{n+1} - X^{n-1}}{2dt} + O\left(dt^2\right) \\ \ddot{X}^n &= \frac{X^{n+1} - 2X^n + X^{n-1}}{dt^2} + O\left(dt^2\right)\end{aligned}$$

Fixed point idea that drove us until now can be extended as follow :

$$A^n x^n - A^0 x^{n+1} + f^n = 0$$

becomes

$$A^n x^n - A^0 x^{n+1} + f^n = M\ddot{x}^n + \beta\dot{x}^n$$

therefore

$$A^0 x^{n+1} = A^n x^n + f^n - \frac{M}{dt^2}\left(x^{n+1} - 2x^n + x^{n-1}\right) - \frac{\beta}{2dt}\left(x^{n+1} - x^{n-1}\right)$$

or :

$$\left(A^0 + \frac{M}{dt^2} + \frac{\beta}{2dt}I\right)x^{n+1} = A^n x^n + f^n + \frac{M}{dt^2}\left(2x^n - x^{n-1}\right) + \frac{\beta}{2dt}x^{n-1} \qquad (15)$$

We make the same block decomposition (see,10), applied to matrix $A^0 + \frac{M}{dt^2} + \frac{\beta}{2dt}I$. Numerical results are given in : 4.3

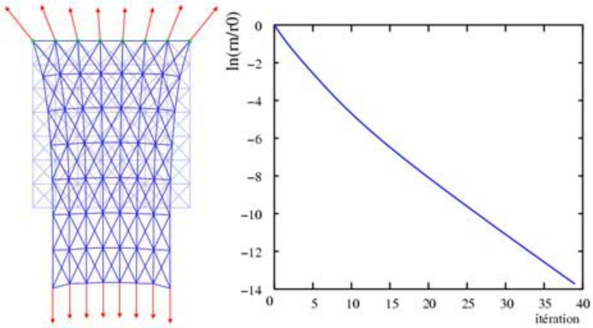
Part III

Numerical results

4.1. 2D numerical results at equilibrium

Stopping criterion is related to residual euclidean norms ratio : $\frac{r^n}{r^0} < 10^{-4}$ with $r^n = \left\|\left(\mathbb{A}^0 - \mathbb{A}^n\right) X^n + F(X)\right\|_2$.

4.1.1. A piece of fabric (2D)



(a) position initiale et filet déformé

(b) Log du résidu

Figure 1: A piece of fabric (8X8 nodes) with uniform force field on a portion of the boundary (iteration 41).

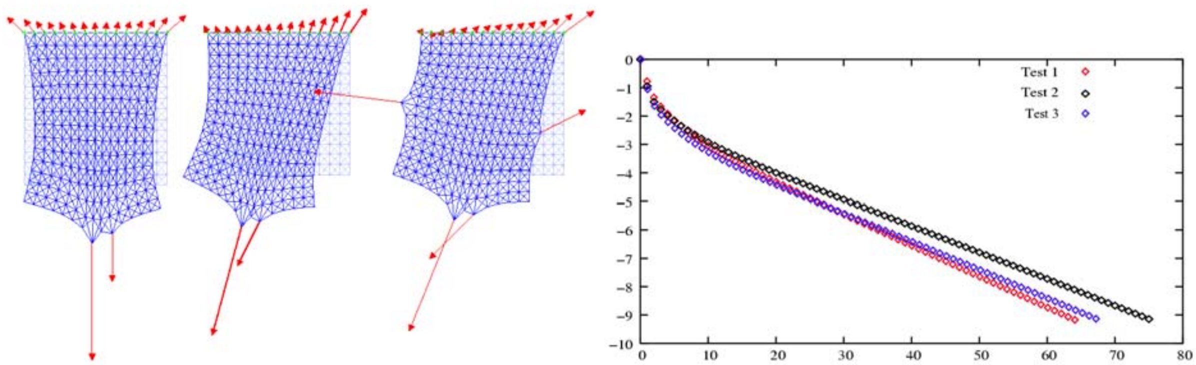


Figure 2: A piece of fabric (15X15 nodes) with non uniform force field on portions of the boundary.

In the simulation that follows, with the same conditions, a non elastic behavior has been imposed on the material. Elasticity coefficients depend on k_{ij} sign of $|X_{ij}| - \left|X_{ij}^{00}\right|$:

- $k_{ij} > 0$ si $|X_{ij}| > \left|X_{ij}^{00}\right|$ (elastic under elongation) and
- $k_{ij} = 0$ si $|X_{ij}| \leq \left|X_{ij}^{00}\right|$ (no resistance under compression).

Notice that this simulation produces images with a realistic 3D appearance, while execution times are drastically reduced if compared with a real 3D simulation.

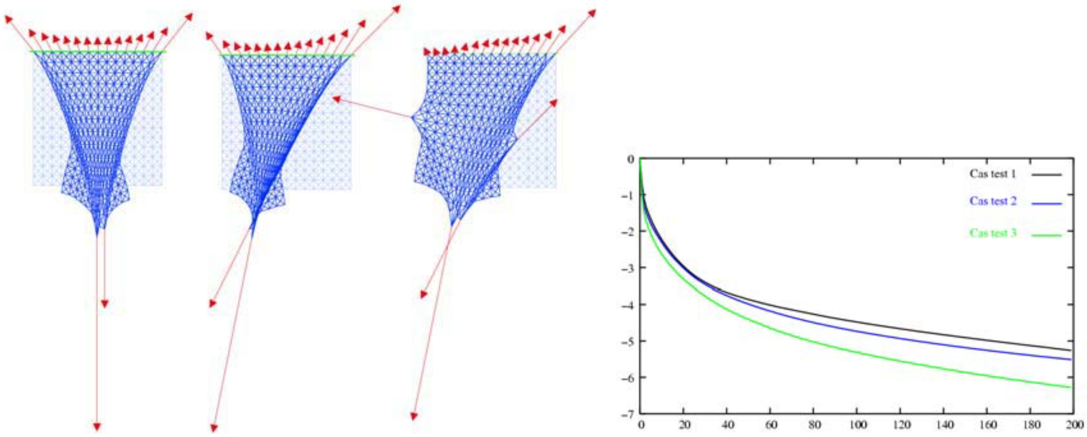


Figure 3: A piece of fabric (15X15 nodes) with non uniform force field on portions of the boundary.

4.1.2. Profile of a paraglider (2D)

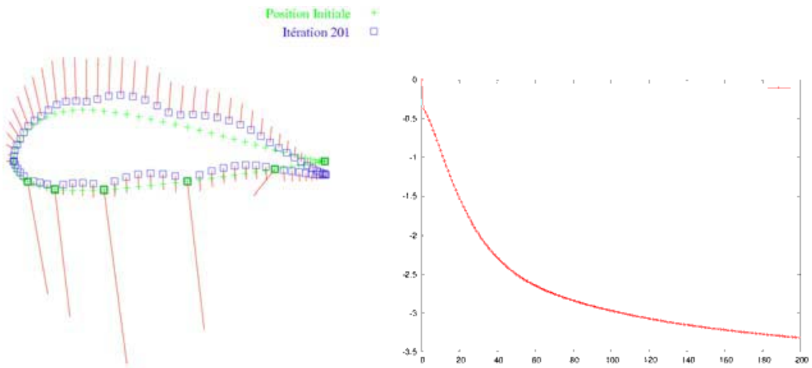


Figure 4: profile deformation : fixed trailing edge.

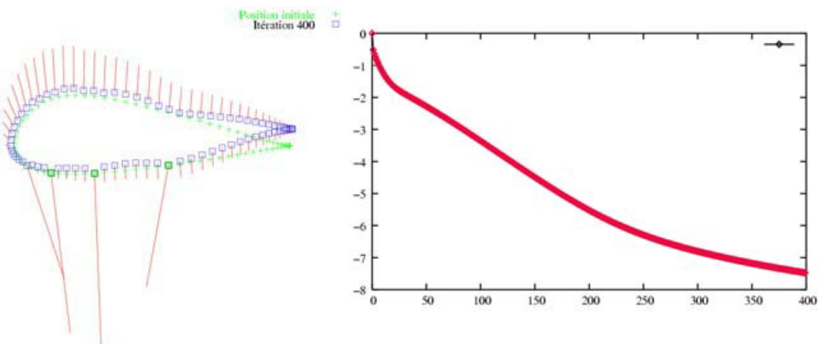


Figure 5: profile deformation : free trailing edge.

4.2. Equilibrium, numerical results (3D).

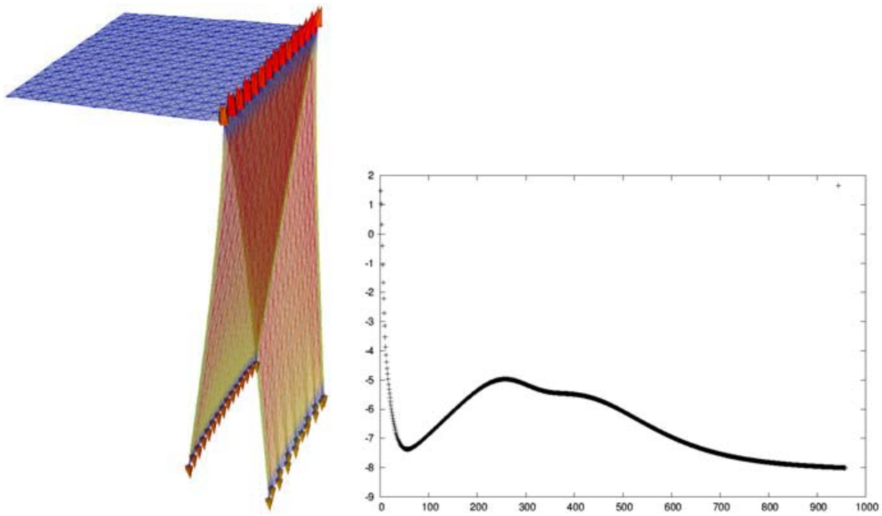


Figure 6: Test case 1 : a piece of fabric (15X15 nodes) subject to its own weight and weighted in its lower boundary, fixed in its upper boundary. Realistic undulations can be seen at the lower part of the fabric. (Like the drape of a curtain).

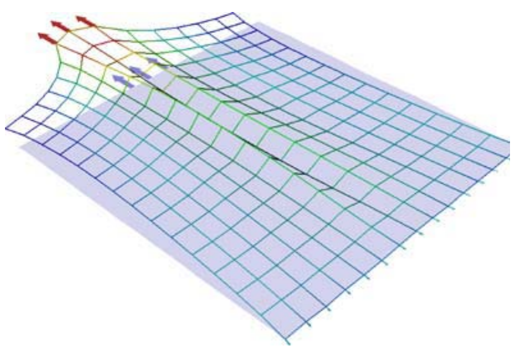


Figure 7: Test case 2 : the same piece of fabric without weight, an external force pulls on a border, the opposite border is fixed.

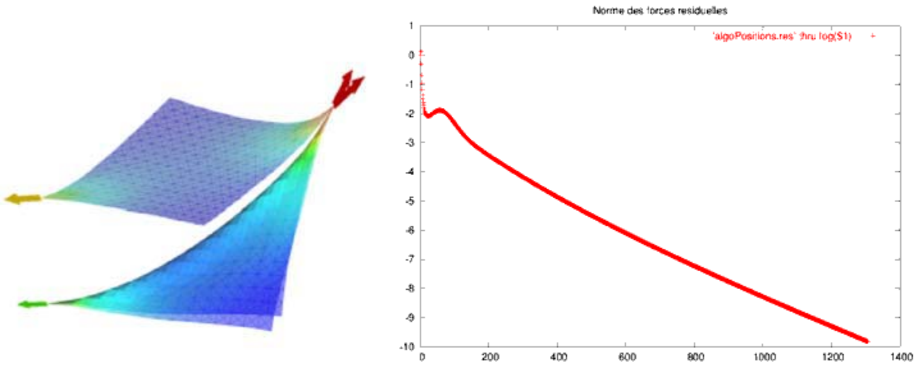


Figure 8: Test case 2 : the same piece of fabric without weight, an external force pulls on a corner, the opposite corner is fixed.

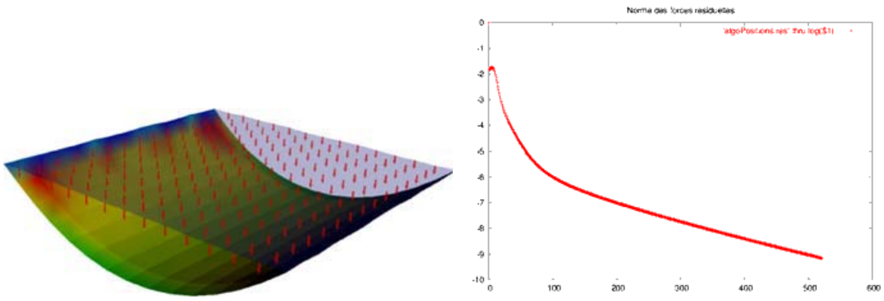


Figure 9: Test case 2 : the same piece of fabric submitted to its own weight, two opposite borders are fixed.

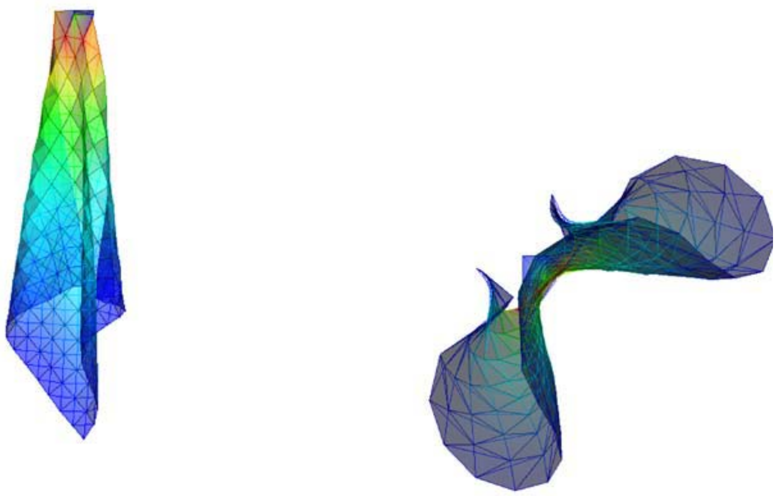


Figure 10: Test case 5 : two views of the same material, hanged on a corner, submitted to its own weight. At equilibrium, the solution is probably not unique.

4.3. Dynamic, numerical results (3D)

4.3.1. Test case 2

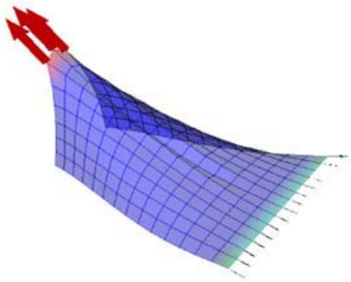


Figure 11: The piece of fabric is submitted to a 3 Newtons force distributed on 3 nodes on a border, and fixed on the opposite border, and finally subjected to its own weight.

Caracteristics

- $t \in [0, 25]\,, \epsilon = 10^{-5}$
- $m_t = 225\,g, K_i = \begin{bmatrix} 3.5 & 10.5 & 3.5 \\ 7 & * & 7 \\ 3.5 & 10.5 & 3.5 \end{bmatrix}$
- stopping criterium : $w_c < \epsilon^2$ or $t > t_{max}$ or $r < \epsilon$

dt	β	it	r	t_{final}	w_c	w_{cg}
5.0	0	5	$1.5e-2$	25	$3.5e-06$	$2.7e-06$
0.5	0	50	$3.0e-3$	25	$5.0e-07$	$5.1e-08$
0.05	0	500	$1.9e-4$	25	$5.2e-9$	$2.8e-11$
0.01	0	1868	$3.13e-5$	18.7	$1.0e-12$	$4.9e-15$
0.005	0	3140	$1.0e-5$	18	$2.2e-08$	$2.7e-11$
0.001	0	12474	$9.9e-06$	12.5	$7.6e-06$	$5.6e-06$
5.0	10^{-3}	5	$1.5e-2$	25	$3.5e-6$	$2.7e-06$
0.5	10^{-3}	50	$3.0e-3$	25	$5.1e-7$	$5.2e-8$
0.05	10^{-3}	500	$1.9e-4$	25	$5.3e-9$	$2.9e-11$
0.01	10^{-3}	1656	$2.1e-5$	16.6	$1.0e-10$	$1.5e-14$
0.05	10^{-6}	500	$1.9e-4$	25.0	$5.2e-09$	$2.7e-11$
0.01	10^{-6}	1868	$2.1e-05$	18.7	$1.0e-10$	$4.9e-15$

Kinetic energy as a function of time

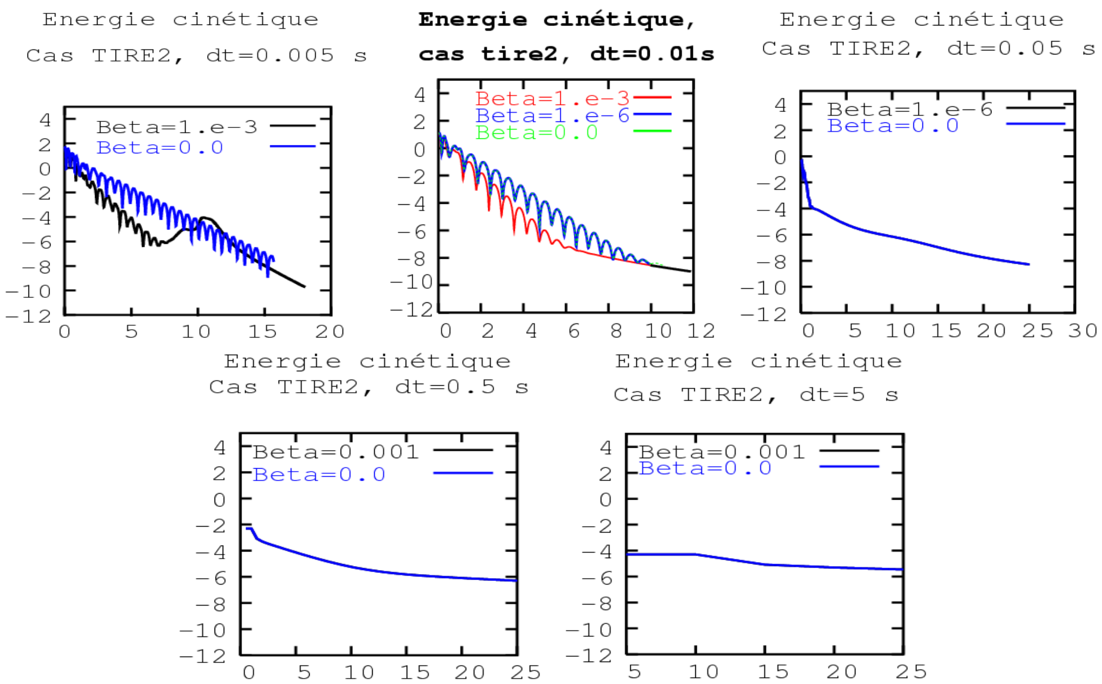


Figure 12: Test case 2, total kinetic energy as a function of time, for various values of time step and of damping coefficient.

4.3.2. Test case tombe1

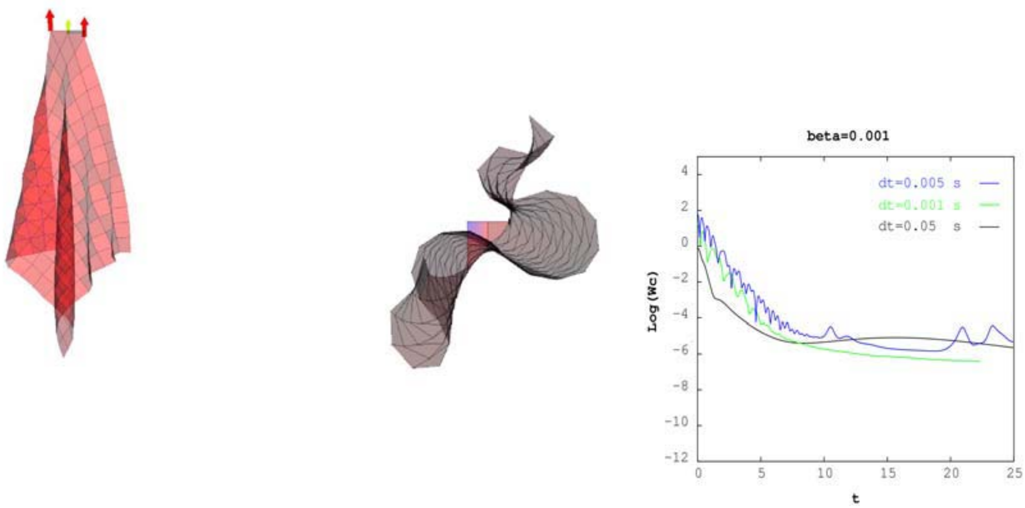


Figure 12: Test case “tombe 1”, two views, kinetic energy history for various values of time step.

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