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ON FEW SHELL MODELS IN NONLINEAR ELASTICITY AND EXISTENCE OF SOLUTIONS

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Abstract. The existence of stable solutions for geometrically nonlinear theory of shells has been widely discussed by mechanicians during the last century. But, certainly because of the difficulty met in the classical three dimensional nonlinear elasticity, few mathematical results have been obtained. A possibility is to apply the polyconvexity introduced in nonlinear elasticity by J. Ball [1] to ad'hoc shell theories. But unfortunately the positive results are restricted to a special class of materials. Another approach consists in using the so-called Γ-convergence. This theory has been suggested by the italian school (E. De Giorgi and G. Dal Maso, [9]), and an application to shell models has been given by H. Ledret and A. Raoult [19]. But the main drawback, in our opinion, is that the solution transgresses the equilibrium equations and the difficulty is to make sense to the model obtained. Therefore, it is not yet possible to use these results for the physical problem to be solved.

In this paper, we suggest another theory based on some nonlinear mathematical tools which have already been used for particular shell models in [12]. The first part gives a formulation of a general geometrically nonlinear shell model based on a full description of the large kinematical movement induced by a Kirchhoff-Love field. The *objectivity property* is checked for a class of materials (energy invariance under the only effect of a nonlinear-rigid body motion). Then the existence of stable solutions is proved using minimizing sequences and some mathematical tricks based on compactness of few nonlinear terms.

Keywords: Shell theory, existence of solutions, stability

AMS classification: 73M05, 74K25, 74B20

§1. Introduction

Following the ideas of W.T. Koiter [16], [17], in linear theory of elasic shells, we discuss the opportunity to introduce new nonlinear shell models which satisfy the fundamental property of energy invariance under the only effect of a rigid body motion. This has been widely discussed in linear theory by W.T. Koiter [16]. It has also been pointed out as a basic point in shell modelling by B. Budiansky and J.L. Sanders [6]. In particular they have suggested a model that they claimed to be the best first order shell model in linear theory of elastic shell. These aspects are discussed in a first part of this paper. They are essential before any discussion on linear or nonlinear shell modelling. This remark is specially true for the existence of solutions for shell models. Therefore we recall few ideas of these fundamental contributions of W.T. Koiter which enable to shunt some awfull complexities in several mathematical papers on shells. Then the extension to nonlinear shell models is discussed and several remarks concerning the validity of such models are given. A particular property of a shell model in non linear elasticity, is to avoid inextensional movements (A.L. Goldenveizer [15], V. V. Novozhilov [21]). Therefore such movements, which are membrane-energy free, are restricted to the linear modelling of the inplane strain tensor of the medium surface of the shell. From the mathematical point of view, they have been analyzed in former publications [10], and creates a *soft* behaviour of the shell operator. Further analysis are contained in the books by P. G. Ciarlet [7]. In particular he used Holmgrem Theorem (in a jointed work with V. Lods [7]), for getting rid of a restriction on the Christoffel symbols which has been used in [10]. The importance in mechanical engineering of inextensional theory is well known known for buckling phenomenon which precisely occurs mainly with an inextensional movement of the shell. Then new branches of solutions are driven by the bending energy at least in the vicinity of the buckling. The point is that the bending is essentially a regularization of the shell models in order to avoid energy free movements in the vicinity of an inextensional movement. Thus it can be suggested to confine the nonlinear formulation of shell model on the extensional part. But nonlinear modelling shows that this remark is no more necessarily true. A simplified explanation of this very important property, is suggested in the preliminary section of this paper. Then we discuss how it can be extended to several shell models. This aspect is not sufficient for the existence of stable solutions. Therefore, a class of models which satisfy the energy invariance and some compactness properties are introduced. Finally some weak semi-continuity are proved in order to derive the existence of a stable solutions. The uniqueness is a local result which is not always true in quasi-static formulations. For instance a buckling can occur. Finally we mention that the polyconvexity can certainly be used for a larger class of models. Nevertheless the developments will be in a further paper with P. G. Ciarlet [8].

§2. Preliminary remark and mathematical aims of the paper

Let us define few notations. First of all we consider an elastic body which occupies in space the open set Ω. The unit outwards normal to the boundary Γ is ν. The points M of Ω are referred to an orthonormal system of axis denoted by $(O; x_1, x_2, x_3)$. Let us now consider a vector field $\mathbf{u} = (u_i)$ defined on Ω and let us assume that the components u_i are elements of the space $W^{1,4}(\Omega)$, such that $\mathbf{u} = (u_i) \in \mathbf{V}$ where:

$$\mathbf{V} = \{\mathbf{v} = (v_i),\ v_i \in W^{1,4}(\Omega),\ \int_\Gamma \mathbf{v}.\nu = 0.\}.$$

It is equiped with the natural norm induced by the space $W^{1,4}(\Omega)^3$. Let us denote by Ω' the new open set obtained from Ω through the displacement field $\mathbf{u}$. The condition on the boundary of Ω traduces that globally the volume of Ω' is the same as the one of Ω, at least in a first order approximation with respect to the displacement magnitude. Let us now consider the the nonlinear expression of the strain operator due to the displacement field $\mathbf{u}$. It is defined on Ω by:

$$\zeta(\mathbf{u}) = \frac{1}{2}(\frac{\partial \mathbf{u}}{\partial M} + \overline{\frac{\partial \mathbf{u}}{\partial M}} + \overline{\frac{\partial \mathbf{u}}{\partial M}}.\frac{\partial \mathbf{u}}{\partial M}),$$

where $\frac{\partial \mathbf{u}}{\partial M}$ is the linear operator from $\mathbb{R}^3$ into itself, (the transpose in $\mathbb{R}^3$ of $\frac{\partial \mathbf{u}}{\partial M}$ is denoted $\overline{\frac{\partial \mathbf{u}}{\partial M}}$), defined as the partial derivative of $\mathbf{u}$ with respect to the points M. Its components in the orthonormal basis are denoted by:

$$(\frac{\partial \mathbf{u}}{\partial M})_{i,j} = \frac{\partial u_i}{\partial x_j} = u_{i,j}.$$

The components of the matrix ζ_{ij} which represents the endomorphism $\zeta(\mathbf{u})$ in the basis $(O; x_1, x_2, x_3)$ are:

$$\zeta_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{i,k}u_{k,j}),$$

where $u_{i,j}$ denotes the partial derivative of u_i with respect to the coordinates x_j. Furthermore, the implicit summation from zero to three is assumed over the repeated latin indices (it will be assumed from one to two for greek indices). Let us now define the trace of an endomorphism $\mathbf{A} = (A_{ij})$ from $\mathbb{R}^3$ into itself, by:

$$Tr_3(\mathbf{A}) = (A_{ii}).$$

The trace of the endomorphism $\zeta(\mathbf{u})$ is therefore:

$$Tr_3(\zeta(\mathbf{u})) = div(\mathbf{u}) + \frac{1}{2}\sum_{i=1,2,3} |\nabla u_i|^2,$$

where $\nabla.$ is the gradient operator. For instance one has (notations): $(\nabla f)_j = \dfrac{\partial f}{\partial x_j} = f_{,j}$. Then by integrating the preceding relation over Ω, using Stokes formula and because of the boundary conditions contained in $\mathbf{V}$, we deduce that (the regular notations are used for the norm and the semi-norm in $W^{m,p}$, ie. $||.||_{m,p,\omega}$, respectively: $|.|_{m,p,\omega}$):

$$\forall \mathbf{u} \in \mathbf{V}, \ \int_\Omega Tr_3(\gamma(\mathbf{u})) = \frac{1}{2}\sum_{j=1,2,3} |u_i|^2_{1,2,\Omega}, \tag{1}$$

(the notation $|.|_{1,2,\Omega}$ is the semi-norm of the first order derivative in $L^2(\Omega)$). If one has, for instance, the inequality:

$$\int_\Omega Tr_3(\gamma(\mathbf{u})) \leq 0,$$

then we can conclude that:

$$\mathbf{u} = 0.$$

This property is a little bit surprising because it is stronger (it requires less assumptions!), that the classical rigid body motion Theorem [14]. In fact the linear version states that if $\mathbf{u} \in \mathbf{V}$, such that: $\forall (i,j) \in \{1,2,3\}, \ u_{i,j} + u_{j,i} = 0,$ then there exist two constant vectors $\mathbf{a}, \mathbf{b}$ in $\mathbb{R}^3$, such that: $\mathbf{u} = \mathbf{a} + \mathbf{b} \wedge OM$, and the boundary conditions in $\mathbf{V}$ are not suficient to conclude that $\mathbf{u} = 0$. This property is underlying in the following analysis. Let us explain how from the mathematical point of view. Let $\mathbf{u}^n$ be a weakly convergent sequences in $\mathbf{V}$ towards an element

$\mathbf{u}^*$. The quantity $u^n_{i,k}u^n_{k,j}$ is also bounded and weakly convergent to an element $h^*_{ij} = h^*_{ji}$ in the space $[L^2(\Omega)]^4$. Furthermore let us set (we use here the notation: $\gamma^n_{ij} = \frac{1}{2}(u^n_{i,j} + u^n_{j,i})$):

$$f^n = \gamma^n_{ii} + \frac{1}{2}u^n_{i,k}u^n_{k,i} = div(\mathbf{u}^n) + \frac{1}{2}\sum_{i=1,2,3}|\nabla u^n_i|^2.$$

If we assume that:

$$f^n \to 0 \ in \ L^2(\Omega) \ as \ n \ \to \infty,$$

then one has, because of the property mentioned above:

$$\begin{cases} u^*_i = 0 \text{ for } i = 1,2,3, \\ \text{and:} \\ \mathbf{u}^n \to 0 \text{ in } [W^{1,4}_0(\Omega)]^3 \text{ strong.} \end{cases} \quad (2)$$

This result was not obvious because the functional:

$$\mathbf{u} \in \mathbf{V} \to ||div(u) + \frac{1}{2}|\nabla u|^2||^2_{0,2,\Omega},$$

is not convex. In fact one has three weak convergences in $\mathbf{V}$ (for the components u^n_i) and one strong convergence in $L^2(\Omega)$ (concerning f^n). A hidden compactness enabled us to get three strong convergences. A similar property is used in the following for getting the lower limit of a non convex functional with a minimizing sequence.

§3. A brief on differential geometry for shells

3.1. The fundamental forms of a surface [23]

Let ω be a surface in $\mathbb{R}^3$ which is described by a smooth enough mapping -say φ- from a two dimensional open set $\hat{\omega}$ into $\mathbb{R}^3$. The coordinates in the plan containing $\hat{\omega}$ are denoted by (ξ_1, ξ_2). At each point m of the surface ω we define two tangent vectors by: $\mathbf{a}_\alpha = \frac{\partial \varphi}{\partial \xi_\alpha}$ for $\alpha = 1, 2$. They are assumed to be linearly independent and the dual basis in this tangent plane -say $\mathbf{a}^\alpha$- is defined by the relation: $\forall \alpha, \ \beta \in \{1,2\}, \ \mathbf{a}^\alpha.\mathbf{a}_\alpha = \delta^\alpha_\beta$, which is the Kronecker symbol and the dot stands for the scalar product in $\mathbb{R}^3$. The metric tensor on ω is defined by: $g_{\alpha\beta} = \mathbf{a}_\alpha.\mathbf{a}_\beta$. Its inverse is: $g^{\alpha\beta} = \mathbf{a}^\alpha.\mathbf{a}^\beta$. Let us now consider the unit normal to ω at point m defined by: $\mathbf{N} = \dfrac{\mathbf{a}_\alpha \wedge \mathbf{a}_\beta}{|\mathbf{a}_\alpha \wedge \mathbf{a}_\beta|}$. The derivative of $\mathbf{N}$ with respect to the coordinates ξ_α are tangent vectors (because $\mathbf{N}$ is unitary). Their components in the basis $\mathbf{a}_\alpha$ are denoted $-b^\beta_\alpha$ such that one has: $\frac{\partial \mathbf{N}}{\partial \xi_\alpha} = \mathbf{N}_{,\alpha} = -b^\beta_\alpha \mathbf{a}_\beta$, or else using the dual basis: $\frac{\partial \mathbf{N}}{\partial \xi_\alpha} = \mathbf{N}_{,\alpha} = -b_{\alpha\beta}\mathbf{a}^\beta$. The following notation will be used for the curvature operator of the surface ω: $\frac{\partial \mathbf{N}}{\partial m} = -b^\beta_\alpha \mathbf{a}_\beta \otimes \mathbf{a}^\alpha$ where the notation $\otimes$ denotes the tensor product between two vectors in $\mathbb{R}^3$. Let us now consider a vector field defined on ω by: $\mathbf{v} = v^\alpha \mathbf{a}_\alpha + v_3\mathbf{N} = v_\alpha \mathbf{a}^\alpha + v_3 \mathbf{N}$. We can associate to it a new surface -say ω'- which is the image of ω under the effect of the vector field $\mathbf{v}$. The new map which describes ω' is therefore: $\varphi'(\xi_1, \xi_2) = \varphi(\xi_1, \xi_2) + \mathbf{v}(\xi_1, \xi_2)$. The new tangent vectors are: $\mathbf{a}'_\alpha = \mathbf{a}_\alpha + \mathbf{v}_{,\alpha}$. We can define a new metric tensor -say $g'_{\alpha\beta}$- and a new curvature tensor -say $b'_{\alpha\beta}$- which are linearized with respect to $\mathbf{v} = v_\alpha \mathbf{a}^\alpha + v_3\mathbf{N}$, by:

$$\left\{\begin{array}{l} g'_{\alpha\beta} = \mathbf{a}'_{\alpha}.\mathbf{a}'_{\beta} = g_{\alpha\beta} + 2\gamma_{\alpha\beta} + \ldots, \\ \text{where } \gamma_{\alpha\beta} = \dfrac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta}u_3, \\ b'_{\alpha\beta} = \mathbf{a}'_{\beta}.\mathbf{N}'_{,\alpha} = b_{\alpha\beta} - \varrho_{\alpha\beta} + \ldots, \\ \text{where } \varrho_{\alpha\beta} = \dfrac{1}{2}(\theta_{\alpha|\beta} + \theta_{\beta|\alpha}) - \dfrac{1}{2}(v_{\alpha|\lambda}b^{\lambda}_{\beta} + v_{\beta|\lambda}b^{\lambda}_{\alpha}) + b^{\lambda}_{\alpha}b_{\lambda\beta}v_3, \\ \text{with the notations :} \\ \theta_{\alpha} = -b^{\lambda}_{\alpha}v_{\lambda} - v_{3,\alpha} \text{ and: } v_{\alpha|\beta} = v_{\alpha,\beta} - \Gamma^{\lambda}_{\alpha\beta}v_{\lambda},\ \Gamma^{\lambda}_{\alpha\beta} = \mathbf{a}_{\alpha,\beta}.\mathbf{a}^{\lambda}. \end{array}\right. \tag{3}$$

Remark 1. Let us define a new surface in $\mathbb{R}^3$ which is parallel to ω and defined for a given value x_3, by: $\omega^{x_3} = \{M \in \mathbb{R}^3,\ M = m + x_3\mathbf{N},\ m \in \omega\}$. The metric tensor on ω^{x_3} is easily obtained by:

$$g^{x_3}_{\alpha\beta} = g_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 b^{\lambda}_{\alpha} b_{\lambda\beta}.$$

Let us call $\delta g_{\alpha\beta}$ (respectively $\delta b_{\alpha\beta}$), the linearized variations of the metric tensor of the surface ω (respectively of the curvature tensor), in a movement of a the surface ω defined by the vector field $\mathbf{v}$. The linear variations of $g^{x_3}_{\alpha\beta}$ with respect to a displacement $\mathbf{v}$ are therefore perfectly defined from the one of $\delta g_{\alpha\beta}$ and $\delta b_{\alpha\beta}$. More precisely, one has:

$$\delta g^{x_3}_{\alpha\beta} = \delta g_{\alpha\beta} - 2x_3\delta b_{\alpha\beta} + x_3^2[b^{\lambda}_{\alpha}\delta b_{\lambda\beta} + \delta b^{\lambda}_{\alpha} b_{\lambda\beta}].$$

Hence if $\delta g_{\alpha\beta} = \delta b_{\alpha\beta} = 0$ one has clearly $\delta g^{x_3}_{\alpha\beta} = 0,\ \forall x_3,$ because $\delta b^{\lambda}_{\alpha} = g^{\lambda\mu}\delta b_{\lambda\alpha} - b_{\xi\alpha}g^{\lambda\mu}g^{\xi\nu}\delta g_{\mu\nu}$. This is the fundamental remark of W.T. Koiter.

3.2. The plurality of the change of curvature tensor

It should be pointed out that there exist several expressions for the first order change of curvature tensor. For instance there is another one, also suggested by W.T. Koiter and mainly B. Budiansky-J.L. Sanders, which is also interesting. It is more accurate for describing the true linear strain tensor for a surface parallel to ω (ie. for $x_3 \neq 0$). But this is not correct for the change of metric of the surface $\omega \times \{x_3\}$. The argument is the following. The change of metric given previously by $g'^{x_3}_{\alpha\beta}$, is applied to tangent vectors to this surface such that $dm = \mathbf{a}_{\alpha}d\xi^{\alpha}$. But the two tangent planes respectively to ω and to $\omega \times \{x_3\}$ are parallel. Thus, one can use two different basis for the tangent vectors: the first one is: $\{\mathbf{a}_{\alpha}\}$. If we set: $dm = \mathbf{a}_{\alpha}d\xi^{\alpha}$, the relative change of length of dm is (after linearization) $g'^{x_3}_{\alpha\beta}d\xi^{\alpha}d\xi^{\beta}$. The second basis is the local one: $\mathbf{a}^{x_3}_{\alpha} = \mathbf{a}_{\alpha} - x_3 b^{\beta}_{\alpha}\mathbf{a}_{\beta}$. In this new basis the tangent vector is $dm = \mathbf{a}^{x_3}_{\alpha}dv^{\alpha}$. One can easily prove check that the change of length of dm can be expressed with respect to the components dv^{α}, with the new change of metric:

$$G'^{x_3}_{\alpha\beta} = \gamma_{\alpha\beta} - 2x_3\bar{\varrho}_{\alpha\beta} + x_3^2 \ldots$$

where we have set:

$$\bar{\varrho}_{\alpha\beta} = \frac{1}{2}(\theta_{\alpha|\beta} + \theta_{\beta|\alpha}) + \frac{1}{2}(b^{\lambda}_{\alpha}v_{\beta|\lambda} + b^{\lambda}_{\beta}v_{\alpha|\lambda}) - b^{\lambda}_{\alpha}b_{\lambda\beta}v_3. \tag{4}$$

A simple exercice enables one to prove the following relation between the two quantities:

$$\varrho_{\alpha\beta} = \bar{\varrho}_{\alpha\beta} - b^{\lambda}_{\alpha}\gamma_{\beta\lambda} - b^{\lambda}_{\beta}\gamma_{\alpha\lambda}. \tag{5}$$

But the expansion of $G'^{x_3}_{\alpha\beta}$, is infinite with respect to x_3. Nevertheless, it is true that for an anisotropical material this last expression is certainly more appropriate for formulating the constitutive relationships, because it is a local expression of the strain through the thickness of the shell. This is one of the reasons evoked by B. Budiansky and J.L. Sanders who claimed [6] that it was the best first order expression for the change of curvature. We refer to [13] for further explanations with complementary arguments.

From Gauss-Bonnet Theorem [5], we can state that the new surface ω' is fully described up to a translation and a rotation by the only knowledge of $g'_{\alpha\beta}$ and $b'_{\alpha\beta}$ but which are nonlinear expressions with respect to the vector field $\mathbf{v}$. This is why another idea, due to W. T. Koiter is usefull as far as the linear formulation is used. Let us recall, in the next section, this fundamental remark of W. T. Koiter [17] which is wellknown among mechanicians in shell structures much less among mathematicians who have rediscovered it more recentlly [3]. I learned it from R. Valid [24] in July 1976, at a summer course on shells organized by EDF-CEA-INRIA.

3.3. Thickening of a surface in $\mathbb{R}^3$

The basic idea is to define a thickening of the surface ω in order to consider a three dimensional body on which it is possible to apply the classical linear rigid body Theorem and then to restrict it to the surface ω. Then let us consider a vector field $\mathbf{v}$ defined on ω, and extended to a three dimensional field $\mathbf{v}^{KL}$ on $\Omega^\eta = \omega\times]-\eta,\eta[$, such that its restriction to ω is equal to $\mathbf{v}$. The transverse strain through the thickness of the volume Ω^η, will be zero. Therefore all the components of the linearized strain tensor are zero as soon as the inplane components will be zero. Therefore, from the three dimensional rigid body Theorem, the displacement field $\mathbf{v}$ will be a (linearized) rigid body motion.
Our goal is to define explicitely such a three dimensional extension of a surface vector field. This is obtained as follows.

Definition 1. Let $\mathbf{v}(m) = v_\alpha \mathbf{a}^\alpha + v_3\mathbf{N}$ a displacement field defined on the surface ω. The components v_α and v_3 are functions defined on ω. We set on $\Omega^\eta = \omega\times]-\eta,\eta[$:

$$\mathbf{v}^{KL}(m,x_3) = \mathbf{v}(m) + x_3\theta(m),\ (m,x_3)\in\Omega^\eta, \tag{6}$$

where at each point $m = \varphi(\xi_1,\xi_2)\in\omega$ one has:

$$\theta = \theta_\alpha \mathbf{a}^\alpha,\ \theta_\alpha = -b^\beta_\alpha v_\beta - v_{3,\alpha}.$$

The displacement field $\mathbf{v}^{KL}$ is called a Kirchhoff-Love one. It has been obtained by solving on Ω^η the equations traducing the nullity of the transverse strains [24].

Theorem 1. *Let $\mathbf{v} = v_\alpha\mathbf{a}^\alpha + v_3\mathbf{N}$ be a vector field defined on ω and $\mathbf{v}^{KL}$ its extension to Ω^η by formulae 6. Then at each point $M = m + x_3\mathbf{N}$ the linearized strain tensor $\zeta = \frac{1}{2}(\frac{\partial\mathbf{v}^{KL}}{\partial M} + \overline{\frac{\partial\mathbf{v}^{KL}}{\partial M}})$ is restricted to its inplane component such that:*

$$\zeta = \zeta_{\alpha\beta}\mathbf{a}^\alpha\otimes\mathbf{a}^\beta$$

where we have set:

$$(\delta_\alpha^\lambda - x_3 b_\alpha^\lambda)\zeta_{\lambda\mu}(\delta_\beta^\mu - x_3 b_\beta^\mu) = [\gamma_{\alpha\beta} + x_3\varrho_{\alpha\beta} + x_3^2\kappa_{\alpha\beta}] = \delta g_{\alpha\beta}^{x_3},$$

with the following notations:

$$\gamma_{\alpha\beta} = \frac{1}{2}(v_{\alpha|\beta} + v_{\beta|\alpha}) - b_{\alpha\beta}v_3, \quad \kappa_{\alpha\beta} = -\frac{1}{2}(b_\alpha^\lambda\theta_{\lambda|\beta} + b_\beta^\lambda\theta_{\lambda|\alpha}),$$

and:

$$\varrho_{\alpha\beta} = \frac{1}{2}(\theta_{\alpha|\beta} + \theta_{\beta|\alpha}) + b_\alpha^\lambda b_{\lambda\beta}v_3 - \frac{1}{2}(v_{\alpha|\lambda}b_\beta^\lambda + v_{\beta|\lambda}b_\alpha^\lambda).$$

A simple computation [13], already mentioned, gives the other expression suggested by B. Budiansky and J. L. Sanders:

$$\zeta_{\alpha\beta} = \gamma_{\alpha\beta} + x_3\bar{\varrho}_{\alpha\beta} + x_3^2 \ldots$$

Let us summarize the basic results of the Koiter shell theory in the following statement.

Theorem 2. *Let* $\mathbf{v} = v_\alpha\mathbf{a}^\alpha + v_3\mathbf{N}$ *be a vector field defined on the surface* ω. *We extend this vector field to a three dimensional one denoted* $\mathbf{v}^{KL}$*, on the open set:* $\Omega^\eta = \omega\times]-\eta,\eta[$ *by the following formulae:*

$$\forall(m,x_3)\in\Omega^\eta,\ \mathbf{v}^{KL}(m,x_3) = \mathbf{v}(m) + x_3\theta(m),$$

where:

$$\theta(m) = \theta_\alpha\mathbf{a}^\alpha,\ \textit{with}\ \theta_\alpha = -b_\alpha^\lambda v_\lambda - v_{3,\alpha}.$$

Then there are several norms which can be used for such three dimensional vector fields. One is the three dimensional one induced by the Korn inequality [14]:

$$\mathbf{v} \to ||\zeta^{\mathbf{L}}(\mathbf{v}^{KL})||_{0,\Omega^\eta},\ \textit{where}\ \zeta^{\mathbf{L}}\ \textit{is the linearized 3D strain,}$$

the most important point is that the following quantity is also a norm on the space $\mathbf{V}^{KL}$*:*

$$\mathbf{v} \to [\sum_{\alpha,\,\beta\in\{1,2\}} ||\gamma_{\alpha\beta}||_{0,\omega}^2 + ||\varrho_{\alpha\beta}||_{0,\omega}^2]^{1/2}.$$

another one is induced by the definition of $\mathbf{v}^{KL}$*:*

$$\mathbf{v} \to [\sum_{\alpha=1,2} ||v_\alpha||_{1,\omega}^2 + ||v_3||_{2,\omega}^2]^{1/2}.$$

Proof. This is straightforwards from Theorem 1. □

The next point concerns the equivalence between the three previous norms. The first proof of this result was given by P.G. Ciarlet and M. Bernadou [2]. They used exactly the same method as the one suggested by G. Duvaut and J. L. Lions [14]. But another one, given for shell [12] and in a joint paper with P.G. Ciarlet for plates [8], can be derived from the three dimensional Korn inequality. The formulation is the following one.

Theorem 3. *The three norms on the space:*

$$\mathbf{V}^{KL} = \{\mathbf{v}^{KL} = \mathbf{v}(m) + x_3\theta(m),\ \mathbf{v}(m) = v_\alpha \mathbf{a}^\alpha,\ v_\alpha \in H^1_0(\omega),\ \theta(m) = \theta_\alpha \mathbf{a}^\alpha,$$

$$\theta_\alpha = -v_{3,\alpha} - b^\beta_\alpha v_\beta,\ v_3 \in H^2_0(\omega)\}$$

which are defined in Theorem 2 are equivalent.

Proof. Let us sketch how it is articulated. Let us introduce the linear mapping j from the space $(H^1_0(\omega))^2 \times H^2_0(\omega)$ into $\mathbf{V}^{KL}$ and such that :

$$\forall v = (v_\alpha, v_3) \in (H^1_0(\omega))^2 \times H^2_0(\omega) \to j(v) = \mathbf{v}^{KL} \in V^{KL}.$$

One has clearly the following properties:
i) j is linear, ii) j is bijective, j is continuous.

Thus, from the Banach endomorphism Theorem [25, Theorem p. 205] one can ensure that j^{-1} is also continuous. Therefore, the equivalence between the norms in $(H^1_0(\omega))^2 \times H^2_0(\omega)$ and the one in V^{KL} is proved. In order to complete the proof it is sufficient to observe that the three dimensional strain (in Ω^η) of a Kirchhoff-Love displacement field is restricted to its inplane components. And because of the expression given by Budiansky-Sanders, the two first norms given in Theorem 2 are clearly equivalent as soon as $\eta < \min_{m\in\omega}(\frac{1}{|R_1(m)|}, \frac{1}{|R_1(m)|})$ and this is always possible because η can be chosen as small as one likes. In fact, one can notice that the result is independent of η and only depends on the surface ω. □

§4. Abstract formulation for a shell model

4.1. A semilinear formulation

Let us consider a vector field $\mathbf{v} \in \mathbf{V}^{KL}$. Let $f(\gamma(\mathbf{v}), \varrho(\mathbf{v}))$ be a scalar function which satisfies the following properties:

i) f is convex and C^1 with respect to γ and ϱ, which are the change of metric and of curvature tensors,
ii) if $||\mathbf{v}||_{\mathbf{V}^{KL}} \to \infty$ then $\lim \int_\omega f(\gamma(\mathbf{v}), \varrho(\mathbf{v})) = \infty$, (coerciveness).

Let us define the elastic energy by:

$$\mathbf{v} \in \mathbf{V}^{KL} \to J(\mathbf{v}) = \int_\omega f(\gamma(\mathbf{v}), \varrho(\mathbf{v})) - F(\mathbf{v}),$$

where $F(.)$ is a linear and continuous form on the space $\mathbf{V}^{KL}$. Then the following problem has a solution in the space $\mathbf{V}^{KL}$.

$$\min_{\mathbf{v}\in\mathbf{V}^{KL}} J(\mathbf{v}) \qquad (7)$$

Furthermore it is unique as soon as J is stictly convex. Let us give an example. We set ($A^{\alpha\beta\mu\lambda}$ and $B^{\alpha\beta\mu\lambda}$ are two positively definite tensors representing respectively the membrane and the bending stiffness of the shell):

$$f(\gamma(\mathbf{v}),\varrho(\mathbf{v}))=\frac{1}{2}\sum_{\alpha\beta\mu\lambda\in\{1,2\}}A^{\alpha\beta\mu\lambda}\gamma_{\alpha\beta}(\mathbf{v})\gamma_{\mu\lambda}(\mathbf{v})+B^{\alpha\beta\mu\lambda}\varrho_{\alpha\beta}(\mathbf{v})\varrho_{\mu\lambda}(\mathbf{v})$$
$$+a|\textstyle\sum_\mu \gamma^\mu_\mu(\mathbf{v})|+b|\sum_\mu \varrho^\mu_\mu(\mathbf{v})|+c|\sum_\mu \kappa^\mu_\mu(\mathbf{v})|.$$

For $a=b=c=0$ one obtains the classical S^t Venant-Kirchhoff model [7]. But as soon as one of the previous coefficient is different from zero, one obtains a nonlinear model similar to what happens in material with a locking effect (non newtonian constitutive relationship). Let us explain why. Let us consider the particular case where $a\neq 0$ and $b=c=0$. The optimality condition traducing that $J(\mathbf{u})$ is minimum is a variational inequality. It can be written:

$$\begin{cases}\mathbf{u}\in\mathbf{V}^{KL},\ \forall\mathbf{v}\in\mathbf{V}^{KL},\ \displaystyle\int_\omega A^{\alpha\beta\mu\lambda}\gamma_{\alpha\beta}(\mathbf{u})\gamma_{\mu\lambda}(\mathbf{v}-\mathbf{u})+\int_\omega B^{\alpha\beta\mu\lambda}\varrho_{\alpha\beta}(\mathbf{u})\varrho_{\mu\lambda}(\mathbf{v}-\mathbf{u})\\ a\displaystyle\int_\omega|\sum_{\alpha=1,2}\gamma^\alpha_\alpha(\mathbf{v})|-a\int_\omega|\sum_{\alpha=1,2}\gamma^\alpha_\alpha(\mathbf{u})|\geq F(v-u).\end{cases}\tag{8}$$

Then choosing $\mathbf{v}=0$ in (8), we obtain, because of the equivalence between the various norm on the space $\mathbf{V}^{KL}$:

$$c_0||\mathbf{u}||^2_{\mathbf{V}^{KL}}+a\int_\omega|\sum_{\alpha=1,2}\gamma^\alpha_\alpha(\mathbf{u})|\leq F(\mathbf{u}).\tag{9}$$

Let us assume that for instance the force applied can be written as follows:

$$F(\mathbf{v})=\int_\omega p\gamma^\alpha_\alpha(\mathbf{v})+F_s(\mathbf{v}),\tag{10}$$

where p is the surface pressure and F_s the shearing component of the applied force. We set : $F_s(\mathbf{v})\leq c_s||\mathbf{v}||_{\mathbf{V}^{KL}}$. A simple upper bounding of the right handside of (9) leads to:

$$[c_0||\mathbf{u}||_{\mathbf{V}^{KL}}-c_s]||\mathbf{u}||_{\mathbf{V}^{KL}}+(a-||p||_{0,\infty,\omega})||\sum_{\alpha=1,2}\gamma^\alpha_\alpha(\mathbf{u})||_{0,1,\omega}\leq 0.\tag{11}$$

Therefore if $c_s=0$ (there is no shear force) and $||p||_{0,\infty,\omega}\leq a$, one has $\mathbf{u}=0$. It means that the shell is not sensitive to a pure -small enough- pressure. It is possible to get rid of this locking by replacing the term $|\sum_{\alpha=1,2}\gamma^\alpha_\alpha(\mathbf{u})|$ by $|\sum_{\alpha=1,2}\gamma^\alpha_\alpha(\mathbf{u})|^q$ where $q>1$. But the mechanical interest of the initial expresssion is also important.

Remark 2. There can exist displacement fields $\mathbf{v}$ such that $\gamma(\mathbf{v})=\gamma_{\alpha\beta}\mathbf{a}^\alpha\otimes\mathbf{a}^\beta=0$ and $\varrho(\mathbf{v})\neq 0$. They are called inextensional. For instance the bending movement of a plate is inextensional. There are also inextensional movements for shells with negative Gaussian curvature [15], [21], [10]. In fact the energy introduced above is not sufficient for eliminating such inextensional displacements. This suggests to use a nonlinear expression of the strain following the ideas of the first section.

4.2. The nonlinear metric for a shell model

Let us consider the exact expression of the new tangent vectors -say $\mathbf{a}^{x_3'}_\alpha$- to the deformed surface $\omega\times\{x_3\}$ by the mapping $(m,x_3)\to(m+\mathbf{v}^{KL}(m,x_3))$ where $\mathbf{v}^{KL}$ is a Kirchhoff-Love vector field defined in (6). The new expression of the full change of metric is now:

$$g^{x_3'}_{\alpha\beta}=g^{x_3}_{\alpha\beta}+2\chi_{\alpha\beta}+2x_3\psi_{\alpha\beta}+2x_3^2\Delta_{\alpha\beta},\tag{12}$$

where we used the following notations (see Theorem 1 for the definition of ϱ, γ and κ):

$$\left\{\begin{array}{l}\chi_{\alpha\beta} = \gamma_{\alpha\beta} + \dfrac{1}{2}\theta_\alpha\theta_\beta + \dfrac{1}{2}(v^\lambda_{|\alpha} - v_3 b^\lambda_\alpha)(v_{\lambda|\beta} - v_3 b_{\lambda\beta}),\\[2ex] \psi_{\alpha\beta} = \varrho_{\alpha\beta} + \dfrac{1}{2}(\theta^\lambda_{|\alpha}(v_{\lambda|\beta} - v_3 b_{\lambda\beta}) + \theta^\lambda_{|\beta}(v_{\lambda|\alpha} - b_{\lambda\alpha}v_3) - \theta^\lambda(\theta_\alpha b_{\lambda\beta} + \theta_\beta b_{\lambda\alpha}))\\[2ex] \Delta_{\alpha\beta} = \kappa_{\alpha\beta} + \dfrac{1}{2}\theta^\lambda_{|\alpha}\theta_{\lambda|\beta} + \dfrac{1}{2}\theta^\lambda\theta^\mu b_{\lambda\alpha}b_{\mu\beta}\ .\end{array}\right. \tag{13}$$

We set: $\chi = \chi^\beta_\alpha \mathbf{a}_\beta \otimes \mathbf{a}^\alpha$, $\psi = \psi^\beta_\alpha \mathbf{a}_\beta \otimes \mathbf{a}^\alpha$, $\Delta = \Delta^\beta_\alpha \mathbf{a}_\beta \otimes \mathbf{a}^\alpha$. It is worth noting that one has the following identity ($-\frac{1}{R_1}$ and $-\frac{1}{R_2}$ are the two eigenvalues of the curvature operator $\frac{\partial \mathbf{N}}{\partial m} = b^\beta_\alpha \mathbf{a}_\beta.\mathbf{a}^\alpha$ such that: $Tr_2(\frac{\partial \mathbf{N}}{\partial m}) = -(\frac{1}{R_1} + \frac{1}{R_2})$ and $\mathbf{v_t} = v^\alpha \mathbf{a}_\alpha$):

$$Tr_2(\chi) = \chi^\mu_\mu = div(\mathbf{v_t}) + Tr_2(\frac{\partial \mathbf{N}}{\partial m})v_3 + \frac{1}{2}|\theta|^2 + \frac{1}{2}|\Omega|^2, \tag{14}$$

where $\Omega = \Omega^\alpha_\beta \mathbf{a}_\alpha \otimes \mathbf{a}^\beta$, and: $\Omega^\alpha_\beta = v^\alpha_{|\beta} - v_3 b^\alpha_\beta$. One has also:

$$Tr_2(\Delta) = \Delta^\mu_\mu = Tr_2(\kappa) + \frac{1}{2}|\nabla_c\theta|^2 + \frac{1}{2}(C\theta.\theta), \tag{15}$$

where C is the *gaussian curvature* operator which can be explicited, in a matrix form in the principal axis of curvature, by [23]:

$$C = \begin{pmatrix} \dfrac{1}{R_1^2} & 0 \\ 0 & \dfrac{1}{R_2^2} \end{pmatrix} \geq 0 \tag{16}$$

and the notation $\nabla_c.$ is the covariant gradient operator [23] such that:

$$|\nabla_c\theta|^2 = g^{\alpha\beta}\theta^\lambda_{|\alpha}\theta_{\lambda|\beta} \tag{17}$$

Let us now consider the principal change of volume -say $\delta(v)$- through the thickness of the shell. It is defined as the trace of the change of metric tensor through the thickness of the shell stucture. Therefore, one introduces the following definition:

$$\delta v(\mathbf{v}) = Tr_2(\chi^\alpha_\alpha + x_3\psi^\alpha_\alpha + x_3^2\Delta^\alpha_\alpha)$$

or else:

$$\delta v(\mathbf{v}) = div(\mathbf{v_t}) - b^\alpha_\alpha v_3 + \frac{|\theta|^2 + |\Omega|^2}{2} + x_3(\varrho^\alpha_\alpha + \theta^\mu_{|\alpha}\Omega^\alpha_\mu - \theta^\alpha\theta^\mu b_{\alpha\mu}) + x_3^2(\kappa^\alpha_\alpha + \frac{|\nabla_c\theta|^2 + (C\theta.\theta)}{2}). \tag{18}$$

A very important point is that the last term (in x_3^2) contains some compactness facilities on θ due to the first order derivative of that term. Nevertheless, the term Ω is much more difficult to handle. It corresponds -for the symmetrical part- to the membrane strain and to the rotation around the normal $\mathbf{N}$ for the unsymmetrical component. From a mechanical point of view it can be suggested to cancel these terms compared to the rotation θ because of the flexibility of thin shells and therefore it is suggested to approximate the principal change of volume by the next one:

$$\delta v(\mathbf{v}) = div(\mathbf{v_t}) + Tr_2(\frac{\partial \mathbf{N}}{\partial m})v_3 + \frac{1}{2}|\theta|^2 + x_3(Tr_2(\varrho) + (\frac{\partial \mathbf{N}}{\partial m}\theta.\theta)) + x_3^2(Tr_2(\kappa) + \frac{1}{2}(|\nabla_c\theta|^2 + (C\theta.\theta))). \tag{19}$$

The integration of this relation on the open set Ω^η occupied by the shell, leads to the following expression (using Stokes formulae [23] and (22)):

$$\delta V = 2\eta \int_\omega [Tr_2(\frac{\partial \mathbf{N}}{\partial m})v_3 + \frac{1}{2}|\theta|^2] + \frac{\eta^3}{3}\int_\omega |\nabla_c\theta|^2 + (C\theta, \theta) + 2Tr_2(\kappa). \tag{20}$$

Consequently, the other fundamental forms of the surface ω are also modified in order to take into account this approximation ($\Omega_\alpha^\beta \simeq 0$). We set ($\Delta$ is unchanged!):

$$\chi(\mathbf{v}) \simeq \gamma(\mathbf{v}) + \frac{1}{2}\theta \otimes \theta, \quad \psi \simeq \varrho + \frac{1}{2}[\frac{\partial \mathbf{N}}{\partial m}\theta \otimes \theta + \theta \otimes \frac{\partial \mathbf{N}}{\partial m}\theta], \tag{21}$$

These approximate expressions will be used in the following of the paper. For instance one has with these new definitions:

$$Tr_2(\Delta) = Tr_2(\frac{\partial \mathbf{N}}{\partial m}\psi) + \frac{1}{2}|\nabla_c\theta|^2 - Tr_2(\frac{\partial \mathbf{N}}{\partial m}\chi\frac{\partial \mathbf{N}}{\partial m}). \tag{22}$$

4.3. A nonlinear shell model in elasticity

The first point is to observe the energetical characterization of the movement.

Theorem 4. *Let us consider a vector field* $\mathbf{v} \in V_0 = (H_0^1(\omega))^2 \times H_0^2(\omega) \equiv \mathbf{V}^{KL}$ *which is such that on the surface* ω *one has:*

$$\chi(\mathbf{v}) = \psi(\mathbf{v}) = \delta V(\mathbf{v}) = 0.$$

Furthermore we assume that the surface ω *is smooth enough; for instance* C^3*, which enables one to make sense to the expressions which appear in the shell models.Then:* $\mathbf{v} = 0$.

Proof. From the relation (22), and because $\chi = \psi = 0$, we deduce that:

$$\nabla_c\theta = 0 \text{ and therefore that } \forall \alpha, \beta \in \{1, 2\},\ \theta_{\alpha|\beta} = 0 \text{ on } \omega.$$

The trace of θ can be defined along each coordinate line (trace Theorem in $H^1(\omega)$). Thus one has for instance [23]: $\frac{d||\theta||^2}{d\xi^\alpha} = 2\theta.\frac{d\theta}{d\xi^\alpha} = 0$. This implies that θ has a constant length on the surface ω, and because of the boundary conditions: $\theta = 0$ inside ω. Finally we can ensure that $\gamma = \varrho = 0$ (which are the linearized expressions of the change of metric and of curvature operators). From Theorem 1, we can conclude the proof of the Theorem 4. □

Remark 3. Theorem 4 can be seen as a nonlinear version of the Koiter and Budiansky-Sanders result. A nonlinear membrane tensor and also a non linear expression of the change of curvature has been considered. Only the nonlinear contribution of $\Omega^\alpha_\beta = v^\alpha_\beta - b^\alpha_\beta v_3$, has been omitted here compared to the full expressions. In this theorem, it has been necessary to take into account the term representing the change of volume of the shell. But many other equivalent conditions could have been introduced. The one used represents a mechanical quantity and therefore it is justified to introduce it in an elastic energy functional.

Let us give an example of a nonlinear elastic energy (we use notations introduced previously).

$$J(\mathbf{v}) = \frac{1}{2}\int_\omega A^{\alpha\beta\mu\lambda}\chi_{\alpha\beta}(\mathbf{v})\chi_{\mu\lambda}(\mathbf{v}) + B^{\alpha\beta\mu\lambda}\psi_{\alpha\beta}(\mathbf{v})\psi_{\mu\lambda}(\mathbf{v}) + a\delta V(\mathbf{v}) - F(\mathbf{v}). \tag{23}$$

Such a functional is not convex and furthermore it is not even obvious that it is bounded from below (because of the term δV). The coerciveness is not clear. Hence the existence of a stable solution must overcome these difficulties.

§5. Existence of a stable solution to a nonlinear shell model

The method that we discuss hereafter is based on a minimizing sequence of the elastic energy in the functional space $\mathbf{W} \equiv \mathbf{V}^{KL}$ defined by:

$$\mathbf{W} = \{\mathbf{v} = (v_i),\ \text{such that } v_\alpha \in H^1_0(\omega),\ v_3 \in H^2_0(\omega)\}.$$

The norm is the one induced by the definition:

$$\mathbf{v} \in \mathbf{W} \to ||\mathbf{v}||_W = \sum_{\alpha=1,2} ||v_\alpha||_{1,2,\omega} + ||v_3||_{2,2,\omega}.$$

The elastic energy is for instance defined by:

$$J(\mathbf{v}) = \frac{1}{2}\int_\omega A^{\alpha\beta\lambda\mu}\chi_{\alpha\beta}\chi_{\lambda\mu} + B^{\alpha\beta\lambda\mu}\psi_{\alpha\beta}\psi_{\lambda\mu} + 2\eta a\int_\omega [\chi^\alpha_\alpha + \frac{\eta^2}{3}\Delta^\alpha_\alpha] - F(\mathbf{v}). \tag{24}$$

In a first step we prove that a minimizing sequence is upper bounded in $\mathbf{W}$ and then, (after extracting a weakly convergent sub-sequence), we prove the semi-continuity of the elastic energy. Finally the existence of a minimizer is established. But the uniqueness is generally false.

5.1. Definition of a minimizing sequence

Let us consider a minimizing sequence of elements $\mathbf{v}^n \in \mathbf{W}$ which is such that:

$$J(\mathbf{v}^{n+1}) \le J(\mathbf{v}^n),\ \text{ and } \lim_{n\to\infty} = \inf_{\mathbf{v}\in\mathbf{W}} J(\mathbf{v}).$$

5.2. The minimizing sequence is upper bounded

First of all, let us point out that the result is not obvious because the energy is not coercive (one can not ensure directly that: $\lim_{||\mathbf{v}||_{\mathbf{W}}\to\infty},\ J(\mathbf{v}) \to \infty$). Therefore let us assume that the minimizing sequence could be such that:

$$||\mathbf{v}^n||_{\mathbf{W}} = \sqrt{\lambda^n} \to \infty, \text{ when } n \to \infty.$$

Let us set:

$$\tilde{\mathbf{v}}^n = \frac{\mathbf{v}^n}{\sqrt{\lambda^n}}.$$

The sequence $\tilde{\mathbf{v}}^n$ is clearly bounded in the space $\mathbf{W}$ and thus one can extract a subsequence denoted by: $\tilde{\mathbf{v}}^{n'}$ and such that:

$$\tilde{\mathbf{v}}^{n'} \rightharpoonup \tilde{\mathbf{v}}^* \text{ in } \mathbf{W} \text{ weak.}$$

Let us now divide the energy J by $\sqrt{\lambda^n}$. From the expression of the membrane strain χ, we deduce that $(\lambda^{n'})^{1/4}\theta(\tilde{\mathbf{v}}^{n'})$ is bounded in the space $L^4(\omega)$, but also, from the last term, in $(H_0^1(\omega))^2$. Thus: $(\lambda^{n'})^{1/4}\theta(\tilde{\mathbf{v}}^{n'}) \rightharpoonup \mathbf{h}^*$ in $H_0^1(\omega)$ weak and in $L^4(\omega)$ strong. From the last term in the energy J which has been divided by $\lambda^{n'}$, and because $|\nabla_c\theta|_{1,\omega} + |\theta|_{0,\omega}$ is a norm on θ equivalent to the one of the space $(H_0^1(\omega))^2$, (see Theorem 4 and (22)), one obtains that (one has $C\theta(\tilde{\mathbf{v}}^{n'}).\theta(\tilde{\mathbf{v}}^{n'}) \geq 0$):

$$\lim_{n'\to\infty} \theta(\tilde{\mathbf{v}}^{n'}) \to \theta(\tilde{\mathbf{v}}^*) = 0 \text{ in } (H_0^1(\omega))^2 \text{ strong when } n' \to \infty. \tag{25}$$

Therefore, from the previous results, one obtains:

$$\begin{cases} \lim_{n'\to\infty} \chi(\tilde{\mathbf{v}}^{n'}) = \lim_{n'\to\infty} \gamma(\tilde{\mathbf{v}}^{n'}) + \dfrac{\sqrt{\lambda^{n'}}}{2}\theta(\tilde{\mathbf{v}}^{n'}) \otimes \theta(\tilde{\mathbf{v}}^{n'}) = \gamma(\tilde{\mathbf{v}}^*) + \dfrac{1}{2}h^* \otimes h^* = 0, \\ \lim_{n'\to\infty} \psi(\tilde{\mathbf{v}}^{n'}) = \varrho(\tilde{\mathbf{v}}^*) + \dfrac{1}{2}(h^* \otimes \dfrac{\partial \mathbf{N}}{\partial m}h^* + \dfrac{\partial \mathbf{N}}{\partial m}h^* \otimes h^*) = 0. \end{cases}$$

In fact, one can check that the second relation is a consequence of the first one and (25). Assuming that there is no generalized Monge-Ampère displacement field on the surface ω, one can conclude that $h^* = 0$. Finally from Theorem 2 one obtains $\mathbf{v}^* = 0$. Therefore, we proved that: $\lim_{n'\to\infty} ||\tilde{\mathbf{v}}^{n'}||_{\mathbf{W}} = 0$ which is a contradiction with $||\tilde{\mathbf{v}}^{n'}||_{\mathbf{W}} = 1$. Thus, the sequence $\mathbf{v}^n$ is bounded in the space $\mathbf{W}$. □

5.3. The semi-continuity of the energy

Because the minimizing sequence is bounded in the space $\mathbf{W}$, one can extract a subsequence denoted by $\mathbf{v}^{n'}$ such that:

$$\mathbf{v}^{n'} \rightharpoonup \mathbf{v}^* \text{ in } \mathbf{W} \text{ weak.}$$

But the component of $\theta(\mathbf{v}^{n'})$ are also bounded in the space $H_0^1(\omega)$. From the compact embedding from $H_0^1(\omega)$ into $L^4(\omega)$, one can state that (for $\alpha = 1, 2$):

$$\lim_{n'\to\infty} \theta_\alpha(\mathbf{v}^{n'}) = \theta(\mathbf{v}^*) \text{ in } L^4(\omega) \text{ strong.}$$

Thus the term:

$$\mathbf{v} \in \mathbf{W} \to \frac{1}{2}\int_\omega A^{\alpha\beta\lambda\mu}\chi_{\alpha\beta}\chi_{\lambda\mu} + \frac{1}{2}\int_\omega A^{\alpha\beta\lambda\mu}\psi_{\alpha\beta}\psi_{\lambda\mu}$$

is clearly weakly lower semi-continuous. The other terms being convex the weakly lower semi-continuity is true. Finally it has been proved that:

$$J(\mathbf{v}^*) \leq \inf_{\mathbf{v}\in\mathbf{W}} J(\mathbf{v}) \leq J(\mathbf{w})\ \forall \mathbf{w} \in \mathbf{W}.$$

One can conclude that the energy J admits at least one stable minimum in the space $\mathbf{W}$ as soon as the following generalized Monge-Ampère condition is satisfied [12]:

$$If\ \exists(\mathbf{h}, \mathbf{v}) \in L^4(\omega)^2 \times \mathbf{W},\ s.t.\ \gamma(\mathbf{v}) + \frac{1}{2}\mathbf{h} \otimes \mathbf{h} = 0\ and\ \theta(\mathbf{v}) = 0\ then\ \mathbf{h} = 0.$$

There are simple examples of shells for which this property is not satisfied and other, for which it is satisfied.

§6. Conclusion

It has been proved in this paper, that the existence of solution for a nonlinear shell model is mainly dependant on two basic properties:

i) The dominant component of the elastic energy should satisfy an energy invariance under nonlinear rigid body motions and there should no generalized Monge-Ampère displacement field on the medium surface ω. This property enables one to bound a minimizing sequence.
ii) The elastic energy should be lower semi-continuous in an ad'hoc functional space.
A large class of functionals satisfying these properties can be handled by the method described here. It appears to have a more realistic mechanical interpretation than the two other stategies met in the litterature (but which have their own advantages): the one of J. Ball [1] which can't ensure that the equilibrium is satisfied by a stable solution because of a restriction on the determinant of the Gramm matrix and the Γ convergence which is a relaxation method and which is still far from the mechanical models.

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