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LINEAR ELASTICITY MODELLING OF THE BEHAVIOUR OF A PRE-STRESSED MATERIAL

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Abstract. In order to study the behavior of the outer envelop of an airship with great payloads, we have to modelize the behaviour of an inflated material structure. Indeed, the envelop is made of a succession of independant longitudinal lobes. Each lobe is made of two rectangular pieces of material stuck on their lengths and widths to the metallic armature of the airship. The inner volume thus obtained is inflated with a perfect gas so as to pre-stress the whole structure. The expected result is to stabilize the geometry of the airship in case of wind, shocks, impacts, turbulences ... and to protect the Helium envelops which are inside. Assuming that the problem is stationnary, that the material is orthotropic, that the outer disturbances are known and that the inner pressure is uniform, we modelize in 3D this fluid-structure interaction problem and we prove the existence and uniqueness of the solution. We explicitely calculate the pre-stress.

§1. Modelling

1.1. Description of one lobe

One lobe is made of two pieces of material. When required the upper layer has the exponent $+$ and the lower layer the exponent $-$. Their length L is very large compared to the width $2l$ which is also very large compared to the thickness e.
Let Ω^+ be the connex bounded open set occupied by the upper material, Ω^- the connex bounded open set occupied by the lower material, and $\Omega = \Omega^+ \cup \Omega^-$, let Γ be the outer free surface and γ be the inner free surface, let Γ_0 be the surface of the thickness in the length and γ_0 the surface of the thickness in the width.

1.2. Assumptions on the materials

The materials are supposed to be elastic and orthotropic. They are modelled in 3D even though one dimension is very small compared to the other ones. The symmetric elasticity tensor, expressed in the axes of orthotropy is $\mathbf{A} = (a_{ijkl})_{1\leq ijkl\leq 3}$ where $a_{ijkl} = \lambda_{ik}\delta_{ij}\delta_{kl} + \mu_{ij}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$. From now on, the directions of orthotropy are indiced with s, r, z or $1, 2, 3$ so that for example, $\lambda_{23} = \lambda_{rz}$. One necessary condition of isotherm stability [Salençon] is that the quadratic form $\mathbf{e} \longmapsto \mathbf{A} : \mathbf{e} : \mathbf{e}$ is positive definite when restricted to space of symmetric tensors.

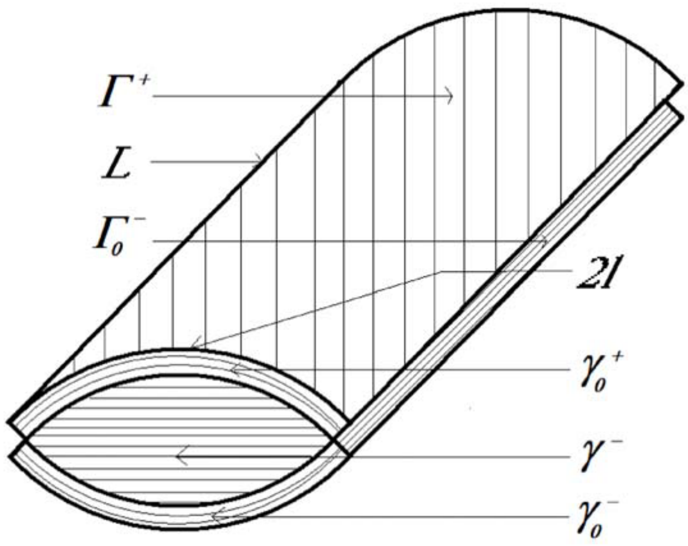

Figure 1: Notations

1.3. Mechanical assumptions

1.3.1. Three states of equilibrium

First of all, we assume the existence of a **quasi naturel state of equilibrium**. This state is physically obtained just before the material stretches when inflating the lobe. In this unstable state, the stress tensor can be neglected compared to the constants of elasticity and the variables have a q exponent.

When this unstable state is reached, the next step is to increase the inflating pressure so as to get a stable state of equilibrium. Each state thus obtained is a **pre-stressed state of equilibrium** where the inner forces depend on the inflating pressure. The variables of this state have a p exponent. The displacement to get it from the quasi natural state is $\mathbf{u}^q$.

Then, the airship is subjected to an outer (known) perturbance. Its shape is deformed. the corresponding displacement of the material is $\mathbf{u}$. The variables of this **deformed state of equilibrium** have no exponent.

1.3.2. Linearized elasticity

Because of the expected result of the pre-stress, we admit that the displacement and the displacement gradient are small. Then, according to [Salençon], the Cauchy stress tensor and the Piola-Kirchoff tensors can be taken one for the other in the equations of equilibrium, whereas, in the behaviour law of an elastic orthotropic material, the linearized strain tensor can replaced by the Green-Lagrange deformation tensor, [Duvaut]. So,

$$\mathbf{\Sigma} = \mathbf{A} : \mathbf{\frac{1}{2}} \left(\nabla \mathbf{u} + \nabla \mathbf{u^T}\right) + \mathbf{\Sigma^P} \text{ and } \mathbf{\Sigma}^p = \mathbf{A} : \frac{1}{2} \left(\nabla \mathbf{u}^q + \nabla \mathbf{u}^{qT}\right) + \mathbf{\Sigma}^q \qquad (1)$$

and

$$\left\{\begin{array}{ll} -\mathbf{div}\left(\boldsymbol{\Sigma}^q\right)=\rho\mathbf{g} & \text{in } \Omega^q \\ \boldsymbol{\Sigma}^q\mathbf{n}=-P\mathbf{n} & \text{on } \Gamma^q \\ \boldsymbol{\Sigma}^q\mathbf{n}=-\pi^q\mathbf{n} & \text{on } \gamma^q \end{array}\right. \quad \left\{\begin{array}{ll} -\mathbf{div}\left(\boldsymbol{\Sigma}^p\right)=\rho\mathbf{g} & \text{in } \Omega^p \ (\Omega^q) \\ \boldsymbol{\Sigma}^p\mathbf{n}=-P\mathbf{n} & \text{on } \Gamma^p \ (\Gamma^q) \\ \boldsymbol{\Sigma}^p\mathbf{n}=-\pi^p\mathbf{n} & \text{on } \gamma^p \ (\gamma^q) \end{array}\right. \quad \left\{\begin{array}{ll} -\mathbf{div}\left(\boldsymbol{\Sigma}\right)=\rho\mathbf{g} & \text{in } \Omega^p \\ \boldsymbol{\Sigma}\mathbf{n}=-P\mathbf{n}+\delta\mathbf{F} & \text{on } \Gamma^p \\ \boldsymbol{\Sigma}\mathbf{n}=-(\pi^p+\delta\pi)\mathbf{n} & \text{on } \gamma^p \end{array}\right. \tag{2}$$

where $\mathbf{n}$ is the outer normal unit vector, ρ the volumic mass of the material, P the outer uniform pressure, π^q the inner uniform pressure, π^p the inflating uniform pressure, $\delta\mathbf{F}$ is an outer perturbation and $\delta\pi$ the resulting variation of inner pressure.

1.4. Relation between the inner pressure and the shape of a lobe

The inflating fluid is a perfect gas so $\pi^p = -\frac{nRT}{V}\mathbf{n}^p_{ext}$ in the pre-stressed configuration. A perturbation causes the displacement $\mathbf{u}$ of the envelop and the variation δV of the inner volume. Assuming it remains uniform, the inner pressure becomes $\pi^p + \delta\pi = -\frac{nRT}{V+\delta V}\mathbf{n}_{ext}$. Since the small perturbations assumptions are fulfilled we can suppose that the outer unit vectors $\mathbf{n}^p_{ext}$ and $\mathbf{n}_{ext}$ merge, so that $\delta\pi = -\left(\frac{nRT}{V+\delta V} - \frac{nRT}{V}\right)\mathbf{n}^p_{ext}$.

After a first order Taylor's development in $\dfrac{\delta V}{V}$ we get $\delta\pi \simeq -\frac{\delta V}{V}\pi_p$. The variation of volume and the displacement are linked by $\delta V = -\int_{\gamma_p}\mathbf{u}.\mathbf{n}^p_{ext}d\gamma$ so, in the end,

$$\delta\pi = -\frac{\pi_p}{V}\left(\int_{\gamma_p}\mathbf{u}.\mathbf{n}^p_{ext}d\gamma\right)\mathbf{n}^p_{ext}$$

1.5. Formulation of the perturbed problem

We suppose that the pre-stressed state is known (shape and stress) and we choose it as the reference state for the calculus. The outer perturbation $\delta\mathbf{F}$ is also supposed to be known and we want to calculate the resulting shape and stress. They are determined by the displacement $\mathbf{u}$. To calculate it, we suppose that the small perturbation assumptions are fulfilled and that the fixation of the material on the structure can be modeled by $\mathbf{u} = \mathbf{0}$ on $\Gamma_0 \cup \gamma_0$.

From (2) we deduce

$$\left\{\begin{array}{ll} -\mathbf{div}\left(\boldsymbol{\Sigma}-\boldsymbol{\Sigma}^p\right)=\mathbf{0} & \text{in } \Omega^p \\ \left(\boldsymbol{\Sigma}-\boldsymbol{\Sigma}^p\right)\mathbf{n}=\delta\mathbf{F} & \text{on } \Gamma^p \\ \left(\boldsymbol{\Sigma}-\boldsymbol{\Sigma}^p\right)\mathbf{n}=-\delta\pi\mathbf{n} & \text{on } \gamma^p \end{array}\right.$$

Because of (1) and because of the boundary conditions we are led to study the problem

$$\left\{\begin{array}{lll} -\mathbf{div}\left(\mathbf{A}:\frac{1}{2}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)\right) & =\mathbf{0} & \text{in } \Omega^p \\ \left(\mathbf{A}:\frac{1}{2}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)\right)\mathbf{n} & =\delta\mathbf{F} & \text{on } \Gamma^p \\ \left(\mathbf{A}:\frac{1}{2}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)\right)\mathbf{n} & =-\dfrac{\pi^p}{V}\left(\int_{\gamma^p}\mathbf{u}.\mathbf{n}d\gamma\right)\mathbf{n} & \text{on } \gamma^p \\ \mathbf{u}=\mathbf{0} & & \text{on } \Gamma_0\cup\gamma_0 \end{array}\right. \tag{3}$$

§2. Existence and uniqueness

2.1. Variational formulation

Let $\mathcal{V} = \left\{\mathbf{u} \in \mathbf{H}^1\left(\Omega_p\right),\ \mathbf{u} = \mathbf{0} \text{ on } \Gamma_0 \cup \gamma_0\right\}$

With the classical techniques, we can establish that the problem (3) is equivalent to the problem

$$\begin{cases} \mathbf{u}\in\mathcal{V}\,,\forall\mathbf{v}\in\mathcal{V} \\ \int_{\Omega^p}-\mathbf{div}\left(\mathbf{A}:\frac{1}{2}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)\right)\ \mathbf{v}d\Omega^p=0 \\ \int_{\Gamma^p}\left(\mathbf{A}:\frac{1}{2}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)\right)\mathbf{n}.\mathbf{v}d\Gamma=\int_{\Gamma^p}\delta\mathbf{F}.\mathbf{v}d\Gamma \\ \int_{\gamma^p}\left(\mathbf{A}:\frac{1}{2}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)\right)\mathbf{n}.\mathbf{v}d\gamma=-\int_{\gamma^p}\frac{\pi^p}{V}\left(\int_{\gamma^p}\mathbf{u}.\mathbf{n}d\gamma\right)\mathbf{n}.\mathbf{v}d\gamma \end{cases}$$

At this stage we use a Green's formula. Then, we let $\int_{\Omega_p}\left(\mathbf{A}:\frac{1}{2}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right):\frac{1}{2}\left(\nabla\mathbf{v}+\nabla\mathbf{v}^T\right)\right)d\Omega_p=b\left(\mathbf{u},\mathbf{v}\right)$. Because of the symmetry of the tensor $\mathbf{A}:\frac{1}{2}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)$ we have $b\left(\mathbf{u},\mathbf{v}\right)=\int_{\Omega_p}\left(\mathbf{A}:\frac{1}{2}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right):\nabla\mathbf{v}\right)d\Omega_p$.
So, taking account of the boundary conditions the previous problem becomes

$$\begin{cases} \mathbf{u}\in\mathcal{V}\,,\forall\mathbf{v}\in\mathcal{V} \\ b\left(\mathbf{u},\mathbf{v}\right)+\frac{\pi^p}{V}\left(\int_{\gamma^p}\mathbf{v}.\mathbf{n}d\gamma\right)\left(\int_{\gamma^p}\mathbf{u}.\mathbf{n}d\gamma\right)=\int_{\Gamma^p}\delta\mathbf{F}.\mathbf{v}d\Gamma \end{cases}$$

2.2. Study of the solution

Theorem 1. *The mapping from $\mathcal{V}$ into $\mathbb{R}$ thus defined* $\mathbf{u}\longmapsto\left(\int_{\Omega_p}\left(\mathbf{A}:\frac{1}{2}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right):\frac{1}{2}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)\right)d\Omega_p\right)^{\frac{1}{2}}$ *is a norm on $\mathcal{V}$, noted $\|\ \|_A$.*

Proof. The positive symmetric bilinear form on $\mathcal{V}\times\mathcal{V}$ $b\left(\mathbf{u},\mathbf{v}\right)$ verifies $b\left(\mathbf{u},\mathbf{u}\right)=0\Longrightarrow\frac{1}{2}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)=\mathbf{0}$. Moreover, because of the boundary condition $\mathbf{u}=\mathbf{0}$ on $\Gamma_0\cup\gamma_0$, we can prove that $\frac{1}{2}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)=\mathbf{0}\implies\mathbf{u}=\mathbf{0}$. □

Theorem 2. *The norms $\|\ \|_A$ and $|\ |_{1,\Omega_p}$ are equivalent on $\mathcal{V}$.*

Proof. Same pattern as the proof of the Korn's inequality given in [Duvaut-Lions]. □

Theorem 3. *Existence and uniqueness of $\mathbf{u}\in\mathcal{V}$ solution of $\forall\mathbf{v}\in\mathcal{V},\ a\left(\mathbf{u},\mathbf{v}\right)=l\left(\mathbf{v}\right)$ with $a\left(\mathbf{u},\mathbf{v}\right)=b\left(\mathbf{u},\mathbf{v}\right)+\frac{\pi^p}{V}\left(\int_{\gamma^p}\mathbf{v}.\mathbf{n}d\gamma\right)\left(\int_{\gamma^p}\mathbf{u}.\mathbf{n}d\gamma\right)$ and $l\left(\mathbf{v}\right)=\int_{\Gamma^p}\delta\mathbf{F}.\mathbf{v}d\Gamma$*

Proof. We check the Lax-Milgram assumptions
- $\left(\mathcal{V},|\ |_{1,\Omega^p}\right)$ Hilbert
- l is a linear mapping continuous because of the continuity of partial trace operator on Γ^p
- a is a bilinear continuous positive $\mathcal{V}$-elliptic mapping because the norms $|\ |_{1,\Omega^p}$ and $\|\ \|_A$ are equivalent on $\mathcal{V}$. □

2.3. Conclusion

As a conclusion, with a 3D elasticity modelling we have established the existence and uniqueness of the displacement caused by an outer disturbance. But since the cost of a 3D mesh is numerically very high, we want to mesh only the inner surface of the lobe. In order to do that we choose to model the displacement so as to integrate it in the thickness. That way, we expect to get rid of the thickness parameter. To have a reallistic model of displacement we solve explicitly the pre-stressed problem, and from the expression of explicit pre-stressed displacement we make a conjecture on the displacement.

§3. Study of the pre-stressed state

3.1. Curvilinear and differential calculus

3.1.1. Cartesian set of coordinates

We associate to each lobe the Cartesian orthonormal basis $(O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ such as the Cartesian representation of the lateral fixations are $\{-d\} \times \{0\} \times \{0 \leq x_3 \leq L\}$ and $\{+d\} \times \{0\} \times \{0 \leq x_3 \leq L\}$ where $2d$ is the distance between the two parallel axes of fixation.

3.1.2. Orthotropic set of coordinates

At a generic point of the material, we want the coefficients of the matrix representing the elasticity tensor to be independent of the shape of the material. To manage that, we write the matrix in a basis of orthotropy. That is, a basis with one vector in the width, one in the thickness and one in the length.

Let $M(x_{M1}, x_{M2}, x_{M3})$ be a generic point of Ω, let $\mathcal{P}_M$ be the perpendicular plane to $\mathbf{e}_3$ going through M. Let $\mathcal{C}_M = \gamma \cap \mathcal{P}_M$ and $x_2 = f(x_1)$, $x_3 = x_{M3}$ be the equations of γ. Let $M_\perp(x_{M1}^\perp, x_{M2}^\perp, x_{M3})$ be the orthogonal projection of M onto $\mathcal{C}_M$ and $\mathbf{t}_M$ and $\mathbf{n}_M$ be the tangent and normal unit vectors to $\mathcal{C}_M$ in $M_\perp$. Let $R_{M_\perp}$ be the curvature radius in $M_\perp$ of $\mathcal{C}_M$. Then,

$$\mathbf{t}_M = \begin{pmatrix} \dfrac{-f'(x_{M1}^\perp)}{\sqrt{1 + (f'(x_{M1}^\perp))^2}} \\ \dfrac{1}{\sqrt{1 + (f'(x_{M1}^\perp))^2}} \\ 0 \end{pmatrix} \text{ and } \mathbf{n}_M = \begin{pmatrix} \dfrac{1}{\sqrt{1 + (f'(x_{M1}^\perp))^2}} \\ \dfrac{f'(x_{M1}^\perp)}{\sqrt{1 + (f'(x_{M1}^\perp))^2}} \\ 0 \end{pmatrix}$$

The orthotropy basis in M is $(\mathbf{t}_M, \mathbf{n}_M, \mathbf{e}_3)$.

The orthotropic set of coordinates of M is (s, r, z) thus defined, $s = \int_0^{x_{M1}^\perp} f(x_1) dx_1$ is the curvilinear abscissa of $M_\perp$ along $\mathcal{C}_M$, r is the abscissa of M on the oriented axis $(M_\perp, \mathbf{n}_M)$, that is $\mathbf{M}_\perp\mathbf{M} = r\mathbf{n}_M$ and $rR_{M_\perp} \leq 0$, and z is x_{M3}.

3.1.3. Partial derivatives operators

Let $\mathbf{u} = u_s \mathbf{t}_M + u_r \mathbf{n}_M + u_z \mathbf{e}_3$, the displacement gradient tensor matrix in M, $(\nabla \mathbf{u})(M)$ in the orthotropy basis $(\mathbf{t}_M, \mathbf{n}_M, \mathbf{e}_3)$ is

$$(\nabla \mathbf{u})(M) = \begin{bmatrix} \dfrac{R_{M_\perp}}{R_{M_\perp} - r} \dfrac{\partial u_s}{\partial s} - \dfrac{1}{R_{M_\perp} - r} u_r & \dfrac{\partial u_s}{\partial r} & \dfrac{\partial u_s}{\partial z} \\ \dfrac{R_{M_\perp}}{R_{M_\perp} - r} \dfrac{\partial u_r}{\partial s} + \dfrac{1}{R_{M_\perp} - r} u_s & \dfrac{\partial u_r}{\partial r} & \dfrac{\partial u_r}{\partial z} \\ \dfrac{R_{M_\perp}}{R_{M_\perp} - r} \dfrac{\partial u_z}{\partial s} & \dfrac{\partial u_z}{\partial r} & \dfrac{\partial u_z}{\partial z} \end{bmatrix} . \tag{4}$$

Let $\mathbf{\Sigma} = \begin{bmatrix} \Sigma_{ss} & \Sigma_{sr} & \Sigma_{sz} \\ \Sigma_{rs} & \Sigma_{rr} & \Sigma_{rz} \\ \Sigma_{zs} & \Sigma_{zr} & \Sigma_{zz} \end{bmatrix}$ in $(\mathbf{t}_M, \mathbf{n}_M, \mathbf{e}_3)$, then the orthotropic coordinates of the vector $\mathbf{div}(\mathbf{\Sigma})$ in M are

$$\begin{pmatrix} div(\mathbf{\Sigma})_s \\ div(\mathbf{\Sigma})_r \\ div(\mathbf{\Sigma})_z \end{pmatrix} = \begin{pmatrix} \frac{R_{M_\perp}}{R_{M_\perp} - r}\frac{\partial \Sigma_{ss}}{\partial s} + \frac{\partial \Sigma_{sr}}{\partial r} + \frac{\partial \Sigma_{sz}}{\partial z} - \frac{1}{R_{M_\perp} - r}\left(\Sigma_{sr} + \Sigma_{rs}\right) \\ \frac{R_{M_\perp}}{R_{M_\perp} - r}\frac{\partial \Sigma_{rs}}{\partial s} + \frac{\partial \Sigma_{rr}}{\partial r} + \frac{\partial \Sigma_{rz}}{\partial z} + \frac{1}{R_{M_\perp} - r}\left(\Sigma_{ss} - \Sigma_{rr}\right) \\ \frac{R_{M_\perp}}{R_{M_\perp} - r}\frac{\partial \Sigma_{zs}}{\partial s} + \frac{\partial \Sigma_{zr}}{\partial r} + \frac{\partial \Sigma_{zz}}{\partial z} - \frac{1}{R_{M_\perp} - r}\Sigma_{zr} \end{pmatrix} \tag{5}$$

3.2. Description of the quasi natural state

We want to know the shape of an object made of two sheets of material, fixed along their lengths to rigid metallic axes, when slowly inflated. Since the length is very large, we will suppose that it is infinite. Then, we assume that the inner fluids tends to occupy the maximum volume. So we have to solve a classical problem of optimization under constraint : find f such as

$$\begin{cases} \forall g \in C_1\left[-d, +d\right], \quad \int_{-d}^{+d} g(x_1)dx_1 \ \leq \int_{-d}^{+d} f(x_1)dx_1 = A(f) \\ L(f) = \int_{-d}^{+d} \sqrt{1 + (f'(x_1))^2}dx_1 = 2l \end{cases}$$

This problem has two solutions (f^+, λ) et (f^-, λ). Their representing curbs (two arcs of circle) are symmetric in relation to $(O, \mathbf{e}_1)$. The radius of the circles is λ solution of

$$l = \lambda Arc\sin(\frac{d}{\lambda})$$

So, in the quasi natural state, each lobe is a cylinder. The generating surface is the intersection of two disks of radius R symmetrical in relation to $(O, \mathbf{e}_1)$.

3.3. Pre-stressed problem

We assume that

$$\mathbf{\Sigma}^q << \mathbf{\Sigma}^p \text{ and } \mathbf{\Sigma}^p\left(s, r, z\right) = \mathbf{\Sigma}^p\left(r\right), \tag{6}$$

As established in the quasi natural state study, the curvature radius R^q is constant. Let $R = |R^q|$.

From (2) written in the reference quasi natural state, and because of (6) we have to solve

$$\begin{cases} \mathbf{div}\left(\mathbf{\Sigma}^p\right) & = 0 & \text{in } \Omega^q \\ \mathbf{\Sigma}^p\mathbf{n} & = 0 & \text{on } \Gamma^q \text{ ie } r = -\frac{R}{R^q}e \\ \mathbf{\Sigma}^p\mathbf{n} & = \left(\pi^q - \pi^p\right)\mathbf{n} & \text{on } \gamma^q \text{ ie } r = 0 \end{cases} .$$

Let $\Pi = \pi^p - \pi^q$. We use (5) to transform these equations into

$$\begin{cases} \dfrac{\partial \Sigma_{sr}^p}{\partial r} - \dfrac{2}{R^q - r}\Sigma_{sr}^p & = 0 \\ \dfrac{\partial \Sigma_{rr}^p}{\partial r} + \dfrac{1}{R^q - r}\left(\Sigma_{ss}^p - \Sigma_{rr}^p\right) & = 0 \\ \dfrac{\partial \Sigma_{zr}^p}{\partial r} - \dfrac{1}{R^q - r}\Sigma_{zr}^p & = 0 \\ \Sigma_{sr}^p\left(-\frac{R}{R^q}e\right) = \Sigma_{rr}^p\left(-\frac{R}{R^q}e\right) = \Sigma_{sr}^p(0) & = 0 \\ \Sigma_{rr}^p(0) & = -\Pi \end{cases}$$

From the first and the third equations and because of the fourth we can state

$$\Sigma_{sr}^p(r) = \Sigma_{zr}^p(r) = 0. \tag{7}$$

Let's now introduce the displacement $\mathbf{u}^q$. The problem to solve becomes

$$\begin{cases} \mathbf{\Sigma}^p(r) & = \mathbf{A} : \frac{1}{2}\left(\nabla \mathbf{u}^q + \nabla \mathbf{u}^{qT}\right) \\ \dfrac{\partial \Sigma_{rr}^p}{\partial r} + \dfrac{1}{R^q - r}\left(\Sigma_{ss}^p - \Sigma_{rr}^p\right) & = 0 \\ \Sigma_{rr}^p\left(-\frac{R}{R^q}e\right) & = 0 \\ \Sigma_{rr}^p(0) & = -\Pi \\ \mathbf{u}^q(l,0,z) = \mathbf{u}^q(-l,0,z) & = 0 \end{cases}$$

Taking account of (4) the first equation is equivalent to the three following ones

$$\Sigma_{ss}^p = \lambda_{ss}\left(\frac{R^q}{R^q - r}\frac{\partial u_s^q}{\partial s} - \frac{1}{R^q - r}u_r^q\right) + \lambda_{sr}\frac{\partial u_r^q}{\partial r} \tag{8}$$

$$\Sigma_{rr}^p = \lambda_{sr}\left(\frac{R^q}{R^q - r}\frac{\partial u_s^q}{\partial s} - \frac{1}{R^q - r}u_r^q\right) + \lambda_{rr}\frac{\partial u_r^q}{\partial r} \tag{9}$$

$$\Sigma_{sr}^p = \mu_{sr}\left(\frac{R^q}{R^q - r}\frac{\partial u_r^q}{\partial s} + \frac{1}{R^q - r}u_s^q + \frac{\partial u_s^q}{\partial r}\right) \tag{10}$$

We first solve the linear system of 2 equations with 2 variables formed by (8) and (9). Then we introduce the auxiliary parameter $\Psi(r) = (r - R_q)\Sigma_{rr}^p$, which verifies $\Psi'(r) = \Sigma_{ss}^p$, and we let $\Lambda = \lambda_{ss}\lambda_{rr} - \lambda_{sr}^2$, so that

$$\frac{\partial u_r^q}{\partial r} = -\frac{1}{\Lambda}\left(\lambda_{sr}\Psi' + \lambda_{ss}\frac{1}{R^q - r}\Psi\right) \tag{11}$$

$$\frac{R^q}{R^q - r}\frac{\partial u_s^q}{\partial s} - \frac{1}{R^q - r}u_r^q = \frac{1}{\Lambda}\left(\lambda_{rr}\Psi' + \lambda_{sr}\frac{1}{R^q - r}\Psi\right) \tag{12}$$

Taking in account (7) and dividing (10) by $R^q - r$, we get

$$\frac{\partial}{\partial r}\left(\frac{1}{R^q - r}u_s^q\right) = -\frac{R^q}{(R^q - r)^2}\frac{\partial u_r^q}{\partial s} \tag{13}$$

We derivate (11) to get $\dfrac{\partial^2 u_r^q}{\partial r \partial s} = 0$. So, we can deduce the existence of 2 mappings f and g such as

$$u_r^q = f(s) + g(r). \tag{14}$$

and, replacing into (13), the existence of a mapping h such as

$$u_s^q = -R^q f'(s) + (R^q - r)\, h(s) \tag{15}$$

so, with (12) we get

$$-(R^q)^2 f''(s) - f(s) + R^q (R^q - r)\, h'(s) - g(r) = \frac{1}{\Lambda}\left(\lambda_{rr}(R^q - r)\,\Psi'(r) + \lambda_{sr}\Psi(r)\right) \tag{16}$$

so, we can state the existence of two constants F and H such as $R^2 f''(s) + f(s) = F$ and $h'(s) = H$. The solutions of these differential equations are $f(s) = a\cos\left(\frac{s}{R}\right) + F$ and $h(s) = Hs$. The next step is to derivate (16). We obtain

$$-R^q H - g'(r) = \frac{1}{\Lambda}\left((\lambda_{sr} - \lambda_{rr})\,\Psi'(r) + \lambda_{rr}(R^q - r)\,\Psi''(r)\right)$$

but, since, according to (11), $g'(r) = -\frac{1}{\Lambda}\left(\lambda_{sr}\Psi'(r) + \lambda_{ss}\frac{1}{R^q - r}\Psi(r)\right)$, we are led to solve the following second order differential equation

$$-\lambda_{rr}(R^q - r)\,\Psi''(r) + \lambda_{rr}\Psi'(r) + \lambda_{ss}\frac{1}{R^q - r}\Psi(r) = R^q H \Lambda$$

Let α^- and α^+ be two undetermined constants. Let $\lambda = \sqrt{\frac{\lambda_{ss}}{\lambda_{rr}}}$ and $H_1 = -\frac{R^q H \Lambda}{\lambda_{ss} - \lambda_{rr}}$. The general solution of the previous equation is

$$\Psi(r) = \alpha^- |R^q - r|^{-\lambda} + \alpha^+ |R^q - r|^{+\lambda} - H_1 (R^q - r).$$

We can now calculate g :

Since, according to (16), $-g(r) = \frac{1}{\Lambda}\left(\lambda_{rr}(R^q - r)\,\Psi'(r) + \lambda_{sr}\Psi(r)\right) + F - R^q (R^q - r)\, H$, then, if we let $\Lambda^- = \sqrt{\lambda_{ss}\lambda_{rr}} - \lambda_{sr}$ and $\Lambda^+ = \sqrt{\lambda_{ss}\lambda_{rr}} + \lambda_{sr}$ (so that $\Lambda = \Lambda^+\Lambda^-$) we have $g(r) = -\frac{\alpha^-}{\Lambda^-}|R^q - r|^{-\lambda} + \frac{\alpha^+}{\Lambda^+}|R^q - r|^{+\lambda} - \left(\frac{\lambda_{ss} - \lambda_{sr}}{\Lambda}\right) H_1 (R^q - r) - F.$

Replacing into (14) and (15), we get the radial and the tangential displacement

$$\begin{cases} u_r^q = a\cos\left(\dfrac{s}{R}\right) - \left(\frac{\lambda_{ss} - \lambda_{sr}}{\Lambda}\right) H_1 (R^q - r) - \dfrac{\alpha^-}{\Lambda^-}|R^q - r|^{-\lambda} + \dfrac{\alpha^+}{\Lambda^+}|R^q - r|^{+\lambda} \\ u_s^q = a\sin\left(\dfrac{s}{R^q}\right) + (R^q - r)\, Hs \end{cases} \tag{17}$$

3.3.1. Calculus of the constants

We can explicitly calculate the constants α^-, α^+, a and H with the boundary conditions

$$\begin{array}{llll} \Sigma_{rr}^p\left(-\frac{R}{R^q}e\right) & = \dfrac{1}{\left(-\frac{R}{R^q}e - R^q\right)}\Psi\left(-\frac{R}{R^q}e\right) & = 0 \\ \Sigma_{rr}^q(0) & = \dfrac{1}{(-R^q)}\Psi(0) & = (\pi^q - \pi^p) \\ \mathbf{u}^q(l,0,z) & = \mathbf{u}^q(-l,0,z) & = \mathbf{0} \end{array}$$

3.3.2. Taylor's developments in $O\left(\frac{e}{R}\right)$ of the constants

$$
\begin{aligned}
&H_1 \simeq \frac{-\Pi}{\left(1-\frac{l}{R}\cot\left(\frac{l}{R}\right)\right)\left(\lambda^2-1\right)}\left(\frac{R}{e}+\frac{\lambda_{sr}}{\lambda_{rr}}-1\right)\\
&H \simeq \frac{\Pi\lambda_{rr}}{R_q\Lambda\left(1-\frac{l}{R}\cot\left(\frac{l}{R}\right)\right)}\left(\frac{R}{e}+\frac{\lambda_{sr}}{\lambda_{rr}}-1\right)\\
&a \simeq -\frac{l\Pi\lambda_{rr}}{\Lambda\left(\sin\left(\frac{l}{R^q}\right)-\frac{l}{R^q}\cos\left(\frac{l}{R}\right)\right)}\left(\frac{R}{e}+\frac{\lambda_{sr}}{\lambda_{rr}}-1\right)\\
&\alpha^- \simeq -\frac{\Pi R^{+\lambda}R^q}{2\lambda}\left(-\left(\frac{R}{e}+\lambda\right)+\frac{\frac{R}{e}+\frac{\lambda_{sr}}{\lambda_{rr}}-1}{\left(1-\frac{l}{R}\cot\left(\frac{l}{R}\right)\right)(\lambda+1)}\right)\\
&\alpha^+ \simeq -\frac{R^{-\lambda}R^q\Pi}{2\lambda}\left(\left(\frac{R}{e}-\lambda\right)+\frac{\frac{R}{e}+\frac{\lambda_{sr}}{\lambda_{rr}}-1}{\left(1-\frac{l}{R}\cot\left(\frac{l}{R}\right)\right)(\lambda-1)}\right)
\end{aligned}
$$

3.3.3. Taylor's development in $O\left(\frac{r}{R^q}\right)$ of Σ^q_{ss}

From the development of $\Sigma^p_{ss}(r)=\Psi'(r)$ we obtain $\Sigma^p_{ss}(r)=\dfrac{\Pi R}{e}+O\left(\dfrac{r}{R^q}\right)$.

3.3.4. Geometry of the pre-stressed state

We want to calculate an approximation of the equations of γ^p. In order to do that, we write that a generic point $M^q(s,0,z)$ of γ^q becomes, after the deformation, M^p, such as $\mathbf{M}^q\mathbf{M}^p=\mathbf{u}^q(s,0,z)$. First of all, let's evaluate $\mathbf{u}^q(s,0,z)$. When $r=0$, (17) becomes

$$
\begin{cases}
u^q_r(s,0,z)=a\cos\left(\frac{s}{R}\right)-\left(\frac{\lambda_{ss}-\lambda_{sr}}{\Lambda}\right)H_1R^q-\frac{\alpha^-}{\Lambda^-}R^{-\lambda}+\frac{\alpha^+}{\Lambda^+}R^{+\lambda}\\
u^q_s(s,0,z)=a\sin\left(\frac{s}{R^q}\right)+R^qHs
\end{cases}
$$

and at the extremities we have

$$
\begin{cases}
u^q_r(l,0,z)=a\cos\left(\frac{l}{R}\right)-\left(\frac{\lambda_{ss}-\lambda_{sr}}{\Lambda}\right)H_1R^q-\frac{\alpha^-}{\Lambda^-}R^{-\lambda}+\frac{\alpha^+}{\Lambda^+}R^{+\lambda}=0\\
u^q_s(l,0,z)=a\sin\left(\frac{l}{R^q}\right)+R^qHl=0
\end{cases}
$$

so the displacement at a generic point of γ^q is

$$
\begin{cases}
u^q_r(s,0,z)=a\cos\left(\frac{s}{R}\right)-a\cos\left(\frac{l}{R}\right)\\
u^q_s(s,0,z)=a\sin\left(\frac{s}{R^q}\right)-a\frac{s}{l}\sin\left(\frac{l}{R^q}\right)
\end{cases}
$$

The Cartesian coordinates of $M^q\in\gamma^{q+}$ are

$$
\begin{cases}
x^q &= R\sin\left(\frac{s}{R}\right)\\
y^q &= R\cos\left(\frac{s}{R}\right)+y_0^+
\end{cases}
$$

and those of M^p are

$$
\begin{cases}
x^p &= R\sin\left(\frac{s}{R}\right)+u^q_s(s,0,z)\cos\left(\frac{s}{R}\right)+u^q_r(s,0,z)\sin\left(\frac{s}{R}\right)\\
y^p &= R\cos\left(\frac{s}{R}\right)-u^q_s(s,0,z)\sin\left(\frac{s}{R}\right)+u^q_r(s,0,z)\cos\left(\frac{s}{R}\right)+y_0^+
\end{cases}.
$$

Since on γ^{q+}, $R^q=-R$, then,

$$\begin{cases} x^p &= a\dfrac{s}{l}\sin\left(\dfrac{l}{R}\right)\cos\left(\dfrac{s}{R}\right)+\left(R-a\cos\left(\dfrac{l}{R}\right)\right)\sin\left(\dfrac{s}{R}\right) \\ y^p &= R\cos\left(\dfrac{s}{R}\right)+a-a\dfrac{s}{l}\sin\left(\dfrac{l}{R}\right)\sin\left(\dfrac{s}{R}\right)-a\cos\left(\dfrac{l}{R}\right)\cos\left(\dfrac{s}{R}\right)+y_0^+ \end{cases}.$$

We notice that $(x^p)^2+\left(y^p-y_0^+-a\right)^2=a^2\left(\dfrac{s}{l}\right)^2\sin^2\left(\dfrac{l}{R}\right)+\left(R-a\cos\left(\dfrac{l}{R}\right)\right)^2$.

Identifying $\sin^2\left(\dfrac{l}{R}\right)\simeq\left(\dfrac{l}{R}\right)^2$ we get

$$(x^p)^2+\left(y^p-y_0^+-a\right)^2=\left(R-a\cos\left(\frac{l}{R}\right)\right)^2+O\left(\left(\frac{s}{R}\right)^2\right)$$

So, we can admit that the pre-stressed inner surface is also constituted by two arcs of circle. The new radius is $(R^p)^2=\left(R-a\cos\left(\dfrac{l}{R}\right)\right)^2$ which is smaller than R^q.

§4. Conclusion

From the explicit displacement obtained in (17), we can conjecture that the general displacement of a generic point is

$$\begin{cases} u_s(s,r,z)=u_s^0(s,z)+(R^q-r)\,u_s^1(s,z) \\ u_r(s,r,z)=u_r^0(s,z)+(R^q-r)\,u_r^1(s,z)+|R^q-r|^{-\lambda}u_r^-(s,z)+|R^q-r|^{+\lambda}u_r^+(s,z) \\ u_z(s,r,z)=u_z^0(s,z) \end{cases}$$

then, in order to solve a 2D problem, we can replace $\mathbf{u}$ in the variationnal formulation and integrate in the thickness.

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